Compactification of Siegel's quotient spaces II*

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In this talk we consider the case of groups that are commensurable to the modular group. The notations Γ, $\mathfrak{G}_r^n$, $\Omega_n^*, \mathfrak{S}_n^*, \dots$ are the same as in the previous talk [3].

1 Additional considerations on the space $\mathfrak{S}_n^*$

Let $\tilde{\Gamma} = \tilde{\Gamma}_n$ be the "transformation group" of $\Gamma = \mathrm{Sp}(n, \mathbb{Z})$, i.e. $\tilde{\Gamma} = \mathrm{Sp}(n, \mathbb{Q})$; we first show how $\tilde{\Gamma}$ acts on $\mathfrak{S}_n^*$.

For this consider the set $\tilde{\mathfrak{S}}_n^*$ constructed by the same method as in [3, Section 2], but using $\tilde{\Gamma}$ instead of Γ, that is the set of points $\tilde{M} \cdot Z$ (classes of pairs $(\tilde{M}, Z)$) with $\tilde{M} \in \tilde{\Gamma}$, $Z \in \mathfrak{S}_r$ $(0 \le r \le n)$; moreover, we can assume that $\tilde{\mathfrak{S}}_n^*$ is endowed with a topology satisfying condition 1° and

$\tilde{2}^\circ$ the actions of $\tilde{M} \in \tilde{\Gamma}$ on $\tilde{\mathfrak{S}}_n^*$ are continuous maps

(for instance, consider the finest topology satisfying conditions 1° and $\tilde{2}^\circ$, defined as in [3, Section 3]); then $\tilde{\mathfrak{S}}_n^*$ contains $\mathfrak{S}_n^*$ as a subset; but in fact they are equal. Indeed, for each $\tilde{M} \in \tilde{\Gamma}$, there is a finite number of $M_i \in \Gamma$ such that

$$\tilde{M}\,\Omega_n \subset \bigcup_i M_i\,\Omega_n$$

(because Ω_n is a "fundamental open set" for the "Minkowskian" group Γ); since the topology on $\tilde{\mathfrak{S}}_n^*$ induces on $\mathfrak{S}_n^*$ a topology satisfying conditions 1° and 2°, we can take the closure with respect to this topology (which we denote $\cdot^*$, notation that does not clash with the notation Ω_n^*) and get

$$\tilde{M}\,\Omega_n^* = \left(\tilde{M}\,\Omega_n\right)^* \subset \bigcup_i M_i\,\Omega_n^*,$$

therefore

$$\tilde{\mathfrak{S}}_n^* = \tilde{\Gamma}\,\Omega_n^* \subset \Gamma\,\Omega_n^* = \mathfrak{S}_n^*;$$

we conclude that $\tilde{\mathfrak{S}}_n^* = \mathfrak{S}_n^*$. In particular, we can view $\tilde{\Gamma}$ as acting on $\mathfrak{S}_n^*$.

If we decompose $\tilde{\Gamma}$ into right cosets for $\tilde{\Gamma} \cap \mathfrak{G}_r^n$, we get a disjoint union decomposition of the space $\tilde{\mathfrak{S}}_n^*$ similar to (2.3) in the previous talk [3]; the fact that $\tilde{\mathfrak{S}}_n^* = \mathfrak{S}_n^*$ means that we can use elements of Γ as representatives of the cosets $\tilde{\Gamma}/\tilde{\Gamma} \cap \mathfrak{G}_r^n$; this means that

$$\tilde{\Gamma} = \Gamma\left(\tilde{\Gamma} \cap \mathfrak{G}_r^n\right) \qquad (0 \le r \le n). \tag{1.1}$$

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This fact was already indicated by Koecher [2] in the special case $r=0$ (and not only for Γ, but for all groups satisfying certain conditions).

The actions of $\tilde M \in \tilde\Gamma$ are $\mathcal{T}^\Gamma$-continuous. Indeed, let F be a $\mathcal{T}^\Gamma$-closed subset of $\mathfrak{S}_n^*$; we prove that for any $\tilde M \in \tilde\Gamma$, $\tilde M F$ is also $\mathcal{T}^\Gamma$-closed, that is that $M\tilde M F \cap \Omega_n^*$ is closed for all $M \in \Gamma$. There exist finitely many $M_i \in \Gamma$ such that

$$\left(M\tilde M\right)^{-1}\Omega_n^* \subset \bigcup_i M_i\,\Omega_n^*;$$

then

$$M\tilde M F \cap \Omega_n^* = \bigcup_i M\tilde M M_i\left(M_i^{-1}F \cap \Omega_n^*\right) \cap \Omega_n^*$$

and, since $M_i^{-1}F \cap \Omega_n^*$ is closed, so is the latter set, by the “continuity” of $M\tilde M M_i$ in Ω_n^* (this follows from [3, Lemma 1]).

The actions of $\tilde M \in \tilde\Gamma$ are also $\mathcal{T}_0^\Gamma$-continuous. It suffices to show that if U is a $\mathcal{T}^\Gamma$-neighborhood of x and Γ_x-saturated, then $\tilde M U$ (for $\tilde M \in \tilde\Gamma$) contains a $\mathcal{T}^\Gamma$-neighborhood of $\tilde M x$ that is $\Gamma_{\tilde M x}$-saturated. This follows immediately from the fact that $\tilde M U$ is $\tilde M\,\Gamma_x\,\tilde M^{-1}$-saturated and that

$$\tilde M\,\Gamma_x\,\tilde M^{-1} = \left(\tilde M\,\Gamma\,\tilde M^{-1}\right)_{\tilde M x}$$

is commensurable to $\Gamma_{\tilde M x}$.

Now let $\Gamma' = \Gamma'_n$ be a group commensurable to Γ; then there are finitely many $M_i \in \Gamma$ such that

$$\Omega'_n = \bigcup_i M_i\,\Omega_n \tag{1.2}$$

is a fundamental open for Γ'; we can take, for instance, those M_i such that

$$\Gamma = \bigcup_i \left(\Gamma \cap \Gamma'\right) M_i.$$

We have then $\mathfrak{S}_n^* = \Gamma'\left(\Omega'_n\right)^*$ (where $\cdot^*$ denotes the closure with respect to any topology that satisfies conditions 1° and 2°). Indeed, for any $M \in \Gamma$, there are finitely many $M'_j \in \Gamma'$ such that

$$M\,\Omega_n \subset \bigcup_j M'_j\,\Omega'_n;$$

hence

$$M\,\Omega_n^* \subset \bigcup_j M'_j\left(\Omega'_n\right)^*,$$

from which we conclude that $\mathfrak{S}_n^* = \Gamma\,\Omega_n^* \subset \Gamma'\left(\Omega'_n\right)^*$.

We now consider conditions 1’, 2’, 3’, and 4’ that are obtained respectively from conditions 1°, 2°, 3°, and 4° by replacing Γ and Ω_n^* by Γ' and $\left(\Omega'_n\right)^*$ (the “natural” topology on $\left(\Omega'_n\right)^*$ is that induced by any topology on $\mathfrak{S}_n^*$ that satisfies conditions 1° and 2°). It is clear that $\mathcal{T}^\Gamma$ satisfies conditions 1’ and 2’. Conversely, we can define the topology $\mathcal{T}^{\Gamma'}$ of $\mathfrak{S}_n^*$ as the finest topology satisfying conditions 1’ and 2’ (we proceed as in [3, Section 3]); then $\mathcal{T}^{\Gamma'}$ satisfies conditions 1° and 2° (the “continuity” of the actions of $\tilde M \in \tilde\Gamma$ on $\left(\Omega'_n\right)^*$); hence $\mathcal{T}^\Gamma = \mathcal{T}^{\Gamma'}$. We realize then by the same argument that the system of Γ'_x-saturated $\mathcal{T}^{\Gamma'}$-neighborhoods of x is equivalent to the system of Γ_x-saturated $\mathcal{T}^\Gamma$-neighborhoods of x; hence, if we define the topology $\mathcal{T}_0^{\Gamma'}$ in the same way as $\mathcal{T}_0^\Gamma$, we have $\mathcal{T}_0^{\Gamma'} = \mathcal{T}_0^\Gamma$. It is easy to see that $\mathcal{T}_0^{\Gamma'} = \mathcal{T}_0^\Gamma$ satisfies conditions 1’, 2’, 3’, and 4’; condition 3’ is proved as follows: let $x, x' \in \mathfrak{S}_n^*$ be two non-Γ'-equivalent points, let

$$\Gamma' = \bigcup_i \big(\Gamma \cap \Gamma'\big)\, M'_i,$$

and, for each i, let U_i, U'_i be neighborhoods of x and $\big(M'_i\big)^{-1} x'$ such that

- $\Gamma\, U_i \cap U'_i = \emptyset$ if x and $\big(M'_i\big)^{-1} x'$ are not Γ-equivalent;
- $\big((\Gamma - \Gamma_x) U_i\big) \cap U_i = \emptyset$, $U'_i = M U_i$, if $\big(M'_i\big)^{-1} x' = M x$ with $M \in \Gamma$;

then, as $\Gamma' \cap M\, \Gamma_x = \emptyset$, we have $\big((\Gamma \cap \Gamma')\, U_i\big) \cap U'_i = \emptyset$; hence, setting

$$U = \bigcap_i U_i, \qquad U' = \bigcap_i M'_i\, U'_i,$$

we have $\big(\Gamma'\, U\big) \cap U' = \emptyset$. If follows that $\mathcal{T}^{\Gamma}$ satisfies condition 3’; we can prove, by the same reasoning as in [3], the uniqueness of the topology satisfying conditions 1’, 2’, 3’, and 4’.

From now on we consider exclusively the topology $\mathcal{T}_0^{\Gamma} = \mathcal{T}_0^{\Gamma'}$; the results obtained above can be stated as follows:

Theorem 1. *The actions of $\tilde{M} \in \tilde{\Gamma}$ on $\mathfrak{S}_n^*$ are $\mathcal{T}_0^{\Gamma}$-continuous maps. For any group Γ' that is commensurable to Γ, the topology $\mathcal{T}_0^{\Gamma}$ satisfies the conditions 1’, 2’, 3’, and 4’, and is entirely determined by these conditions.*

2 The structure of the compactified spaces $\Gamma' \backslash \mathfrak{S}_n^*$

First, it is clear that Theorem 1 implies the following:

Theorem 2. *The quotient space $\Gamma' \backslash \mathfrak{S}_n^*$ is Hausdorff and compact.*

If Γ'' is a finite index subgroup of Γ', obviously there is a canonical map

$$\pi_{\Gamma',\Gamma''} :\ \Gamma'' \backslash \mathfrak{S}_n^* \longrightarrow \Gamma' \backslash \mathfrak{S}_n^* \tag{2.1}$$

that is a “ramified covering” (which we make more precise below); $\pi_{\Gamma',\Gamma''}$ is continuous and maps open sets to open sets and closed sets to closed sets. If moreover Γ'' is a **normal** subgroup of Γ', then $\Gamma'' \backslash \mathfrak{S}_n^*$ is “Galois” over $\Gamma' \backslash \mathfrak{S}_n^*$, which means that the finite group Γ'/Γ'' acts on $\Gamma'' \backslash \mathfrak{S}_n^*$ and we have

$$\big(\Gamma'/\Gamma''\big) \backslash \big(\Gamma'' \backslash \mathfrak{S}_n^*\big) = \Gamma' \backslash \mathfrak{S}_n^*. \tag{2.2}$$

We now study the structure of the space $\Gamma' \backslash \mathfrak{S}_n^*$. For this we decompose $\tilde{\Gamma}$ into left Γ'- and right $\big(\tilde{\Gamma} \cap \mathfrak{G}_r^n\big)$-cosets as follows:

$$\tilde{\Gamma} = \bigcup_\lambda \Gamma'\, M_{r,\lambda}\, \big(\tilde{\Gamma} \cap \mathfrak{G}_r^n\big), \tag{2.3}$$

where by (1.1) the number of cosets is finite; we have then the corresponding decomposition of $\mathfrak{S}_n^*$:

$$\mathfrak{S}_n^* = \bigcup_r \bigcup_\lambda \Gamma'\, M_{r,\lambda}\, \mathfrak{S}_r, \tag{2.4}$$

and therefore

$$\Gamma' \backslash \mathfrak{S}_n^* = \bigcup_r \bigcup_\lambda \Gamma' \backslash \big(\Gamma'\, M_{r,\lambda}\, \mathfrak{S}_r\big);$$

if we set

$$\Gamma'_{r,\lambda} = \varpi_r \left(M_{r,\lambda}^{-1} \, \Gamma' \, M_{r,\lambda} \cap \mathfrak{G}_r^n \right), \tag{2.5}$$

it is easy to see that $\Gamma'_{r,\lambda}$ is a discrete subgroup of $\mathrm{Sp}(r, \mathbb{R})$ that is commensurable to Γ_r, and that the quotient space $\Gamma' \backslash \big(\Gamma' \, M_{r,\lambda} \mathfrak{S}_r \big)$ is canonically identified with $\Gamma'_{r,\lambda} \backslash \mathfrak{S}_r$; hence the last relation can be written

$$\Gamma' \backslash \mathfrak{S}_n^* = \bigcup_{r,\lambda} \Gamma'_{r,\lambda} \backslash \mathfrak{S}_r. \tag{2.6}$$

We should note that if we set in (1.2)

$$M_i = M'_i \, M_{r,\lambda_i} \, L_i, \quad M'_i \in \Gamma', \, L_i \in \tilde{\Gamma} \cap \mathfrak{G}_r^n,$$

then

$$\Omega'_{r,\lambda} = \bigcup_{i \,:\; \lambda_i = \lambda} \varpi_r(L_i) \, \Omega_r \tag{2.7}$$

is a fundamental open for $\Gamma'_{r,\lambda}$; this is an immediate consequence of the fact that $\mathfrak{S}_n^* = \Gamma' \big(\Omega'_n \big)^*$.

We consider the relation between $\Gamma'_{r,\lambda} \backslash \mathfrak{S}_r^*$ and $\Gamma' \backslash \mathfrak{S}_n^*$. We first note that there is a canonical injective map from $\mathfrak{S}_r^*$ to $\mathfrak{S}_n^*$ given by

$$M.Z \longmapsto \iota_n(M).Z \qquad \big(M \in \tilde{\Gamma}_r, Z \in \mathfrak{S}_s, 0 \leq s \leq r \big),$$

because $(M, Z) \sim (M', Z')$ is obviously equivalent to $\big(\iota_n(M), Z \big) \sim \big(\iota_n(M'), Z' \big)$; this map, clearly a homeomorphism with respect to $\mathcal{T}^\Gamma$ or $\mathcal{T}_0^\Gamma$, allows us to identify $\mathfrak{S}_r^*$ with the closure of $\mathfrak{S}_r$ in $\mathfrak{S}_n^*$.

Given Γ', there exists also a map $\psi_{r,\lambda}$ from $\Gamma'_{r,\lambda} \backslash \mathfrak{S}_r^*$ to $\Gamma' \backslash \mathfrak{S}_n^*$ given by

$$\psi_{r,\lambda} \big(M.Z \pmod{\Gamma'_{r,\lambda}} \big) = M_{r,\lambda} \, \iota_n(M).Z \pmod{\Gamma'}, \tag{2.8}$$

since, if $M.Z$ and $M'.Z'$ are $\Gamma'_{r,\lambda}$-equivalent, there exists $M'_0 \in \Gamma'$ such that

$$\varpi_r \left(M_{r,\lambda}^{-1} M'_0 \, M_{r,\lambda} \right) M.Z = M'.Z',$$

that is

$$\tilde{M}_0 = (M')^{-1} \varpi_r \left(M_{r,\lambda}^{-1} M'_0 \, M_{r,\lambda} \right) M \in \mathfrak{G}_s^r \quad \text{and} \quad \varpi_s \big(\tilde{M}_0 \big) Z = Z',$$

which implies that

$$\iota_n(M')^{-1} M_{r,\lambda}^{-1} M'_0 \, M_{r,\lambda} \, \iota_n(M) \in \mathfrak{G}_s^n$$

and

$$\varpi_s \left(\iota_n(M')^{-1} M_{r,\lambda}^{-1} \, M'_0 \, M_{r,\lambda} \, \iota_n(M) \right) = \varpi_s \big(\tilde{M}_0 \big),$$

that is that $M_{r,\lambda} \, \iota(M).Z$ and $M_{r,\lambda} \iota_n(M').Z'$ are Γ'-equivalent.

As the following diagram is commutative, it is clear that the map $\psi_{r,\lambda}$ is **continuous**:

$$\begin{array}{ccc} \mathfrak{S}_r^* & \xrightarrow{\;M_{r,\lambda}\;} & \mathfrak{S}_n^* \\ \big\downarrow & & \big\downarrow \\ \Gamma'_{r,\lambda} \backslash \mathfrak{S}_r^* & \xrightarrow{\;\psi_{r,\lambda}\;} & \Gamma' \backslash \mathfrak{S}_n^* \end{array} \tag{2.9}$$

but, as we are about to see, it is in general **not injective**.

Indeed, let $s < r < n$ and consider double coset decompositions

$$\begin{aligned}\tilde\Gamma_n &= \bigcup_\lambda \Gamma' M_{r,\lambda}\big(\tilde\Gamma_n \cap \mathfrak{G}_r^n\big)\\ &= \bigcup_\mu \Gamma' M_{s,\mu}\big(\tilde\Gamma \cap \mathfrak{G}_s^n\big),\\ \tilde\Gamma_r &= \bigcup_\nu \Gamma'_{r,\lambda}\, M_{s,\nu}^{(r,\lambda)}\big(\tilde\Gamma_r \cap \mathfrak{G}_s^r\big);\end{aligned} \tag{2.10}$$

we have then

$$\tilde\Gamma_n \cap \mathfrak{G}_r^n = \bigcup_\nu \big(M_{r,\lambda}^{-1}\Gamma' M_{r,\lambda} \cap \mathfrak{G}_r^n\big)\iota_n\big(M_{s,\nu}^{(r,\lambda)}\big)\big(\tilde\Gamma_n \cap \mathfrak{G}_r^n \cap \mathfrak{G}_s^n\big),$$

since $\tilde\Gamma_r = \varpi_r\big(\tilde\Gamma_n \cap \mathfrak{G}_r^n\big)$ and $\varpi^{-1}(\mathfrak{G}_s^r) = \mathfrak{G}_r^n \cap \mathfrak{G}_s^n$; therefore

$$\tilde\Gamma_n = \bigcup_{\lambda,\nu} \Gamma' M_{r,\lambda}\, \iota_n\big(M_{s,\nu}^{(r,\lambda)}\big)\big(\tilde\Gamma_n \cap \mathfrak{G}_r^n \cap \mathfrak{G}_s^n\big); \tag{2.11}$$

this is the decomposition of $\tilde\Gamma_n$ into left-Γ' and right-$\tilde\Gamma_n \cap \mathfrak{G}_r^n \cap \mathfrak{G}_s^n$ double cosets, a refinement of the second decomposition of (2.10). We write

$$(\lambda,\nu) \longrightarrow \mu \quad \text{if} \quad M_{r,\lambda}\, \iota_n\big(M_{s,\nu}^{(r,\lambda)}\big) \in \Gamma' M_{s,\mu}\big(\tilde\Gamma_n \cap \mathfrak{G}_s^n\big). \tag{2.12}$$

Then the function $\psi_{r,\lambda}$ maps

$$\Gamma'_{r,\lambda}\backslash\Gamma'_{r,\lambda}\, M_{s,\nu}^{(r,\lambda)}\mathfrak{S}_s \longmapsto \Gamma'\backslash\Gamma' M_{r,\lambda}\, \iota_n\big(M_{s,\nu}^{(r,\lambda)}\big)\mathfrak{S}_s = \Gamma'\backslash\Gamma' M_{s,\mu}\mathfrak{S}_s,$$

or, setting

$$\begin{aligned}\big(\Gamma'_{r,\lambda}\big)_{s,\nu} &= \varpi_s\left(M_{s,\nu}^{(r,\lambda)^{-1}}\Gamma'_{r,\lambda}M_{s,\nu}^{(r,\lambda)} \cap \mathfrak{G}_s^r\right),\\ M_{r,\lambda}M_{s,\nu}^{(r,\lambda)} &= M' M_{s,\mu} L, \quad M' \in \Gamma', L \in \tilde\Gamma_n \cap \mathfrak{G}_s^n,\end{aligned}$$

$\psi_{r,\lambda}$ maps

$$\big(\Gamma'_{r,\lambda}\big)_{s,\nu}\backslash\mathfrak{S}_s \longrightarrow \Gamma'_{s,\mu}\backslash\mathfrak{S} \quad \text{via} \qquad Z \pmod{\big(\Gamma'_{r,\lambda}\big)_{s,\nu}} \longmapsto L.Z \pmod{\Gamma'_{s,\mu}}, \tag{2.13}$$

for $(\lambda,\nu) \longrightarrow \mu$. It is possible that two distinct pairs (λ,ν) and (λ',ν') (even with $\lambda = \lambda'$) correspond to the same μ; on the other hand, we have

$$\begin{aligned}\big(\Gamma'_{r,\lambda}\big)_{s,\nu} &= \varpi_s\left(M_{s,\nu}^{(r,\lambda)^{-1}}\Gamma'_{r,\lambda}M_{s,\nu}^{r,\lambda} \cap \mathfrak{G}_s^r\right)\\ &= \varpi_s\left(\iota_n\big(M_{s,\nu}^{(r,\lambda)}\big)^{-1}M_{r,\lambda}^{-1}\Gamma' M_{r,\lambda}\, \iota_n\big(M_{s,\nu}^{r,\lambda}\big) \cap \mathfrak{G}_r^n \cap \mathfrak{G}_s^n\right)\\ &\subset \varpi_s\left(\iota_n\big(M_{s,\nu}^{(r,\lambda)}\big)^{-1}M_{r,\lambda}^{-1}\Gamma' M_{r,\lambda}\, \iota_n\big(M_{s,\nu}^{r,\lambda}\big) \cap \mathfrak{G}_s^n\right)\\ &= \varpi_s\left(L^{-1}M_{s,\mu}^{-1}\Gamma' M_{s,\mu} L \cap \mathfrak{G}_s^n\right)\\ &= \varpi_s(L)^{-1}\Gamma'_{s,\mu}\varpi_s(L);\end{aligned}$$

and it is possible that $\big(\Gamma'_{r,\lambda}\big)_{s,\nu}$ is strictly smaller than $\varpi_s(L)^{-1}\Gamma'_{s,\mu}\varpi_s(L)$. These two possible cases mean that, in general, $\psi_{r,\lambda}$ is not injective.

Example Let us consider the case of the "Hauptkongruenzgruppe":

$$\Gamma_n(q) = \{M : \ M \in \Gamma_n, M \equiv E_n \pmod q\}.$$

In this case any $\Gamma'_{r,\lambda}$ is equal to $\Gamma_r(q)$, hence we are **not** in the second case

$$(\Gamma'_{r,\lambda})_{s,\nu} \subsetneq \varpi_s(L)^{-1}\Gamma'_{s,\mu}\varpi_s(L));$$

let us compute the "multiplicity" $\nu_{n,r}$ of $\Gamma_r(q)$. We have obviously

$$\begin{aligned}\nu_{n,r} &= [\Gamma_n \colon \Gamma_n(q)(\Gamma_n \cap \mathfrak{G}^n_r)]\\ &= [\Gamma_n \colon \Gamma_n(q)]/[\Gamma_n \cap \mathfrak{G}^n_r \colon \Gamma_n(q) \cap \mathfrak{G}^n_r]\\ &= [\Gamma_n \colon \Gamma_n(q)]/[\Gamma_r \colon \Gamma_r(q)] \cdot [\Gamma_n \cap \mathfrak{N}^n_r \colon \Gamma_n(q) \cap \mathfrak{N}^n_r],\end{aligned}$$

where $\mathfrak{N}^n_r$ denotes the kernel of ϖ_r. But $\mathfrak{N}^n_r$ decomposes as a semi-direct product

$$\mathfrak{N}^n_r = \mathfrak{U}^n_r \ltimes \mathfrak{T}^n_r, \tag{2.14}$$

where

$$\mathfrak{U}^n_r = \left\{ \begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{pmatrix} \ : \ U = \begin{pmatrix} E_r & U_{12} \\ 0 & U_2 \end{pmatrix}, \det(U_2) \neq 0 \right\}, \tag{2.15}$$

$$\mathfrak{T}^n_r = \left\{ \begin{pmatrix} E & T \\ 0 & E \end{pmatrix} \ : \ T = \begin{pmatrix} 0 & T_{12} \\ {}^tT_{12} & T_2 \end{pmatrix}, T_2 \text{ symmetric} \right\}, \tag{2.16}$$

the latter being a normal subgroup of $\mathfrak{N}^n_r$. As

$$\Gamma_n(q) \cap \mathfrak{N}^n_r = (\Gamma_n(q) \cap \mathfrak{U}^n_r) \ltimes (\Gamma_n(q) \cap \mathfrak{T}^n_r)$$

and

$$\Gamma_n(q) \cap \mathfrak{U}^n_r = \left\{ \begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{pmatrix} \ : \ U = \begin{pmatrix} E_r & U_{12} \\ 0 & U_2 \end{pmatrix} \equiv E_n \pmod q, U \text{ unimodular} \right\},$$

$$\Gamma_n(q) \cap \mathfrak{T}^n_r = \left\{ \begin{pmatrix} E & T \\ 0 & E \end{pmatrix} \ : \ T = \begin{pmatrix} 0 & T_{12} \\ {}^tT_{12} & T_2 \end{pmatrix} \equiv E_n \pmod q, T \text{ integral, symmetric} \right\},$$

we have

$$\begin{aligned}[\Gamma_n \cap \mathfrak{N}^n_r \colon \Gamma_n(q) \cap \mathfrak{N}^n_r] &= [\Gamma_n \cap \mathfrak{N}^n_r \colon \Gamma_n(q) \cap \mathfrak{U}^n_r] \cdot [\Gamma_n \cap \mathfrak{T}^n_r \colon \Gamma_n(q) \cap \mathfrak{T}^n_r]\\ &= (2)[\gamma_{n-r} \colon \gamma_{n-r}(q)]q^{r(n-r)}q^{\frac{(n-r)(n-r+1)}{2}+r(n-r)},\end{aligned}$$

where

$$\gamma_{n-r} = \mathrm{SL}(n-r,\mathbb{Z}), \quad \gamma_{n-r}(q) = \{U \in \gamma_{n-r} : \ U \equiv E_{n-r} \pmod q\},$$

and the factor (2) appears if $n-r \geq 1$ and $q > 2$. It is well-known that

$$[\gamma_n \colon \gamma_n(q)] = q^{n^2-1} \prod_{p|q} \prod_{2\leq k\leq n} (1-p^{-k}),$$

$$[\Gamma_n \colon \Gamma_n(q)] = q^{n(2n+1)} \prod_{p|q} \prod_{1\leq k\leq n} (1-p^{-2k})$$

(see [1]). We obtain

$$\begin{aligned}\nu_{n,r} &= \frac{q^{n(2n+1)-r(2r+1)} \prod_{p|q} \prod_{r+1\leq k\leq n} (1-p^{-2k})}{(2)q^{\frac{(n-r)(n-r+1)}{2}+2r(n-r)+(n-r)^2-1} \prod_{p|q} \prod_{2\leq k\leq n-r} (1-p^{-k})}\\ &= q^{\frac{1}{2}(n-r)(n+3r+1)+1}(2)^{-1} \prod_{p|q} \frac{\prod_{r+1\leq k\leq n} (1-p^{-2k})}{\prod_{2\leq k\leq n-r} (1-p^{-k})}\end{aligned} \tag{2.17}$$

for $r<n$. If $s<r<n$, we have therefore

$$\frac{\nu_{n,r}\nu_{r,s}}{\nu_{n,s}} = q^{(n-r)(r-s)+1}(2)^{-1}\prod_{p|q}\frac{\prod_{2}^{n-s}\left(1-p^{-k}\right)}{\prod_{2}^{n-r}\left(1-p^{-k}\right)\prod_{2}^{r-s}\left(1-p^{-k}\right)} > 1,$$

which shows that the map $\Gamma_r(q)\backslash\mathfrak{S}_r^* \longrightarrow \Gamma_n(q)\backslash\mathfrak{S}_n^*$ is certainly not injective if $0<r<n$.

3 A connectedness theorem

Finally we add a theorem that will be useful later.

Theorem 3. *Every point of $\mathfrak{S}_n^*-\mathfrak{S}_n$ has a base of $\mathcal{T}_0^\Gamma$-neighborhoods whose intersection with $\mathfrak{S}_n$ is connected and open.*

Proof. We can assume that the point in question is $Z_0 \in \Omega_r$ ($r<n$). Let U_r be a connected, $\left(\Gamma_r\right)_{Z_0}$-saturated neighborhood of Z_0 in Ω_r and let $U_s = V^{(s)}(U_r, K)$ ($r\leq s\leq n$) be the set defined in [3, p. 3], that is the set of $Z\in\Omega_s$ such that

$$Z=\begin{pmatrix} Z_1^{(r)} & Z_{12} \\ {}^tZ_{12} & Z_2\end{pmatrix} = X+iY, \quad Y={}^tWDW, \quad D=\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_s\end{pmatrix}$$

with $Z_1^{(r)}\in U_r$, $d_{r+1}>K$; then

$$U=\bigcup_{r\leq s\leq n} U_s$$

is a neighborhood of Z_0 in Ω_n^* and therefore $\tilde{U}=\Gamma_{Z_0}U$ is a $\mathcal{T}_0^\Gamma$-neighborhood of Z_0 in $\mathfrak{S}_n^*$. We will prove that $\tilde{U}=\Gamma_{Z_0}U$ intersects $\mathfrak{S}_n$ in a connected set, that is that $\Gamma_{Z_0}U_n$ is connected. It is easy to see that Γ_{Z_0} is *finitely generated* (note that $\Gamma\cap\mathfrak{N}_r^n$ is a finite index subgroup of Γ_{Z_0} and is finitely generated); let $\{M_i\}$ be a finite generating set for Γ_{Z_0} that contains the identity E; we may assume in addition that M_i is of one of the following forms:

$$M_i=\begin{pmatrix} E_r & 0 & 0 & 0 \\ 0 & {}^tU_2 & 0 & 0 \\ 0 & 0 & E_r & 0 \\ 0 & 0 & 0 & U_2^{-1}\end{pmatrix}, \qquad U_2 \text{ unimodular,} \tag{3.1}$$

$$M_i=\begin{pmatrix} A_1 & 0 & B_1 & B_{12} \\ A_{21} & E_{n-r} & B_{21} & B_2 \\ C_1 & 0 & D_1 & D_{12} \\ 0 & 0 & 0 & E_{n-r}\end{pmatrix}. \tag{3.2}$$

(Indeed, Γ_{Z_0} is the semi-direct product of subgroups consisting of matrices of respective forms (3.1) and (3.2), with the latter subgroup being normal.) Decomposing $Z'=M_iZ$, $Z\in U_n$ as above, we see easily that if M_i is of the form (3.2), then Z' belongs to $\Omega_n(u')$, where $u'\geq u$ depends only on U_r, K, and M_i; therefore $M_iU_n\subset\Omega_n(u')$ for u' sufficiently large. Now let M_i be of the form (3.1); for an arbitrary but fixed matrix Z_i in U_n we can again take u' such that $Z_i'=M_iZ_i\in\Omega_n(u')$; in this case we have the relations

$$X_1'=X_1, \qquad X_{12}'=X_{12}U_2, \qquad X_2'={}^tU_2X_2U_2, \qquad W_1'=W_1,$$
$$W_{12}'=W_{12}U_2, \qquad D_1'=D_1, \qquad {}^tW_2'D_2'W_2'={}^tU_2{}^tW_2D_2W_2U_2;$$

so if we denote by $Z_i(\lambda)$ and $Z_i'(\lambda)$ the matrices obtained by replacing D_2 and D_2' by λD_2 and $\lambda D_2'$ ($\lambda \geq 1$) inside Z_i, respectively Z_i', we see easily that $Z_i(\lambda) \in U_n$, $Z_i'(\lambda) \in \Omega_n(u')$, $Z_i'(\lambda) = M_i Z_i(\lambda)$ and hence $Z_i'(\lambda)$ ($\lambda \geq 1$) belongs to $M_i U_n \cap \Omega_n(u')$. Let u' be such that all the above conditions are satisfied; we define a neighborhood

$$U' = \bigcup_{r \leq s \leq n} U_s'$$

of Z_0 in $\Omega_n(u')^*$ exactly as above with $U_r' = U_r$ and K' large enough that $U' \subset \tilde{U}$; then $M_i U_n \cap U_n' = \emptyset$ for all i; indeed, if M_i is of the form (3.2), we can take, for a given K', K_1 such that $M_i V^{(n)}(U_r, K_1) \subset U_n'$; on the other hand, if M_i is of the form (3.1), $Z_i'(\lambda)$ belongs to $M_i U_n \subset U_n'$ for sufficiently large λ. Then, since U_n and U_n' are connected, so is $U_n \cup U_n'$, and $(U_n \cup U_n') \cap M_i(U_n \cup U_n') \neq \emptyset$ for all i, from which we conclude the connectedness of $\Gamma_{Z_0} U_n = \Gamma_{Z_0}(U_n \cup U_n')$. □

Bibliographic note

The two results of Koecher referred to in this talk can be found in [1, 2].

References

[1] Max Koecher. Zur Theorie der Modulformen n-ten Grades. I. *Math. Z.*, 59:399–416, 1954.

[2] Max Koecher. Zur Theorie der Modulformen n-ten Grades. II. *Math. Z.*, 61:455–466, 1955.

[3] Ichiro Satake. Compactification des espaces quotients de Siegel I. In *Séminaire Henri Cartan*, volume 10. E.N.S., 1957–1958. No. 2, Talk no. 12, 13 p.