Compactification of Siegel's quotient spaces I*

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3 March 1958

[*Translator's note*: We have attempted, as much as possible, to keep the notation in the original article. Some items may be confusing to a modern reader:

- E is an identity matrix, E_n if we want to make the size explicit;
- "neighborhood" means "open neighborhood".]

Let $\mathfrak{S}_n$ be the Siegel space and Γ_n the Siegel modular group; we aim to construct a compactification of the quotient space $\Gamma_n\backslash\mathfrak{S}_n$. Of course, there are several possible compactifications; but, as we shall see, it is natural to consider a compactification of the form

$$\left(\Gamma_n\backslash\mathfrak{S}_n\right)^* = \Gamma_n\backslash\mathfrak{S}_n \cup \Gamma_{n-1}\backslash\mathfrak{S}_{n-1} \cup \cdots \cup \Gamma_0\backslash\mathfrak{S}_0,$$

where $\mathfrak{S}_0$ denotes a single point, and Γ_0 is the trivial group. The aim of this talk is to give the topological construction of this compactification. We then show, in the following talks, that $\left(\Gamma_n\backslash\mathfrak{S}_n\right)^*$, endowed with a canonically defined ringed space structure, is a normal analytic space that can be realized as a normal algebraic subvariety of a projective space; we will consider at the same time the corresponding problems for all the groups commensurable to the group Γ_n.

To describe our method, recall the case $n = 1$; in this case, it is well-known that the classical fundamental domain for Γ_1 has a single cusp (point at infinity), so that the quotient space $\Gamma_1\backslash\mathfrak{S}_1$ can be compactified by adjoining a single point P_∞ corresponding to this point, or more precisely to the class of this point; the compactified space $\left(\Gamma_1\backslash\mathfrak{S}_1\right)^*$ is a compact Riemann surface, whose local parameter around the point P_∞ is given by $e^{2\pi iz}$, which maps the subset $y > c$ of the upper half plane $\mathfrak{S}_1$ onto a neighborhood of P_∞ in $\left(\Gamma_1\backslash\mathfrak{S}_1\right)^*$. But the orbit of the point at infinity under Γ_1 consists precisely of the rational points on the real axis, and the images of the set $y > c$ under Γ_1 are horocycles at these points (i.e. cycles tangential to the real axis). Therefore the compactification $\left(\Gamma_1\backslash\mathfrak{S}_1\right)^*$ is obtained as follows: first let the space $\mathfrak{S}_1^*$ be the disjoint union of the upper half plane $\mathfrak{S}_1$ and all its rational points, then topologize it by taking the horocycles to be the neighborhoods of the rational points, and finally take the quotient $\Gamma_1\backslash\mathfrak{S}_1^*$ of $\mathfrak{S}_1^*$ by Γ_1. Our objective is to prove that this method generalizes to the case of arbitrary n.

*This paper appeared in *Séminaire Henri Cartan*, pages 12-01 to 12-13 (1957/58).
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1 Preliminary considerations

Let $\mathfrak{S}_n$ be the Siegel space; we always denote an element of $\mathfrak{S}_n$ as

$$Z = X + iY, \quad X = \bigl(x_{ij}\bigr), \quad Y = \bigl(y_{ij}\bigr), \quad Y = {}^tWDW,$$

with a diagonal matrix $D = \bigl(d_i\delta_{ij}\bigr)$ and a strictly upper triangular matrix $W = \bigl(w_{ij}\bigr)$. Denote by $\Omega_n(u)$ $(u > 0)$ the set of $Z \in \mathfrak{S}_n$ satisfying

(i) $|x_{ij}| < u$,

(ii) $|w_{ij}| < u \quad (1 \le i < j \le n)$,

(iii) $1 < ud_1$, $d_i < ud_{i+1} \quad (1 \le i \le n-1)$.

We already know ([1, Section 5]) that the collection of $\Omega_n(u)$ for sufficiently large $u > 0$ is a collection of "fundamental open sets" for the modular group Γ_n. (We deviate here from the definition given in [1]; but setting

$$M_0 = \begin{pmatrix} e_n & 0 \\ 0 & e_n \end{pmatrix}, \quad e_n = \bigl(\delta_{i,n+1-j}\bigr),$$

it is easy to see that the collection defined in [1] is equivalent to the collection $\bigl\{M_0\Omega_n(u)\bigr\}$ in the current notation.)

Let $0 \le r \le n$; we decompose matrices into $(r, n-r)$ blocks:

$$Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad W = \begin{pmatrix} W_1 & W_{12} \\ 0 & W_2 \end{pmatrix}, \dots$$

Then $Z \in \Omega_n(u)$ implies that $Z_1 \in \Omega_r(u)$, given the relation

$$(1.1) \qquad {}^tWDW = \begin{pmatrix} {}^tW_1D_1W_1 & {}^tW_1D_1W_{12} \\ 0 & {}^tW_{12}D_1W_{12} + {}^tW_2D_2W_2 \end{pmatrix}.$$

From now on, we fix a number u such that $\Omega_r(u)$ is a fundamental open set of Γ_r for all $r \le n$ and we write Ω_r instead of $\Omega_r(u)$.

Consider the set

$$(1.2) \qquad \Omega_n^* = \overline{\Omega}_n \sqcup \overline{\Omega}_{n-1} \sqcup \cdots \sqcup \overline{\Omega}_0$$

(disjoint union in the abstract sense), where $\overline{\Omega}_r$ denotes the closure of Ω_r in $\mathfrak{S}_r$ and $\Omega_0 = \mathfrak{S}_0$ (a one-point set). We introduce the following "natural" topology: let U be a neighborhood of $Z_0 \in \overline{\Omega}_r$ in $\overline{\Omega}_r$ and K a positive number; we denote by $V^{(s)}(U, K)$ $(r \le s \le n)$ the set of $Z \in \overline{\Omega}_s$ such that $Z_1 \in U$ and $d_{r+1} > K$, where Z_1 is as above the matrix of degree r in the $(r, s-r)$ block decomposition of Z, and d_{r+1} is the $(r+1)$-st diagonal element of D such that $Z = X + iY$, $Y = {}^tWDW$; then a neighborhood of Z in Ω_n^* is given by the union

$$\bigcup_{r \le s \le n} V^{(s)}(U, K);$$

in other words, a sequence (Z_ν) contained in $\overline{\Omega}_s$ converges to Z_0 in $\overline{\Omega}_r$ if and only if $Z_{\nu,1} \longrightarrow Z_0$ and $d_{\nu,r+1} \longrightarrow \infty$. It is clear that these definitions give a Hausdorff topology on Ω_n^* inducing the original topology on each $\overline{\Omega}_r$. It is also clear that any sequence (Z_ν) contained in $\overline{\Omega}_s$ has a subsequence that converges in our sense (for an appropriately chosen r); hence Ω_n^* is a Hausdorff and **compact** space.

Let $0 \le r \le n$; we decompose

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R})$$

as follows:

$$A = \begin{pmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_{12} \\ B_{21} & B_2 \end{pmatrix}, \dots$$

into $(r, n-r)$ blocks. We consider the subgroup $\mathfrak{G}_r^n$ of $\mathrm{Sp}(n,\mathbb{R})$ consisting of matrices of the form

$$M = \begin{pmatrix} A_1 & 0 & B_1 & B_{12} \\ A_{21} & A_2 & B_{21} & B_2 \\ C_1 & 0 & D_1 & D_{12} \\ 0 & 0 & 0 & D_2 \end{pmatrix}. \tag{1.3}$$

It is trivial that the set of all matrices of this form is in fact a subgroup; we note that simplecticity implies that the conditions $A_{12} = 0$, $C_{12} = 0$, C_0 (or the conditions $C_{21} = 0$, $C_2 = 0$, $D_{21} = 0$) are equivalent to conclude that an element M of $\mathrm{Sp}(n,\mathbb{R})$ belongs to $\mathfrak{G}_r^n$.

It also follows that

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{G}_r^n$$

implies that

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}(r,\mathbb{R})$$

and that the map

$$\varpi_r : \ M \in \mathfrak{G}_r^n \longmapsto M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}(r,\mathbb{R}) \tag{1.4}$$

is a homomorphism from $\mathfrak{G}_r^n$ to $\mathrm{Sp}(r,\mathbb{R})$. On the other hand, let ι_n be the canonical embedding of $\mathrm{Sp}(r,\mathbb{R})$ to $\mathrm{Sp}(n,\mathbb{R})$ defined by

$$\iota_n : \ M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \longmapsto M = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & E & 0 & 0 \\ C_1 & 0 & D_1 & 0 \\ 0 & 0 & 0 & E \end{pmatrix}. \tag{1.5}$$

We have then $\varpi_r \circ \iota_n = 1$ (the identity), which means that ϖ_r is surjective, and letting $\mathfrak{N}_r^n$ denote the kernel of ϖ_r, we can decompose $\mathfrak{G}_r^n$ into a semidirect product as follows:

$$\mathfrak{G}_r^n = \iota_n\big(\mathrm{Sp}(r,\mathbb{R})\big) \ltimes \mathfrak{N}_r^n. \tag{1.6}$$

We note that for the modular group Γ_n we have the relation

$$\Gamma_n \cap \mathfrak{G}_r^n = \iota_n\big(\Gamma_r\big) \ltimes \big(\Gamma_n \cap \mathfrak{N}_r^n\big). \tag{1.7}$$

The significance of the group $\mathfrak{G}_r^n$ is shown by the following lemma:

Lemma 1 (Godement)**.** *Let (Z_ν), (Z'_ν) be sequences in $\overline{\Omega}_n$.*

1° (Z_ν) converges to $Z_0 \in \overline{\Omega}_r$ if and only if

$$(Z_\nu^{-1}) \quad \text{converges to} \quad \begin{pmatrix} Z_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

in the usual sense.

2° If (Z_ν) and (Z'_ν) converge to $Z_0 \in \overline{\Omega}_r$, respectively $Z'_0 \in \overline{\Omega}_{r'}$, and if $Z'_\nu = MZ_\nu$ ($\nu = 1, 2, \dots$) for a matrix $M \in \mathrm{Sp}(n,\mathbb{R})$, then we have $r = r'$, $M \in \mathfrak{G}_r^n$, and $Z'_0 = \varpi_r(M)Z_0$.

Proof. Suppose (Z_ν) converges to Z_0 and set

$$\begin{aligned} Z_\nu &= X_\nu + iY_\nu, \\ Y_\nu &= {}^tW_\nu D_\nu W_\nu, \\ D_\nu &= \begin{pmatrix} D_{\nu,1} & 0 \\ 0 & D_{\nu,2} \end{pmatrix}, \\ X_\nu &= \begin{pmatrix} X_{\nu,1} & X_{\nu,12} \\ {}^tX_{\nu,12} & X_{\nu,2} \end{pmatrix}, \\ W_\nu &= \begin{pmatrix} W_{\nu,1} & W_{\nu,12} \\ 0 & W_{\nu,2} \end{pmatrix}, \\ Z_0 &= X_0 + iY_0, \\ W_0 &= {}^tW_0 D_0 W_0; \end{aligned}$$

then $X_{\nu,1} \longrightarrow X_0$, $W_{\nu,1} \longrightarrow W_0$, $D_{\nu,1} \longrightarrow D_0$.

By passing to a subsequence, we can moreover assume that (X_ν) and (W_ν) (and not only $(X_{\nu,1})$ and $(W_{\nu,1})$) converge, because for $Z \in \overline{\Omega}_n$ all the coefficients of X and of W are bounded; we have therefore

$$\begin{aligned} Z_\nu^{-1} &= W_\nu^{-1} D_\nu^{-1/2} \Big(iE + D_\nu^{-1/2}\, {}^tW_\nu^{-1} X_\nu W_\nu^{-1} D_\nu^{-1/2} \Big)^{-1} D_\nu^{-1/2}\, {}^tW_\nu^{-1} \\ &\longrightarrow \begin{pmatrix} W_0^{-1} & * \\ 0 & * \end{pmatrix} \begin{pmatrix} D_0^{-1/2} & 0 \\ 0 & 0 \end{pmatrix} \Big(iE + M_0 \Big)^{-1} \begin{pmatrix} D_0^{-1/2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {}^tW_0^{-1} & 0 \\ * & * \end{pmatrix}, \end{aligned}$$

where

$$M_0 = \begin{pmatrix} D_0^{-1/2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {}^tW_0^{-1} & 0 \\ * & * \end{pmatrix} \begin{pmatrix} X_0 & * \\ * & * \end{pmatrix} \begin{pmatrix} W_0^{-1} & * \\ 0 & * \end{pmatrix} \begin{pmatrix} D_0^{-1/2} & 0 \\ 0 & 0 \end{pmatrix}.$$

This is equal to

$$\begin{aligned} &\begin{pmatrix} W_0^{-1} D_0^{-1/2} & 0 \\ 0 & 0 \end{pmatrix} \left(iE + \begin{pmatrix} D_0^{-1/2}\, {}^tW_0^{-1} X_0 W_0^{-1} D_0^{-1/2} & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} D_0^{-1/2}\, {}^tW_0^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} W_0^{-1} D_0^{-1/2} (iE_r + D_0^{-1/2}\, {}^tW_0^{-1} X_0 W_0^{-1} D_0^{-1/2})^{-1} D_0^{-1/2}\, {}^tW_0^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} Z_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

whence the first statement in 1°. The converse follows immediately from this and the fact that every sequence in $\overline{\Omega}_n$ has a converging subsequence.

Now let (Z'_ν) be another sequence converging to $Z'_0 \in \overline{\Omega}_r$, and let

$$Z'_\nu = \big(AZ_\nu + B\big)\big(CZ_\nu + D\big)^{-1} \quad \text{with } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R});$$

without loss of generality $r' \le r$. We have then

$$(Z'_\nu)^{-1} = \big(DZ_\nu^{-1} + C\big)\big(BZ_\nu^{-1} + A\big)^{-1}$$

and, by passage to the limit,

$$\begin{aligned} &\begin{pmatrix} (Z'_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} B_1 & B_{12} \\ B_{21} & B_2 \end{pmatrix} \begin{pmatrix} Z_0^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{pmatrix} \right) \\ &\qquad = \begin{pmatrix} D_1 & D_{12} \\ D_{21} & D_2 \end{pmatrix} \begin{pmatrix} Z_0^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} C_1 & C_{12} \\ C_{21} & C_2 \end{pmatrix}, \end{aligned}$$

where the blocks are $(r, n-r)$. By comparing the corresponding coefficients, we get the relations

$$\begin{pmatrix} (Z_0')^{-1} & 0 \\ 0 & 0 \end{pmatrix} \big(B_1 Z_0^{-1} + A_1\big) = D_1 Z_0^{-1} + C_1, \tag{1.8}$$

$$\begin{pmatrix} (Z_0')^{-1} & 0 \\ 0 & 0 \end{pmatrix} A_{12} = C_{12}, \tag{1.9}$$

$$0 = D_{21} Z_0^{-1} + C_{21}, \tag{1.10}$$

$$0 = C_2. \tag{1.11}$$

As the imaginary part Y_0 of Z_0 is $\gg 0$, it follows from (1.10) that $C_{21} = D_{21} = 0$, which, together with (1.11), shows that $M \in \mathfrak{G}_r^n$; we have therefore that

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}(r, \mathbb{R})$$

and then (1.8) shows that $\begin{pmatrix} (Z_0')^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ has rank r; hence $r = r'$ and $Z_0' = M_1 Z_0$. $\square$

2 Construction of the space $\mathfrak{S}_n^*$

We now construct the space $\mathfrak{S}^*$ which is a generalization to the case of arbitrary n of the $\mathfrak{S}_1^*$ stated above. We could use the bounded model of the Siegel space, i.e. the space of complex symmetric matrices W of degree n such that $\overline{W}W \ll E_n$. But we will instead construct directly the space corresponding to the half-plane $\mathfrak{S}_n$.

Let $\Gamma = \Gamma_n$ be the Siegel modular group; consider the set of pairs (M, Z) with $M \in \Gamma$, $Z \in \mathfrak{S}_r$ $(0 \leq r \leq n)$; take the equivalence relation defined by

$$(M, Z) \sim (M', Z'), Z \in \mathfrak{S}_r, Z' \in \mathfrak{S}_{r'} \iff r = r', (M')^{-1}M \in \mathfrak{G}_r^n, Z' = \varpi_r\big((M')^{-1}M\big)Z.$$

This is clearly an equivalence relation; we write $M.Z$ for the equivalence class of (M, Z) and we denote by $\mathfrak{S}_n^*$ the set of equivalence classes. We can view $\mathfrak{S}_r$ as a subset of $\mathfrak{S}_n^*$ via the natural injective map $Z \longmapsto 1.Z$; similarly we can make Γ act on $\mathfrak{S}_n^*$ via the obvious formula $M_1\,(M.Z) = (M_1 M).Z$, since $(M, Z) \sim (M', Z')$ obviously implies that $(M_1 M, Z) \sim (M_1 M', Z')$. All of this agrees with the usual notations when $n = r$.

We have therefore

$$\mathfrak{S}_n^* = \bigcup_{0 \leq r \leq n} \Gamma \mathfrak{S}_r. \tag{2.1}$$

More precisely, if we decompose Γ into right cosets for $\Gamma \cap \mathfrak{G}_r^n$:

$$\Gamma = \bigcup_i M_{r,i}\big(\Gamma \cap \mathfrak{G}_r^n\big), \tag{2.2}$$

we have the following decomposition of $\mathfrak{S}_n^*$:

$$\mathfrak{S}_n^* = \bigsqcup_{r,i} M_{r,i} \mathfrak{S}_r. \tag{2.3}$$

Note that we can consider $\Omega_n^* \subset \mathfrak{S}_n^*$ and obtain $\mathfrak{S}_n^* = \Gamma \Omega_n^*$.

We now define a topology on $\mathfrak{S}_n^*$; we are interested in a topology $\mathcal{T}$ on $\mathfrak{S}_n^*$ satisfying

1° $\mathcal{T}$ induces the “natural” topology on Ω_n^*.

2° The actions of $M \in \Gamma$ on $\mathfrak{S}_n^*$ are continuous maps.

3° If two points x, x' of $\mathfrak{S}_n^*$ are not Γ-equivalent, there exist neighborhoods U of x and U' of x' such that $\Gamma U \cap U' = \emptyset$.

4° Each point $x \in \mathfrak{S}_n^*$ has a system of open neighborhoods $\{U\}$ such that $\Gamma_x U = U$ and if $MU \cap U \neq \emptyset$ then $M \in \Gamma_x$, where Γ_x is the stabilizer of x in Γ.

Our main results consist of the following theorems:

Theorem 1. *Among the topologies satisfying conditions 1° and 2°, there is a finest one, denoted $\mathcal{T}^\Gamma$; it also satisfies condition 3°.*

Theorem 2. *There exists a unique topology, denoted $\mathcal{T}_0^\Gamma$, satisfying conditions 1°, 2°, 3°, and 4°.*

Before giving the proofs, we discuss the consequences of these theorems. We start by considering the quotient space $\Gamma\backslash\mathfrak{S}_n^*$ with the topology induced by $\mathcal{T}^\Gamma$: the open sets of $\Gamma\backslash\mathfrak{S}_n^*$ are the images of the $\mathcal{T}^\Gamma$-open sets of $\mathfrak{S}_n^*$ under the canonical projection $\pi_n^* : \mathfrak{S}_n^* \longrightarrow \Gamma\backslash\mathfrak{S}_n^*$. We have then

Theorem 3. *The quotient space $\Gamma\backslash\mathfrak{S}_n^*$ is Hausdorff and compact.*

Proof. The space is Hausdorff by condition 3° above; it is compact since it is the continuous image of the compact space Ω_n^*. □

We now have

$$\Gamma\backslash\mathfrak{S}_n^* = \bigcup_{0\le r\le n} \Gamma\backslash\Gamma\mathfrak{S}_r$$

by (2.1); as the stabilizer of $\mathfrak{S}_r$ in Γ is $\Gamma \cap \mathfrak{G}_r^n$ and the action of $\Gamma \cap \mathfrak{G}_r^n$ on $\mathfrak{S}_r$ is the same as that of $\varpi_r(\Gamma \cap \mathfrak{G}_r^n) = \Gamma_r$, $\Gamma\backslash\Gamma\mathfrak{S}_r$ is canonically identified with $\Gamma_r\backslash\mathfrak{S}_r$; we have therefore

$$\Gamma\backslash\mathfrak{S}_n^* = \bigcup_{0\le r\le n} \Gamma_r\backslash\mathfrak{S}_r. \tag{2.4}$$

There are several topologies satisfying conditions 1° and 2°; but they all induce the same topology on any finite union of $M_i\Omega_n^*$ ($M_i \in \Gamma$). If they also satisfy condition 3°, they induce the same topology on the quotient space $\Gamma\backslash\mathfrak{S}_n^*$, so that we can assume in Theorem 3 that the topology on $\Gamma\backslash\mathfrak{S}_n^*$ is defined by any topology on $\mathfrak{S}_n^*$ satisfying conditions 1°, 2°, and 3°.

3 Proof of Theorems 1 and 2

We first define the topology $\mathcal{T}^\Gamma$, as follows: we declare a subset F of $\mathfrak{S}_n^*$ to be $\mathcal{T}^\Gamma$-closed if and only if for all $M \in \Gamma$ we have that $MF \cap \Omega_n^*$ is closed in the "natural" topology on Ω_n^*. It is clear that this defines a topology $\mathcal{T}^\Gamma$ and that the latter satisfies condition 2°. To verify condition 1°, it suffices to prove that if F is closed in Ω_n^* (in the "natural" topology), then $MF \cap \Omega_n^*$ is also closed, for all $M \in \Gamma$; but this follows immediately from Lemma 1. It is clear that $\mathcal{T}^\Gamma$ is the finest topology satisfying conditions 1° and 2°.

To prove the last statement of Theorem 1, we need several lemmas:

Lemma 2. *For each r there exists a finite number of $M_i^{(r)} \in \mathfrak{G}_r^n$ such that the relations $M\overline{\Omega}_r \cap \overline{\Omega}_r \neq \emptyset$ for $M \in \Gamma$ (and hence $M \in \mathfrak{G}_r^n$) imply that $\varpi_r(M) = \varpi_r\left(M_i^{(r)}\right)$ for some i.*

This is an immediate consequence of the fact that Ω_r is a "fundamental open" of Γ_r. We note in fact that, if $r < n$, there are **infinitely many** $M \in \Gamma$ such that $M\overline{\Omega}_r \cap \overline{\Omega}_r \neq \emptyset$.

Lemma 3. *For each $Z \in \overline{\Omega}_r$, there exists a neighborhood U of Z in Ω_n^* such that*

1° if $M \in \Gamma$ and $MU \cap \Omega^* \neq \emptyset$, then $M \in \mathfrak{G}_r^n$ and $MZ \in \overline{\Omega}_r$;

2° if $M \in \Gamma$ and $MU \cap U \neq \emptyset$, then $M \in \Gamma_Z$, the stabilizer of Z in Γ.

Proof. Suppose $M \in \Gamma$ is fixed. It is clear that if $MU \cap \Omega_n^* \neq \emptyset$ for all neighborhoods U of Z in Ω_n^*, then $MZ \in \Omega_n^*$ and hence $M \in \mathfrak{G}_r^n$, $MZ \in \overline{\Omega}_r$. Therefore we can take a neighborhood U of Z such that

$$U \subset \bigcup_{r \leq s \leq n} \overline{\Omega}_s$$

and that the statement of the Lemma holds for all $M_i^{(s)}$ ($r \leq s \leq n$) stated in Lemma 2. We then prove that the statement of the Lemma holds for all $M \in \Gamma$. Indeed, if $MU \cap \Omega_n^* \neq \emptyset$, there exists s ($r \leq s \leq n$) such that $MU \cap \overline{\Omega}_s \neq \emptyset$; by Lemma 2 we then have $M \in \mathfrak{G}_s^n$ and $\varpi_s(M) = \varpi_s\left(M_i^{(s)}\right)$. Hence $M_i^{(s)} \cap \overline{\Omega}_s \neq \emptyset$ and by our choice of U we have $M_i^{(s)} \in \mathfrak{G}_r^n$, $M_i^{(s)} Z \in \overline{\Omega}_r$; next $\varpi_s(M) = \varpi_s\left(M_i^{(s)}\right) \in \mathfrak{G}_r^s$, hence $M \in \mathfrak{G}_r^n$, $\varpi_r(M) = \varpi_r\left(M_i^{(s)}\right)$ and so $MZ = M_i^{(s)} Z \in \overline{\Omega}_r$, which proves the first statement in the Lemma. The second statement can be proved similarly. □

Lemma 4. *Let $Z \in \Omega_r$; if U is a neighborhood of Z in Ω_n^*, then $\tilde{U} = \Gamma_Z U$ is a $\mathcal{T}^\Gamma$-neighborhood of Z.*

Proof. We may assume that U satisfies property 1° stated in Lemma 3; therefore if $M\tilde{U} \cap \Omega_n^* \neq \emptyset$ with $M \in \Gamma$ then $M \in \mathfrak{G}_r^n$, $MZ \in \overline{\Omega}_r$. So there are only finitely many possibilities for M (up to right multiplication by Γ_Z) such that $MU \cap \Omega_n^* \neq \emptyset$. Hence it suffices to prove that $M\tilde{U} \cap \Omega_n^*$ is a neighborhood of MZ in Ω_n^* for these finitely many representatives M modulo Γ_Z. Let $r \leq s \leq n$, $U_s = U \cap \overline{\Omega}_s$; then

$$M\tilde{U} \cap \Omega_n^* = \bigcup_{r \leq s \leq n} M\Gamma_Z U_s \cap \overline{\Omega}_s.$$

But as $\Gamma_Z \supset \Gamma \cap \mathfrak{N}_r^n$ and $\Gamma = \iota_n(\Gamma_r) \ltimes \left(\Gamma \cap \mathfrak{N}_r^n\right)$, we can take M such that $M \in \iota_n(\Gamma_r)$. Then $M\Gamma_Z U_s \cap \overline{\Omega}_s$ contains all the matrices $Z^{(s)} \in \overline{\Omega}_s$ such that

$$Z^{(s)} = \begin{pmatrix} Z_1^{(r)} & Z_{12} \\ {}^t Z_{12} & Z_2 \end{pmatrix} = X + iY, \quad Y = {}^t W D W, \quad D = (d_i \delta_{ij}),$$

with $Z_1^{(r)}$ close enough to MZ and d_{r+1} sufficiently large. (This follows from the Proposition proven in the Appendix.) Therefore $M\tilde{U} \cap \Omega_n^*$ is a neighborhood of MZ in Ω_n^*. □

We now prove the last statement of Theorem 1. Let x, x' be two points of $\mathfrak{S}_n^*$ that are not Γ-equivalent; we need to construct neighborhoods $\tilde{U}$ and $\tilde{U}'$ of x, respectively x', that are Γ-saturated and disjoint; it suffices to do this for two points

$$Z, Z' \in \bigcup_{0 \leq r \leq n} \Omega_r.$$

Let U, U' be respective neighborhoods of Z and Z' in Ω_n^* such that $M_i^{(r)} U \cap U' = \emptyset$ for all $M_i^{(r)}$ from Lemma 2; it is clear then that $MU \cap U' = \emptyset$ for all $M \in \Gamma$. Let $\tilde{U} = \Gamma U$, $\tilde{U}' = \Gamma U'$; by Lemma 4 these are $\mathcal{T}^\Gamma$-neighborhoods of Z and Z' in $\mathfrak{S}_n^\star$, and they are Γ-saturated and disjoint, from which we deduce the desired statement.

We now prove Theorem 2. We define the topology $\mathcal{T}_0^\Gamma$ as follows: we say that U is a $\mathcal{T}_0^\Gamma$-neighborhood of $x \in \mathfrak{S}_n^*$ if and only if U is a Γ_x-saturated $\mathcal{T}^\Gamma$-neighborhood of x. For

$$Z \in \bigcup_{0 \le r \le n} \Omega_r$$

such a neighborhood always contains a neighborhood $\tilde{U} = \Gamma_Z U$ as given in Lemma 4; taking U sufficiently small so that condition 2° of Lemma 3 is satisfied, $\tilde{U} = \Gamma_Z U$ is a $\mathcal{T}_0^\Gamma$-neighborhood of Z satisfying condition 4°; it follows immediately that the conditions for the systems of neighborhoods are satisfied for $\mathcal{T}_0^\Gamma$; it is then clear that $\mathcal{T}_0^\Gamma$ is a topology satisfying conditions 1°, 2°, 3°, and 4°; condition 1° is satisfied since for $\tilde{U} = \Gamma_Z U$, we can make

$$\tilde{U} \cap \Omega_n^* = \bigcup_{M_i^{(s)} \in \Gamma_Z} M_i^{(s)} U \cap \Omega_n^*$$

as small as we want by taking U to be sufficiently small.

Finally, we prove the uniqueness of the topology satisfying conditions 1°, 2°, 3°, and 4°. Let $\mathcal{T}$ be such a topology and let $\tilde{U}_1$ be a $\mathcal{T}$-neighborhood of $Z \in \Omega_r$ satisfying condition 4°; setting $U = \tilde{U}_1 \cap \Omega_n^*$, $\tilde{U} = \Gamma_Z U$, we get a $\mathcal{T}_0^\Gamma$-neighborhood $\tilde{U}$ of Z, clearly contained in $\tilde{U}_1$; conversely let $\tilde{U} = \Gamma_Z U$ be a $\mathcal{T}_0^\Gamma$-neighborhood of $Z \in \Omega_r$; we may assume that $\tilde{U}$ is contained in a $\mathcal{T}$-neighborhood of $\tilde{U}_1$ of Z satisfying condition 4°; let $\tilde{U}_2 = \Gamma U$; $\tilde{U}_2$ is a $\mathcal{T}$-neighborhood of Z, because it is a Γ-saturated $\mathcal{T}_0^\Gamma$-neighborhood of Z, and because $\mathcal{T}$ and $\mathcal{T}_0^\Gamma$ define the same topology on the quotient space $\Gamma\backslash\mathfrak{S}_n^*$ due to conditions 1°, 2°, and 3°; we have then

$$\tilde{U}_1 \cap \tilde{U}_2 = \tilde{U}_1 \cap \bigcup_{M_i \in \Gamma/\Gamma_Z} M_i \tilde{U} = \tilde{U}_1 \cap \tilde{U} = \tilde{U},$$

hence $\tilde{U}$ is a $\mathcal{T}$-neighborhood of Z, which proves our statement.

The classical topology of $\mathfrak{S}_1^*$ is $\mathcal{T}_0^\Gamma$; we see easily that the two topologies $\mathcal{T}^\Gamma$ and $\mathcal{T}_0^\Gamma$ are in fact different; we note also that these topologies are not locally compact. We also note that the topologies $\mathcal{T}^\Gamma$ and $\mathcal{T}_0^\Gamma$ induce the same topology on $\mathfrak{S}_r$ ($0 \le r \le n$), namely the original topology on $\mathfrak{S}_r$.

Appendix

We complete here the proof of Lemma 4. By changing notation, this involves the following setup: let U_r and U_r' be neighborhoods of $Z_0 \in \Omega_r$ in Ω_r; let K and K' be positive numbers, $U_s = V^{(s)}\big(U_r, K\big)$ the set of all matrices $Z \in \Omega_s$ such that

$$Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix} = X + iY, \quad Y = {}^tWDW, \quad D = (d_i\delta_{ij}),$$

with $Z_1 \in U_r$, $d_{r+1} > K$. Let U_s' the analogue of U_s obtained by replacing Ω_s, U_r, and K by $M_0^{-1}\Omega_s$, U_r', and K', where M_0 is an element of $\iota_s(\Gamma_r)$. We have to prove that, given U_r, K, and M_0, we can choose U_r' and K' such that

$$U_s' \subset (\Gamma_s)_{Z_0} U_s. \tag{3.1}$$

Then it suffices to take K'' such that

$$M_0^{-1} V^{(s)}\big(M_0 U_r', K''\big) \subset U_s',$$

which is possible thanks to the continuity of M_0^{-1} in $\Omega_s^* \cup M_0^{-1}\Omega_s^*$.

We will use the following result:

Proposition. *With the above notation, given M_0 and a bounded U'_r, we can choose K' such that U'_s is contained in $\big(\Gamma_s \cap \mathfrak{N}_r^s\big)\Omega_s$.*

The statement we want to prove follows from the Proposition: indeed we get finitely many $M_i \in \Gamma_s \cap \mathfrak{N}_r^s$ such that

$$U'_s \subset \bigcup_i M_i \Omega_s$$

and hence, modifying U'_r and K' so that

$$(M_i^{-1} U'_s) \cap \Omega_s \subset U_s \qquad \text{for all } i$$

(which is possible thanks to the continuity of M_i^{-1} in Ω_s^*), we get

$$U'_s \subset \bigcup_i M_i U_s,$$

and therefore (3.1).

So it remains to prove the Proposition. Let

$$Z' = \begin{pmatrix} Z'_1 & Z'_{12} \\ {}^t Z'_{12} & Z'_2 \end{pmatrix} \in U'_s.$$

We will show that, if we take K' sufficiently large, there exists $M \in \Gamma_s \cap \mathfrak{N}_r^s$ such that $MZ' \in \Omega_s$. But the group $\Gamma_s \cap \mathfrak{N}_r^s$ is generated by transformations of the form

(i) $M = \begin{pmatrix} {}^t U & 0 \\ 0 & U^{-1} \end{pmatrix}$, with $U = \begin{pmatrix} E & 0 \\ 0 & U_2 \end{pmatrix}$, U_2 being integral and unimodular;

(ii) $M = \begin{pmatrix} {}^t U & 0 \\ 0 & U^{-1} \end{pmatrix}$, with $U = \begin{pmatrix} E & U_{12} \\ 0 & E \end{pmatrix}$, U_{12} integral;

(iii) $M = \begin{pmatrix} E & T \\ 0 & E \end{pmatrix}$, with $T = \begin{pmatrix} 0 & T_{12} \\ {}^t T_{12} & T_2 \end{pmatrix}$, where T_{12} and T_2 are integral, and $T_2 = {}^t T_2$.

If we set

$$Z' = X' + iY', \qquad X' = \begin{pmatrix} X'_1 & X'_{12} \\ {}^t X'_{12} & X'_2 \end{pmatrix},$$

$$Y' = {}^t W' D' W', \qquad W' = \begin{pmatrix} W'_1 & W'_{12} \\ 0 & W'_2 \end{pmatrix}, \qquad D' = \begin{pmatrix} D'_1 & 0 \\ 0 & D'_2 \end{pmatrix},$$

then these transformations act as follows:

(i) $W'_{12} \longmapsto W'_{12} U'_2, \quad {}^t W'_2 D'_2 W'_2 \longmapsto {}^t U_2 \, {}^t W'_2 D'_2 W'_2 U_2$;

(ii) $W'_{12} \longmapsto W'_{12} + W'_1 U_{12}, \quad {}^t W'_2 D'_2 W'_2$ unchanged;

(iii) $X'_{12} \longmapsto X'_{12} + T_{12}, \quad X'_2 \longmapsto X'_2 + T_2, \quad Y'$ unchanged;

and these transformations do not change X'_1, W'_1, and D'_1. By setting $Z'' = MZ'$ for some M of type (i), we can arrange that ${}^t W''_2 D''_2 W''_2 \in S'(u)$ (in the notation of [6, 5]), that is that

$$|w''_{ij}| < u \quad (\text{for } r+1 \le i < j \le s), \quad d''_i < u d''_{i+1} \quad (\text{for } r+1 \le i < s).$$

Next, using a transformation of type (ii), we can arrange that

$$|w''_{ij}| \le \frac{1}{2} \quad (\text{for } 1 \le i \le r,\ r+1 \le j \le s).$$

Finally, using a transformation of type (iii), we can arrange that

$$|x''_{ij}| \leq \frac{1}{2} \quad \text{(for } r+1 \leq j \leq s\text{, any } i\text{).}$$

Under all these transformations $Z''_1 = Z'_1$ does not change. Finally, we see that we can choose $M \in \Gamma_s \cap \mathfrak{N}^s_r$ so that $Z'' = MZ'$ satisfies all the conditions of belonging to Ω_s, with the exception of

$$d''_r < ud''_{r+1}.$$

But $d''_r = d'_r$ is bounded as Z'_1 ranges through U'_r. Moreover, there are only finitely many transformations of type (i) as Z' ranges through $M_0^{-1}\Omega_s$ such that $Z'_1 \in U'_r$ (indeed, for $Z' = M_0^{-1}Z \in M_0^{-1}\Omega_s$ and $Z'_1 \in U'_r$, all the coefficients of $Y'_2 - Y_2$ are bounded). We can therefore choose K' depending only on U'_r, u, and M_0, in such a way that, for all $Z' \in U'_s$, $Z'' = MZ'$ also satisfies $d''_r < ud''_{r+1}$, that is $MZ' \in \Omega_s$. This concludes the proof of the Proposition.

Bibliographic note

The above compactification was given in [3] (but without using the space $\mathfrak{S}^*_n$). Lemma 2 of [3] corresponds to Lemma 1 of the present talk, but the proof is much simplified by an idea of Godement. In any case the introduction of the space $\mathfrak{S}^*_n$ is preferable, especially in view of its usefulness in the consideration of groups commensurable to Γ.

Other methods of compactification can be found in [2, 4].

References

[1] Henri Cartan. Ouverts fondamentaux pour le groupe modulaire. In *Séminaire Henri Cartan*, volume 10. E.N.S., 1957–1958. No. 1, Talk no. 3, 12 p.

[2] Ichiro Satake. On Siegel's modular functions. In *Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955*, pages 107–129. Science Council of Japan, Tokyo, 1956.

[3] Ichiro Satake. On the compactification of the Siegel space. *J. Indian Math. Soc. (N.S.)*, 20:259–281, 1956.

[4] Carl Ludwig Siegel. Zur Theorie der Modulfunktionen n-ten Grades. *Comm. Pure Appl. Math.*, 8:677–681, 1955.

[5] André Weil. Groupes des formes quadratiques indéfinies et des formes bilinéaires alternées. In *Séminaire Henri Cartan*, volume 10. E.N.S., 1957–1958. No. 1, Talk no. 2, 14 p.

[6] André Weil. Réduction des formes quadratiques. In *Séminaire Henri Cartan*, volume 10. E.N.S., 1957–1958. No. 1, Talk no. 1, 9 p.