AN ELEMENTARY INTRODUCTION TO SIEGEL MODULAR FORMS

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Abstract. Siegel modular forms can be thought of as modular forms in more than one variable. Introduced in the 1930's by Siegel in his analytic study of quadratic forms, they nowadays occur naturally in many unexpected places. We develop the basic theory from scratch, assuming only that the listener/reader has seen some rudiments of modular forms in one variable. We list some of the many applications and indicate some fundamental questions that are still open.

1. Definitions

1.1. **Classical modular forms.** Recall the major ingredients in the classical definition of modular forms: the group $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$ acts on the upper half-plane

$$\mathscr{H}_1 := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$$

via fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}.$$

We define a *modular form* of weight $k \in \mathbb{Z}$ and level Γ_1 to be a holomorphic function $f : \mathscr{H}_1 \to \mathbb{C}$ satisfying

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1,$$

as well as the following growth condition (called *holomorphicity at infinity*): there exists $A > 0$ such that have

$$|f(x + iy)| \ll y^A \quad \text{for } y \geq 1.$$

Suppose now that we want to generalize this to functions of more than one complex variable. We need to generalize Γ_1, $\mathscr{H}_1$, the action, the modular condition and (a priori) the growth condition.

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1.2. **Several variables.** In order to motivate the following definitions, one needs to discuss the relation between quadratic forms and modular forms. Since that is done most naturally after introducing Fourier expansions, we postpone this motivational discussion until §2.2.

Let $g \geq 1$ be an integer.

Set

$$J := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

We define the symplectic group

$$\begin{aligned} \Gamma_g := \mathrm{Sp}_{2g}(\mathbb{Z}) &= \{\gamma \in \mathrm{GL}_{2g}(\mathbb{Z}) : {}^t\gamma J \gamma = J\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2g}(\mathbb{Z}) : {}^tAC = {}^tCA, {}^tBD = {}^tDB, {}^tAD - {}^tCB = I_g \right\}. \end{aligned}$$

Next we define the *Siegel upper half space*

$$\begin{aligned} \mathscr{H}_g &:= \{\text{symmetric } g \times g \text{ complex matrices with positive definite imaginary part}\} \\ &= \{\tau \in \mathrm{Mat}_{g\times g}(\mathbb{C}) : {}^t\tau = \tau, \mathrm{Im}(\tau) > 0\}. \end{aligned}$$

The natural way to attempt defining an action of Γ_g on $\mathscr{H}_g$ is then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tau := (A\tau + B)(C\tau + D)^{-1}.$$

Of course, one must first show that $C\tau + D$ is invertible (this was not much of an issue in the classical case $g = 1$). The proof can be found for instance in Proposition 1, Section 1 of (Kli90).

In the $g = 1$ case, some mileage can be obtained from working explicitly with a fundamental domain for the action of Γ_1 on $\mathscr{H}_1$. In the general case, such an approach would be very cumbersome. Although fundamental domains were constructed by Siegel for all g, already for $g = 2$ their boundary is made of 28 algebraic surfaces!

It remains to generalize the modularity condition. This requires an extra step: instead of being a mere integer, the *weight* will now be a rational representation

$$\rho : \mathrm{GL}(g, \mathbb{C}) \to \mathrm{GL}(V),$$

where V is a finite-dimensional complex vector space. We will talk more about this notion of weight at the end of this section.

1.3. **Siegel modular forms.** To simplify the notation, we introduce the *slash operator*: for any $\gamma \in \Gamma_g$, set

$$(f|\gamma)(\tau) := (\rho(C\tau + D))^{-1} f(\gamma\tau).$$

We say that a holomorphic function $f : \mathscr{H}_g \to V$ is a *Siegel modular form* of degree g, weight ρ and level Γ_g if[1]

$$f|\gamma = f \quad \text{for all } \gamma \in \Gamma_g.$$

We denote the vector space of Siegel modular forms of weight ρ by $M_\rho(\Gamma_g)$, and we feel free to drop the level when it is understood. Note that a Siegel modular form of degree g is a holomorphic function in $\frac{g(g+1)}{2}$ complex variables.

[1]If $g = 1$, we must also impose the condition of holomorphicity at infinity given in §1.1. This is not necessary when $g > 1$, thanks to the Köcher principle (see §2.3).

In arithmetic applications one often works with other levels, such as

$$\Gamma_g(N) := \{\gamma \in \Gamma_g : \gamma \equiv I_{2g} \pmod N\}.$$

One advantage is that if $N \geq 3$, $\Gamma_g(N)$ acts *freely* on $\mathscr{H}_g$ (Serre's Lemma, see (Ser61)).

1.4. **Weights.** We close this section with some remarks on weights for Siegel modular forms. If a weight ρ decomposes as a direct sum of representations

$$\rho = \rho_1 \oplus \rho_2,$$

then it is easily seen that

$$M_\rho = M_{\rho_1} \oplus M_{\rho_2}.$$

Therefore we may assume without loss of generality that weights are irreducible representations.

In the case $g = 1$, we are looking at irreducible representations of $\mathbb{C}^\times$; these representations must be one-dimensional (because $\mathbb{C}^\times$ is abelian), and since they are by assumption rational the only possibilities are

$$z \mapsto z^k$$

for some integer k. Hence we recover precisely the notion of weight of a classical modular form.

Suppose now $g = 2$. Let std denote the standard representation of $\mathrm{GL}_2(\mathbb{C})$, i.e. std is the identity map from $\mathrm{GL}_2(\mathbb{C})$ to $\mathrm{GL}(\mathbb{C}^2)$. Then to any pair of integers (j, k) we can associate the following representation of $\mathrm{GL}_2(\mathbb{C})$:

$$\rho_{j,k} := \mathrm{Sym}^j(\mathrm{std}) \otimes \det(\mathrm{std})^k.$$

These are actually *all* the irreducible representations of $\mathrm{GL}_2(\mathbb{C})$.

In general, irreducible representations of $\mathrm{GL}_g(\mathbb{C})$ are classified by g-tuples of integers, called *highest weight vectors* (this correspondence is more complicated to describe – see (FH91)). If we only consider weights of the form

$$\rho = \det(\mathrm{std})^k,$$

then we get the notion of *scalar-valued* Siegel modular forms.

This gives us two graded rings:

$$\begin{aligned} M^{\mathrm{scalar}}(\Gamma_g) &:= \bigoplus_{k\in\mathbb{Z}} M_k(\Gamma_g) \\ M(\Gamma_g) &:= \bigoplus_{\rho \text{ irrep}} M_\rho(\Gamma_g). \end{aligned}$$

2. Some fundamental results and open questions

2.1. **Fourier expansion.** When $g = 1$, a classical modular form satisfies in particular the equality

$$f(\tau + 1) = f(\tau) \quad \text{for all } \tau \in \mathscr{H}_1.$$

In other words, f is periodic of period 1, and as such has a Fourier expansion

$$f(\tau) = \sum_{n\in\mathbb{Z}} a(n) e^{2\pi i n\tau},$$

where

$$a(n) = \int_{x \bmod 1} f(\tau) e^{-2\pi i n \tau} dx, \quad \tau = x + iy.$$

One sets traditionally $q = e^{2\pi i \tau}$ and calls the result the *q-expansion* of f:

$$f(q) = \sum_{n \in \mathbb{Z}} a(n) q^n.$$

The condition that f be holomorphic at infinity implies that $a(n) = 0$ for negative n.

For general g, we can consider the following $2g \times 2g$ matrix:

$$\gamma = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix},$$

where S is a symmetric $g \times g$ matrix with integer entries. It is easily seen that $\gamma \in \Gamma_g$.

If f is a Siegel modular form, it satisfies

$$(f|\gamma)(\tau) = f(\tau) \quad \text{for all } \tau \in \mathscr{H}_g, \gamma \in \Gamma_g.$$

But for our cleverly chosen γ, this relation becomes

$$f(\tau + S) = f(\tau) \quad \text{for all } \tau \in \mathscr{H}_g.$$

This means in particular that f is a periodic function in its $g(g+1)/2$ variables τ_{ij}, where $\tau = (\tau_{ij})$.

Setting $q_{ij} = e^{2\pi i \tau_{ij}}$, we get a multivariate q-expansion of the form

$$f(q_{11}, \ldots, q_{gg}) = \sum_{n_{11}, \ldots, n_{gg} \in \mathbb{Z}} a(n_{11}, \ldots, n_{gg}) q_{11}^{n_{11}} \ldots q_{gg}^{n_{gg}}. \tag{1}$$

This notation obscures some of the features of the Fourier expansion, so people prefer to use a different notation, which we explain now.

Let N be a symmetric $g \times g$ matrix; we say N is *half-integral* if $2N$ has integral entries with even integers on the diagonal. Given a half-integral matrix N, and $\tau \in \mathscr{H}_g$, one easily checks that

$$\mathrm{Tr}(N\tau) = \sum_{i=1}^{g} N_{ii}\tau_{ii} + 2 \sum_{1 \le i < j \le g} N_{ij}\tau_{ij}.$$

Running over all half-integral matrices N will yield all possible linear combinations of the τ_{ij}'s with integer coefficients, which is exactly what our expansions are made of! Therefore we can write the Fourier expansion as

$$f(\tau) = \sum_{N \text{ h.-i.}} a(N) e^{2\pi i \,\mathrm{Tr}(N\tau)},$$

which is equivalent to but more compact than (1), and looks very much like the 1-variable case.

Once again, we can recover the Fourier coefficients using the formula

$$a(N) = \int_{x \bmod 1} f(\tau) e^{-2\pi i \,\mathrm{Tr}(N\tau)} dx, \quad \tau = x + iy,$$

where dx is the standard measure on $\mathbb{R}^g$.

If $u \in \mathrm{GL}_g(\mathbb{Z})$, then

$$a({}^t u N u) = \rho({}^t u) a(N).$$

This can be used to show that $M_k = 0$ if $kg \equiv 1 \pmod 2$.

2.2. **Quadratic forms and theta series.** As a motivation for the definitions given in §1.2, we take a quick detour and discuss the relation between quadratic forms and modular forms. Since entire books can (and have) been written on the subject, we will focus on some examples.

Consider the quadratic form $Q(a,b,c,d) = a^2 + b^2 + c^2 + d^2$. Set

$$\theta_Q(\tau) := \sum_{a,b,c,d \in \mathbb{Z}} q^{a^2+b^2+c^2+d^2}.$$

It can be shown that θ is a modular form of weight 2 (and level $\Gamma_0(4)$), and it is easy to see that its n-th Fourier coefficient is precisely the number of ways of writing n as a sum of 4 squares, where the order and the sign matter. In other words, if we write

$$\theta_Q(\tau) = \sum_{n \geq 0} a(n) q^n = 1 + 8q + 24q^2 + \ldots,$$

then $a(n)$ is the number of representations of n by the quadratic form $Q(a,b,c,d)$. For instance:

$$\begin{aligned}
a(0) = 1: \quad 0 &= 0^2 + 0^2 + 0^2 + 0^2 \\
a(1) = 8: \quad 1 &= 1^2 + 0^2 + 0^2 + 0^2 = 0^2 + 1^2 + 0^2 + 0^2 \\
&= 0^2 + 0^2 + 1^2 + 0^2 = 0^2 + 0^2 + 0^2 + 1^2 \\
&= (-1)^2 + 0^2 + 0^2 + 0^2 = 0^2 + (-1)^2 + 0^2 + 0^2 \\
&= 0^2 + 0^2 + (-1)^2 + 0^2 = 0^2 + 0^2 + 0^2 + (-1)^2.
\end{aligned}$$

The moral of the story is that a positive-definite integer-valued quadratic form Q gives rise to a modular form whose q-expansion is the generating series for the number of representations by Q.

Similarly, if we define

$$\theta_Q^{(2)}(\tau) = \sum_{s \in \mathrm{Mat}_{2\times 4}(\mathbb{Z})} e^{2\pi i \operatorname{Tr}(Q\, {}^t s \tau s)},$$

then this turns out to be a Siegel modular form of degree 2 and weight 2 (of a certain level). Its Fourier expansion is

$$\theta_Q^{(2)}(\tau) = \sum_{N \geq 0, \text{h.-i.}} \alpha(N,Q) e^{2\pi i \operatorname{Tr}(N\tau)},$$

where

$$\alpha(N,Q) = \#\{r \in \mathrm{Mat}_{4\times 2}(\mathbb{Z}) : N = {}^t r Q r\}$$

is the number of representations of the quadratic form N by the quadratic form Q.

In general, given a integers g and ℓ and a positive-definite integer-valued quadratic form Q in ℓ variables,

$$\theta_Q^{(g)}(\tau) = \sum_{s \in \mathrm{Mat}_{g\times \ell}(\mathbb{Z})} e^{2\pi i \operatorname{Tr}(Q\, {}^t s \tau s)}$$

is a Siegel modular form of degree g and weight $\ell/2$, and its Fourier expansion is the generating series for the number of representations of positive-definite integer-valued quadratic forms N by the fixed form Q. For details, see Section 8 in (Kli90).

2.3. **The Köcher principle.** This is a general property satisfied by several kinds of automorphic functions in more than one variable (not just Siegel modular forms). In our case, it can be stated as follows:

Theorem (Köcher principle)**.** *Let f be a Siegel modular form of weight ρ and level Γ_g, with $g > 1$. Then f is bounded on any subset of $\mathscr{H}_g$ of the form*

$$\{\tau \in \mathscr{H}_g : \operatorname{Im}(\tau) > c \cdot I_g\}, \quad \textit{where } c > 0.$$

For the proof, see Theorem 1, Section 4 of (Kli90).

A consequence of the Köcher principle is that if $g > 1$, a Siegel modular form has Fourier expansion

$$f(\tau) = \sum_{N \geq 0,\ \text{h.-i.}} a(N) e^{2\pi i \operatorname{Tr}(N\tau)}.$$

With some more work, this can be used to bound the dimension of the space $M_k(\Gamma_g)$: there is a constant c_g (depending only on g) such that

$$\dim M_k(\Gamma_g) \leq c_g k^{g(g+1)/2} \quad \text{for } k > 0.$$

2.4. **The Siegel Φ-operator.** An important feature of the theory of Siegel modular forms is the Φ-operator, which maps forms of degree g into forms of degree $g - 1$. It is possible to define it in full generality, but for simplicity of the exposition we restrict ourselves to scalar-valued forms.

We define the linear map

$$\Phi : M_k(\Gamma_g) \to M_k(\Gamma_{g-1})$$

by the rule

$$\Phi f(\tau') := \lim_{t \to \infty} f \begin{pmatrix} \tau' & 0 \\ 0 & it \end{pmatrix} \quad \text{for } \tau' \in \mathscr{H}_{g-1}.$$

For a proof that this is a well-defined map, see Proposition 1, Section 5 of (Kli90).

In terms of the Fourier expansion, if f can be written as

$$f(\tau) = \sum_{N \geq 0,\ \text{h.-i.}} a(N) e^{2\pi i \operatorname{Tr}(N\tau)},$$

then we have

$$\Phi f(\tau') = \sum_{N' \geq 0,\ \text{h.-i.}} a \begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix} e^{2\pi i \operatorname{Tr}(N'\tau')}.$$

2.5. **Siegel cusp forms.** An element $f \in M_\rho(\Gamma_g)$ is called a *Siegel cusp form* if it belongs to the kernel of Φ. Equivalently, f is a cusp form if its Fourier coefficients satisfy: $a(N) = 0$ for all half-integral matrices N which are not positive definite. We denote the space of Siegel cusp forms of weight ρ by $S_\rho(\Gamma_g)$.

In the case $g = 1$, there is an explicit formula for the dimension of the space of cusp forms, namely

$$\dim S_k(\Gamma_1) = \begin{cases} 0 & \text{if } k < 0 \text{ or } k \text{ odd and } > 0, \\ \left[\frac{k}{4}\right] + \left[\frac{k}{3}\right] - \frac{k}{2} & \text{if } k \geq 0 \text{ even.} \end{cases}$$

It is usually obtained by using the formula

$$\dim M_{2k}(\Gamma_1) = \dim S_{2k}(\Gamma_1) + 1 \qquad \text{for } k > 1,$$

together with the Riemann-Roch theorem for computing $\dim M_{2k}(\Gamma_1)$; see for instance Theorem 4.9 and Example 4.14 in (Mil97).

The problem of computing the dimensions of the spaces $S_k(\Gamma_g)$ for general g is much more difficult and far from being settled. The state of the art in this area is contained in (PY02), (PY01), and (PY00); for example, they prove that

$$\dim S_{10}(\Gamma_4) = 1 \quad \text{and} \quad \dim S_{2k+1}(\Gamma_4) = 0 \text{ for } k = 0, 1, \ldots, 6.$$

The cases $g = 2$ and $g = 3$ are completely solved thanks to (Igu64) (see also (Has83)), resp. (Tsu86). There is an explicit yet rather complicated *conjectural* expression for the dimension in the general case; see (IS92).

A related (and even more difficult) problem is finding a basis for the space $M_k(\Gamma_g)$, or a set of generators for the graded ring $M(\Gamma_g)$. Once again this is known for $g = 1$:

$$M(\Gamma_1) = \mathbb{C}[E_4, E_6],$$

where $E_{2k} \in M_{2k}(\Gamma_1)$ denotes the Eisenstein series of weight $2k$.

For $g = 2$, (Igu62) proved that

$$M(\Gamma_2) = \mathbb{C}[E_4, E_6, \chi_{10}, \chi_{12}, \chi_{35}] / \left(\chi_{35}^2 = P(E_4, E_6, \chi_{10}, \chi_{12})\right),$$

where E_{2k} is the Siegel-Eisenstein series of weight $2k$, and χ_k are Siegel cusp forms of weight k which can be expressed explicitly in terms of Siegel-Eisenstein series.

The case $g = 3$ was solved by (Tsu86). He exhibits a set of 34 generators for $M(\Gamma_3)$, some of which cannot be written in terms of Siegel-Eisenstein series. I think it is still unknown whether this set of generators is minimal.

3. A panoply of applications

(a) Algebraic number theory

- (FK02): let $K := \mathbb{Q}(\alpha)$, where

  $$\alpha := \zeta + \zeta^3 + \zeta^9, \quad \zeta := e^{2\pi i/13}.$$

  They consider various abelian extensions of K, and show how special values of the L-functions of these extensions can be computed explicitly using Siegel modular forms. They can also show that these special values are units.
- (FK03): similar results for $K := \mathbb{Q}(e^{2\pi i/5})$, get so-called Minkowski units.
- (dSG97), (GL04): similar techniques applied to K *any* quartic CM field; they get S-units for a finite set of primes S effectively computable for any given K.

(b) Arithmetic geometry

- (SU02): let f be a cuspidal eigenform of weight $2k - 2 \geq 2$ and level 1, let p be an ordinary prime and let V_f be the p-adic Galois representation associated to f. If the L-function of f vanishes to odd order at $s = k - 1$, then the Selmer group of the representation V_f is infinite.
  This is a small partial result towards the conjectures of Beilinson and Bloch-Kato, which are generalizations of the Birch and Swinnerton-Dyer conjecture.
  The proof uses in a critical way Siegel modular forms and their associated Galois representations.
- (Dum01): uses Siegel modular forms to construct non-trivial elements in certain Shafarevich-Tate groups.

- (JK98): study heights of abelian varieties, find out that the archimedean local height is "almost" the logarithm of the Petersson norm of a Siegel modular form.

(c) Cohomology of some arithmetic groups
- (HW03): compute explicitly the cohomology of the arithmetic group $\Gamma_2(3)$:

$$\begin{aligned}
\mathrm{H}^0(\Gamma_2(3),\mathbb{Z}) &= \mathbb{Z}\\
\mathrm{H}^1(\Gamma_2(3),\mathbb{Z}) &= 0\\
\mathrm{H}^2(\Gamma_2(3),\mathbb{Z}) &= \mathbb{Z}^{21}\oplus(\mathbb{Z}/3)^{10}\oplus\mathbb{Z}/2\\
\mathrm{H}^3(\Gamma_2(3),\mathbb{Z}[1/6]) &= \mathbb{Z}[1/6]^{139}\\
\mathrm{H}^4(\Gamma_2(3),\mathbb{Z}[1/3]) &= \mathbb{Z}[1/3]^{81}\\
\mathrm{H}^i(\Gamma_2(3),\mathbb{Z}) &= 0 \quad \text{for } i>4.
\end{aligned}$$

  They also explicitly decompose $\mathrm{H}^i(\Gamma_2(3),\mathbb{Q})$ into irreducible representations for the finite group $\mathrm{PSp}_4(\mathbb{F}_3)$.

(d) Coding theory
- (Duk93) and (Run96): describe a correspondence between Siegel modular forms and self-dual codes.
- (Our97): computes the dimension of the ring of code polynomials in genus 4.
- (CK01): give a correspondence between a certain type of codes over $\mathbb{Z}/2m$ and Siegel modular forms.

(e) Elliptic cohomology
- (Gri99): the second quantized elliptic genus of a Calabi-Yau manifold[2] is "almost" a Siegel modular form; namely

$$(\text{fudge})\cdot\prod_{m,n,\ell}\left(\frac{1}{1-q^m y^\ell t^n}\right)^{c(mn,\ell)}$$

  is a Siegel modular form.
- Book by (Tho99): the last chapter describes an idea of Hopkins and Morava for generalizing elliptic cohomology by using $K3$ surfaces[3] instead of elliptic curves. Such a theory should be related to Siegel modular forms.

(f) Physics
- (Tui01): expresses the partition function of the genus two conformal field theory in terms of Siegel modular forms.
- (Cur98): studies $N=2$ string-string duality and holomorphic couplings. The moduli space of a heterotic STUV model is locally given by $\mathscr{H}_2$, so leads naturally to working with Siegel modular forms with $g=2$.

4. What else is there?

The limited scope and length of both the talk and these notes forced us to leave out quite a few topics that are of great importance in the theory of Siegel modular forms. In this final

[2]Complex manifold admitting a Kähler metric and having trivial canonical bundle.

[3]Surfaces with trivial canonical bundle and $\dim \mathrm{H}^1(X,\mathscr{O}_X)=0$.

section, we try to say a few words about these and provide appropriate references for further reading.

4.1. **Hecke operators.** These operators exist, and their theory is similar but more complicated than in the classical case. To give a quick illustration of this, consider the case $g = 1$: for each prime ℓ there is only one Hecke operator T_ℓ, namely the one associated with the double coset

$$\Gamma_1 \begin{pmatrix} 1 & \\ & \ell \end{pmatrix} \Gamma_1.$$

When $g = 2$, there are two Hecke operators at ℓ, T_ℓ and T_{ℓ^2}, corresponding to the double cosets

$$\Gamma_2 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{pmatrix} \Gamma_2 \qquad \text{and} \qquad \Gamma_2 \begin{pmatrix} 1 & & & \\ & \ell & & \\ & & \ell^2 & \\ & & & \ell \end{pmatrix} \Gamma_2.$$

The reference for general g is (AZ95).

4.2. **Siegel-Eisenstein series.** We hinted at the existence of Eisenstein series for higher g in § 2.5, when we gave Igusa's description of $M(\Gamma_2)$. They can be defined in a fairly general way as follows.

Let $0 \le r \le g$, an even positive weight k and a Siegel cusp form $f \in S_k(\Gamma_r)$. We define the Siegel-Eisenstein series with these parameters to be

$$E_{g,r,k}(f)(\tau) := \sum_{\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in P_r \backslash \Gamma_g} \frac{f((\gamma\tau)^*)}{(\det(C\tau + D))^k}$$

where

- for any $\tau \in \mathscr{H}_g$, we write
$$\tau = \begin{pmatrix} \tau^* & z \\ z & w \end{pmatrix} \quad \text{with } \tau^* \in \Gamma_r,$$
- the subgroup $P_r \subset \Gamma_g$ is given by
$$P_r = \left\{ \begin{pmatrix} a' & 0 & b' & * \\ * & u & * & * \\ c' & 0 & d' & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \in \Gamma_g : \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_r, u \in \mathrm{GL}_{g-r}(\mathbb{Z}) \right\}.$$

See Section 5 of (Kli90) for a proof of convergence and other properties, including the formula

$$\Phi^{g-r}(E_{g,r,k}(f)) = f.$$

4.3. ***L*- and zeta-functions.** If f is a classical cusp form of weight k which is a Hecke eigenform for all Hecke operators with eigenvalues a_ℓ, one can associate an L-function

$$L(f, s) = \prod_{\ell \text{ prime}} \frac{1}{1 - a_\ell \ell^{-s} + \ell^{k-1} \cdot \ell^{-2s}}.$$

This can still be done for higher g, but there are several types of L- and ζ-functions that can be considered, and the expressions get rather complicated. As an example, to a Hecke eigenform $f \in S_k(\Gamma_2)$ with eigenvalues a_ℓ (for T_ℓ) and a_{ℓ^2} (for T_{ℓ^2}), one can associate the *spinor zeta-function*

$$Z(f,s) = \prod_{\ell \text{ prime}} \frac{1}{1 - a_\ell \ell^{-s} + (a_\ell^2 - a_{\ell^2} + \ell^{2k-4})\ell^{-2s} - a_\ell \ell^{2k-3} \cdot \ell^{-3s} + \ell^{4k-6} \cdot \ell^{-4s}}.$$

4.4. **Saito-Kurokawa and Ikeda lifts.** There exists a linear map (called the *Saito-Kurokawa lifting*)

$$S_{2k-2}(\Gamma_1) \longrightarrow S_k(\Gamma_2)$$

which sends Hecke eigenforms to Hecke eigenforms and such that if the image of f is F, then

$$Z(f,s) = \zeta(s-k+1)\zeta(s-k+2)L(f,s),$$

where ζ is the Riemann zeta-function.

This was generalized by (Ike01) to a map

$$S_{2k-g}(\Gamma_1) \longrightarrow S_k(\Gamma_g),$$

provided that $k \equiv g \pmod 2$.

Acknowledgments

As this is a survey, we felt justified in borrowing from existing expositions; in particular, we made extensive use of notes taken at recent lectures by G. (vdG04), and of the textbook by H. (Kli90). Another reference we have used in the past but which was not available at the time these notes were written is (Fre83); we recommend it to the interested reader.

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