Letter[*] from J-P. Serre to J. Tate, 7 August 1987

Jean-Pierre Serre

Dear Tate,

I feel as though I understand a little bit better the modular forms (mod p), as well as our dear $W_k = M_k/M_{k-(p-1)}$ from 1973 and 1974.

I started with the following problem: how should one interpret in an adelic manner the modular forms (mod p) of all levels and all weights? (This is question 2, p. 198, of my article in *Duke Math. J.*, t. 54—the one “aimed at optimists”.) More precisely, we are interested in the eigenvalues (a_ℓ) of the Hecke operators T_ℓ ($\ell \neq p$, ℓ coprime to the level) coming from these modular forms. Here is the answer (or in any case *an* answer. . .):

Let D be the quaternion algebra over $\mathbb{Q}$ ramified at $\{p, \infty\}$ and let $D_{\mathbb{A}}^\times$ be the group of adelic points of the multiplicative group $D^\times$. Then:

Theorem. *The systems of eigenvalues* (a_ℓ) *(with* $a_\ell \in \overline{\mathbb{F}}_p$*) coming from the modular forms (mod* p*) are* the same *as those coming from the locally constant functions* $f\colon D_{\mathbb{A}}^\times/D_{\mathbb{Q}}^\times \to \overline{\mathbb{F}}_p$.

(The action of the T_ℓ’s on these functions is defined in a more or less obvious manner, except for a factor of $1/\ell$ multiplying the naive Hecke operator.)

The functions f described above can also be seen as functions $f\colon D_{\mathbb{A}}^\times \to \overline{\mathbb{F}}_p$ such that

$$f(ux\gamma) = f(x) \tag{1}$$

for all $\gamma \in D_{\mathbb{Q}}^\times$ and all u in an open subgroup of $D_{\mathbb{A}}^\times$. Note that any open subgroup contains the real component $D_{\mathbb{R}}^\times$, which is connected. We can therefore remove $D_{\mathbb{R}}^\times$ if we so desire, i.e. work with the ring of finite adeles $\mathbb{A}_f$.

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[*]This appeared in the *Israel Journal of Mathematics*, Vol. 95, 1996 (bundled with a subsequent letter from Serre to Kazhdan), under the title *Two letters on quaternions and modular forms (mod* p*)*.
Translated from the original French by Alexandru Ghitza <aghitza@alum.mit.edu>.
For the historical and mathematical context, see the prefacing comments by R. Livné in the above reference. In particular, note that before publication, Serre removed a few short paragraphs (indicated by “. . . ” in the text), as well as inserted a few comments (enclosed within brackets [ ] in the text).

Proof of the Theorem. Fix a level $N \geq 3$, coprime to p, and work with modular forms (mod p) of level N, in the manner of Katz in Anvers[1] III (LN 350). The corresponding modular curve is not absolutely irreducible; too bad! By definition, a form of weight k, with coefficients in $\overline{\mathbb{F}}_p$, associates to any pair (E, α), where E is an elliptic curve and α an N-level structure on E, an element $f(E, \alpha)$ of $\omega^k(E)$, i.e. an (invariant) differential k-form on E. It is also, if you want, a section of a certain sheaf $\mathcal{M}_k$ on the modular curve $X(N)$. I will denote $M_k(N)$, or simply M_k, the space of global sections:

$$M_k = H^0(X(N), \mathcal{M}_k).$$

According to Swinnerton-Dyer (for $p \geq 5$) and Katz (for $p = 2, 3$), there is a natural embedding $M_{k-(p-1)} \to M_k$ given by multiplication by a certain form A of weight $p-1$ (namely E_{p-1} if $p \geq 5$, b_2 if $p = 3$ and a_1 if $p = 2$).

In 1973–1974, we were very interested in the structure of the quotient

$$W_k = M_k / M_{k-(p-1)},$$

seen as a module over the Hecke operators T_ℓ, $\gcd(\ell, pN) = 1$.

From a sheaf point of view, this involves considering the exact sequence

$$0 \to \mathcal{M}_{k-(p-1)} \xrightarrow{A} \mathcal{M}_k \to \mathcal{S}_k \to 0,$$

where $\mathcal{S}_k$ is the cokernel of multiplication by A. As A vanishes at the supersingular points with multiplicity 1, the structure of the sheaf $\mathcal{S}_k$ is clear: it is 0 away from the supersingular points ("$\mathcal{S}$"="supersingular"), and of dimension 1 at these points. Let S_k be the space of global sections of $\mathcal{S}_k$. We have the exact sequence

$$0 \to M_{k-(p-1)} \to M_k \to S_k \to H^1(\mathcal{M}_{k-(p-1)}) \to H^1(\mathcal{M}_k) \to 0,$$

or even:

$$0 \to W_k \to S_k \to H^1(\mathcal{M}_{k-(p-1)}) \to H^1(\mathcal{M}_k) \to 0. \tag{2}$$

We have therefore embedded W_k into a slightly larger space S_k; the two spaces are by the way equal if $k > p+1$ as in this case $H^1(\mathcal{M}_{k-(p-1)})$ is 0 (by duality).

The space S_k is much easier to describe concretely than its subspace W_k: by its very construction, it is the space of functions

elliptic curve over $\overline{\mathbb{F}}_p$ with level N structure	$\mapsto$	invariant differential k-form on the curve.

The action of the Hecke operators T_ℓ on S_k is just as obvious. If $f(E, \alpha)$ is a function as above (with E supersingular) we have

$$(f \mid T_\ell)(E, \alpha) = \frac{1}{\ell} \sum_C f(E/C, \alpha_C), \tag{3}$$

[1] Serre presumably means Antwerp III.

where C ranges over the $\ell+1$ subgroups of order ℓ of E, where α_C denotes the level N structure on E/C induced by α, and where I identify the differential forms on E/C to those on E, via the isogeny $E \to E/C$. In short, it is as usual!

Of course, this action of T_ℓ on S_k extends the action on W_k.

Remarks:

(4) *The S_k only depend on k* modulo p^2-1 (and on N and $p\ldots$).

Indeed any supersingular curve over $\overline{\mathbb{F}}_p$ has a canonical (and functorial) $\mathbb{F}_{p^2}$-structure, namely the one where the Frobenius is equal to $-p$. Then the tangent space to E also has a canonical $\mathbb{F}_{p^2}$-structure, and its (p^2-1)-st tensor power has a canonical basis. This basis allows us to identify $\omega^k(E)$ and $\omega^{k+p^2-1}(E)$, and this identification is compatible with isogenies, hence with the operators T_ℓ. (We already knew this result for W_k for large enough k; in fact S_k is the "stabilization" of W_k, as topologists would say.)

[Canonical basis for $\omega^{p^2-1}(E)$ for E supersingular:

Let's write E in the standard form:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

and let $\omega = dx/(2y + a_1x + a_3)$.

If $p = 2$, the canonical basis of $\omega^{p^2-1}(E)$ is $a_3\omega^{\otimes 3}$.

If $p = 3$, it is $b_4^2\omega^{\otimes 8}$, where $b_4 = a_1a_3 + 2a_4$.

If $p \geq 5$, it is $B^{p-1}\omega^{\otimes(p^2-1)}$, where B is the Eisenstein series E_{p+1}.]

Another useful formula (which I do not need at the moment):

(5) $$S_{k+p+1} \cong S_k[1], \qquad \text{where } [1] \text{ denotes a "Tate twist".}$$

This formula will become obvious later, from the quaternionic point of view. We knew it already—but only up to semisimplification—for the W_k with k large enough. According to G. Robert (*Invent. math.* **61** (1980), p. 123), the isomorphism $S_k[1] \to S_{k+p+1}$ is given by multiplication by $B = E_{p+1}$ if $p \geq 5$. There are analogous constructions for $p = 2$ and $p = 3$.

[If $p = 2$ we choose in M_3 an element A_3 whose image in $S_3 = M_3/M_2$ is the element "a_3" given above (such an element exists because $N \geq 3$); multiplication by A_3 gives the desired isomorphism $S_k[1] \to S_{k+3}$.

If $p = 3$, it is the same thing using the element $b_4 = a_1a_3 + 2a_4$ of S_4.]

(6) *Any system (a_ℓ) of eigenvalues of the T_ℓ that comes from an M_k also comes from an $S_{k'}$ and vice-versa.*

(The weight k' may be different from k, but in any case we have

$$k' \equiv k \pmod{p-1}.)$$

This is clear: if (a_ℓ) comes from $f \in M_k$, we write f as $A^m g$, with g not divisible by A; the image of g in $S_{k'}$, where $k' = k - m(p-1)$, is nonzero and corresponds to (a_ℓ). Conversely, if (a_ℓ) comes from S_k, we may, thanks to the periodicity of the S_k's, assume that $k \geq p+1$, in which case S_k is a quotient of M_k and therefore (a_ℓ) comes from M_k.

Conclusion: instead of looking at the T_ℓ's over the M_k's for $k = 0, 1, \ldots$, it suffices to look at them over the S_k's where k ranges over the integers modulo $(p^2 - 1)$. This suggests constructing the direct sum

(7)
$$S(N) = \bigoplus_{k \mod p^2-1} S_k(N).$$

(8) You see now what we are about to do: we will interpret $S(N)$ as *a space of functions on* $D^\times_{\mathbb{A}}/D^\times_{\mathbb{Q}}$, using the well-known correspondence between supersingular curves and quaternions.

More precisely, choose a maximal order $D_\mathbb{Z}$ of $D = D_\mathbb{Q}$, and set:

$O_p = \mathbb{Z}_p \otimes D_\mathbb{Z} =$ the unique maximal order of $D_p = \mathbb{Q}_p \otimes D_\mathbb{Z}$;

$O_p^\times =$ the multiplicative group of O_p;

$O_p^\times(1) =$ the kernel of $O_p^\times \to \mathbb{F}^\times_{p^2}$, that is the kernel of reduction (mod π), where π is a uniformizer of O_p;

$O_\ell = \mathbb{Z}_\ell \otimes D_\mathbb{Z}$, isomorphic to the matrix algebra $M_2(\mathbb{Z}_\ell)$, $\ell \neq p$;

$O_\ell^\times =$ the multiplicative group of $O_\ell \cong \mathrm{GL}_2(\mathbb{Z}_\ell)$;

$O_\ell^\times(N) =$ the subgroup of the latter consisting of the elements $\equiv 1 \pmod{\ell^n}$, where ℓ^n is the largest power of ℓ that divides N;

$U(1, N) = D^\times_\mathbb{R} \times O_p^\times(1) \times \prod_{\ell \neq p} O_\ell^\times(N)$, an open subgroup of $D^\times_\mathbb{A}$.

Consider the finite set $\Omega_N = U(1, N)\backslash D^\times_\mathbb{A}/D^\times_\mathbb{Q}$. The following statement will not surprise you:

(9) *There is a bijection* (almost but not quite canonical, see below) *between* Ω_N *and the set of isomorphism classes of triples* (E, ω, α)*, where* E *is a supersingular elliptic curve over* $\overline{\mathbb{F}}_p$*,* ω *is a nonzero invariant differential form on* E *rational over* $\mathbb{F}_{p^2}$*, and* α *is a level* N *structure on* E*.* (Moreover, this bijection is compatible with loads of more or less obvious operators, in particular the correspondences T_ℓ.)

Let's admit (9), which is a mere exercise (see below). We deduce from it:

(10) *The space* $S(N) = \bigoplus S_k(N)$ *defined in* (7) *is isomorphic to the space of functions on* Ω_N*, and this isomorphism is compatible with*

a) *the action of the* T_ℓ*'s, for* $\ell \nmid pN$*;*

b) *the decomposition with respect to the weight* mod $(p^2 - 1)$.

(On the side of Ω_N, the "weight" comes from the natural action of $O_p^\times / O_p^\times(1) = \mathbb{F}_{p^2}^\times$ on Ω_N.)

In other words, we can interpret $S(N)$ as the space of functions $f\colon D_{\mathbb{A}}^\times \to \overline{\mathbb{F}}_p$ such that $f(ux\gamma) = f(x)$ if $u \in U(1,N)$, $\gamma \in D_{\mathbb{Q}}^\times$. And the union of the $S(N)$'s with varying N can be identified with the space V_1 of locally constant functions on $D_{\mathbb{A}}^\times / D_{\mathbb{Q}}^\times$ that are invariant under $O_p^\times(1)$.

To finish the proof of the theorem stated at the start, it remains to explain why the condition of invariance under $O_p^\times(1)$ is irrelevant. This is simply because $O_p^\times(1)$ is an *invariant pro-p-subgroup* in $D_p^\times$, therefore also in $D_{\mathbb{A}}^\times$. We have the following lemma:

Lemma. *Let* G *be a pro-*p *group acting continuously on a vector space* V *over* $\overline{\mathbb{F}}_p$*, and let* T_ℓ *be a set of endomorphisms of* V *that commute with* G*. Let* (a_ℓ) *be a system of eigenvalues for the* T_ℓ*'s corresponding to a common eigenvector* $v \neq 0$ *in* V*. We can then choose* v *to be invariant under* G *(without changing the* (a_ℓ)*).*

(If V_a is the eigenspace of V corresponding to (a_ℓ), then V_a is $\neq 0$ and stable under G, hence contains a vector $\neq 0$ that is fixed by G.)

This concludes, more or less, the proof of the theorem. To complete it, I have to give some details about the proof of (9). This is a bit annoying, but essentially trivial. One way to proceed is to interpret the elements of $\Omega_N = U(1,N)\backslash D_{\mathbb{A}}^\times / D_{\mathbb{Q}}^\times$ as *isomorphism classes of projective* $D_{\mathbb{Z}}$*-modules of rank* 1*, endowed with "level* πN *structures".* (If $\mathfrak{a}$ is a nonzero two-sided ideal of $D_{\mathbb{Z}}$, a "level $\mathfrak{a}$ structure" on a projective $D_{\mathbb{Z}}$-module P is simply a *basis* of $P/\mathfrak{a}P$ as a $D_{\mathbb{Z}}/\mathfrak{a}$-module.) We then choose a triple (E, ω, α) with $\mathrm{End}(E) = D_{\mathbb{Z}}$ and note that, if P is a projective $D_{\mathbb{Z}}$-module of rank 1 with level πN structure, then the elliptic curve $E_P = E \otimes_{D_{\mathbb{Z}}} P$ is automatically endowed with an ω and an α. The map

$$\text{class of } P \mapsto \text{ class of } (E_P, \omega, \alpha)$$

is bijective, as is easily seen (the main point is, of course, that two supersingular curves are isogenous.) I can't be bothered to give further details.

$\square$

Some complements:

(11) The action of $D_p^\times$ on the space $S(N)$ is of "dihedral type"; in particular, a uniformizer π of $D_p^\times$ *interchanges* S_k and S_{pk}, which are therefore isomorphic as (T_ℓ)-modules (we already knew this, thanks to the operator V of the usual theory). We can also see this in terms of projective modules with level πN structure: to such a module P we associate its unique submodule of index p^2, endowed with the obvious level πN structure (not entirely obvious, for the level π part of the structure... one needs to think a little).

(12) We can use the *action of the center of* $D_{\mathbb{A}}^\times$ to decompose the space of functions on $D_{\mathbb{A}}^\times/D_{\mathbb{Q}}^\times$ just as we do in the complex case. The central characters that appear here are trivial at infinity. They are characters $\varpi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \overline{\mathbb{F}}_p^\times$. We decompose them as $\chi^k\varepsilon$, where χ is the usual cyclotomic character (mod p) and ε has conductor coprime to p; the integer k is defined mod $(p-1)$, and has the same parity as ε if $p \neq 2$. If (a_ℓ) is given by an eigenfunction with character $\varpi = \chi^k\varepsilon$, the corresponding Galois representation ρ_a satisfies

$$\det \rho_a = \chi^{-1}\varpi = \chi^{k-1}\varepsilon.$$

(There is therefore a "twist by χ^{-1}" compared to what we would get from a correspondence *à la* Langlands. In Deligne's terminology (LN 349, pp. 99–100) it is a correspondence "*à la* Hecke", unless it is "*à la* Tate"...)

(13) If $\psi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \overline{\mathbb{F}}_p^\times$ is an arbitrary character, by composing ψ with the reduced norm $\mathrm{Nrd}\colon D_{\mathbb{A}}^\times \to \mathbb{A}^\times$, we get a function on $D_{\mathbb{A}}^\times/D_{\mathbb{Q}}^\times$ that I will denote ψ_D. It is an eigenfunction for the T_ℓ's with eigenvalues $(1+\ell^{-1})\psi(\ell)$, for ℓ coprime to the conductor of ψ; the corresponding Galois representation is $\chi^{-1}\psi \oplus \psi$, of Eisenstein type. The central character is ψ^2.

The function ψ_D can be used to *twist* a system of eigenvalues. Indeed, if f is a locally constant function on $D_{\mathbb{A}}^\times/D_{\mathbb{Q}}^\times$, we have:

$$(f.\psi_D) \mid T_\ell = \psi(\ell)(f \mid T_\ell).\psi_D.$$

The case $\psi = \chi$ is particularly interesting: the corresponding function χ_D belongs to S_{p+1}, and the above formula shows that *the map* $f \mapsto f.\chi_D$ *is an isomorphism from* $S_k[1]$ *to* S_{k+p+1}, as stated in (5). (This proof is very closely related to that of G. Robert, *loc.cit.*, p. 124, Lemma 7.)

(14) I return to the exact sequence (2) from the start:

$$0 \to W_k \to S_k \to H^1(\mathcal{M}_{k-(p-1)}) \to H^1(\mathcal{M}_k) \to 0. \tag{2}$$

We can determine the H^1's via duality: $H^1(\mathcal{M}_k)$ is dual to $H^0(\Omega \otimes \mathcal{M}_{-k})$. As Ω is isomorphic to the sheaf $\mathcal{M}_2^0$ of cusp forms of weight 2, $\Omega \otimes \mathcal{M}_{-k}$ is isomorphic to $\mathcal{M}^0_{2-k}$. We therefore transform (2) into the exact sequence

(2') $$0 \to W_k \to S_k \to \text{the dual of } M^0_{p+1-k} \to \text{the dual of } M^0_{2-k} \to 0.$$

What is the structure of (T_ℓ)-module of the dual of M^0_{p+1-k} that is compatible with this exact sequence? One would want to say (but I do not know how to prove it) that this module is, perhaps up to semisimplification, a *twist* of M^0_{p+1-k}, the only reasonable twist being, by the way:

(15) $$M^0_{p+1-k}[k-1].$$

You had yourself obtained a similar result when you proved that any system of eigenvalues can be obtained, up to twist, in weight $\leq p+1$. (Conversely, if a formula as above were true, it would give an easy proof of this twist result: using (5), we place ourselves in an S_k with $1 \leq k \leq p+1$ and then use (2').)

To prove (15), one needs the courage to describe the behavior of the duality theorem in relation to correspondences. Not amusing! I will do without for now.

I want to tell you now about the *problems* that come up. There are plenty of these. Here are the main ones:

(16) *How can one describe the subspace* W_k *of* S_k *(*$0 < k \leq p+1$*) in quaternion terms*, i.e. in terms of functions on the space $D^\times_{\mathbb{A}}/D^\times_{\mathbb{Q}}$? One would want to say that the W_k's, and their images under the action of $D^\times_{\mathbb{A}}$, generate a pretty $D^\times_{\mathbb{A}}$-submodule, but how to characterize it? Must we involve functions that are invariant not under $O^\times_p(1)$, but under $O^\times_p(n)$, $n \geq 2$? I don't see it.

A related question is to define Atkin's "U_p" operator in quaternion terms. Note that U_p cannot be defined on all of S_k, as it is stably zero; but one should be able to define it on W_k for $1 < k \leq p+1$.

(17) One would like to know how to define directly the Galois representation

$$\rho_a\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\overline{\mathbb{F}}_p)$$

attached to a system (a_ℓ) of eigenvalues of the T_ℓ's. It is not clear that this is a reasonable question. But in any case we would like to know this: if a system (a_ℓ) comes from an eigenfunction $f \in S_k$, is it true that it can only come from an $S_{k'}$ if we have

(18) $$k' \equiv k \text{ or } pk \pmod{p^2-1}?$$

Alas, (18) appears to be false for a system (a_ℓ) of Eisenstein type, i.e. corresponding to a reducible representation ρ_a. But I hope it is true when ρ_a is irreducible. If that were the case, ρ_a would determine the pair (p, pk) mod (p^2-1), which would be a "multiplicity 1 theorem" for the p-component. Moreover, if $k(\rho_a)$ denotes the weight attached to ρ_a by the somewhat quirky rules in *Duke Math. J.*, t. 54, we would have:

(19) $k(\rho_a) =$ one of the two integers (or the unique integer) in the interval $[1, p^2 - 1]$ congruent to k or pk mod $(p^2 - 1)$.

This would explain why the weights in *Duke* are $\leq p^2 - 1$ (careful, for $p = 2$, we have to modify the definition in *Duke* by replacing 4 by 3).

Of course, we would want to make (19) more precise, and pinpoint which of the two integers in question is equal to $k(\rho_a)$; this requires knowledge of the subspaces W_k of the S_k's, i.e. one must first know how to answer (16).

(20) A question unrelated to quaternions, but natural in the context of the weights:

Start with $f \in M_k$, with $k = 1$, an eigenfunction for the Hecke operators, and let ρ be the corresponding Galois representation. *Is it true that ρ is unramified at p?* This is clear if f lifts to a weight 1 form in characteristic 0, but we are dealing here with forms "*à la* Katz", which have no reason to lift to characteristic 0. One therefore needs a different proof. How to go about it? The question is linked to the special case $k = 1$ of (16): how to characterise $W_1 = M_1$ inside the much larger space S_1?

Of course, we would want the converse to be true: if ρ is unramified at p, it should come from M_1. Unfortunately I lack numerical examples for this type of situation. Even the dihedral case (for instance when $p = 2$ and $\mathrm{Im}(\rho) = \mathrm{GL}_2(\mathbb{F}_2) = S_3$) is not obvious.

[This question has been mostly answered by B. Gross (*Duke Math. J.*, **61** (1990), 445–517) and R.F. Coleman–J.F. Voloch (*Invent. math.*, **110** (1992), 263–281). See also B. Edixhoven, *Invent. math.* **109** (1992), 563–594.]

(21) This brings us to consider *the structure of $D_{\mathbb{A}}^{\times}$-module* on the space F of locally constant functions (with values in $\overline{\mathbb{F}}_p$) on the homogeneous space $D_{\mathbb{A}}^{\times}/D_{\mathbb{Q}}^{\times}$. I know too little of the complex theory to be sure of the right questions to ask. In any case, we can fix a central character ω, and restrict our attention to the subspace F_ω of F consisting of functions f such that $f(xy) = \omega(x)f(y)$ for all x in the center of $D_{\mathbb{A}}^{\times}$. The direct sum of the F_ω's is not F, but that is not a big deal: any simple submodule of F is contained in an F_ω. Regarding the F_ω's, we would like them to contain "sufficiently many" simple modules. For instance:

(22) *Does every nonzero $D_{\mathbb{A}}^{\times}$-submodule of F_ω contain a simple submodule?*

[The answer is: no. The only simple submodules of F are the dimension 1 subspaces generated by the ψ_D, cf. (13). See the letter to Kazhdan.]

...

(24) If an (a_ℓ) comes from a level N_1, as well as from another level N_2, *does it come from level* $\gcd(N_1, N_2)$ (assuming this $\gcd \geq 3$, to avoid trouble)? This would be a theorem "*à la* Ribet". One should be able to prove this, provided one has good answers to the questions asked in (21).

(25) *Links with Eichler's theory*. One way to attack the space F described above (that of locally constant functions on $D_{\mathbb{A}}^\times / D_{\mathbb{Q}}^\times$) is to view it as the reduction (mod p) of the space of functions (complex, if we like—or integer-valued, if that is preferred) that are locally constant on the same space. Up to changing the level, this boils down to looking at the T_ℓ's as "Brandt matrices", or rather as the reduction (mod p) of Brandt matrices. Thanks to Eichler, we know that this produces the same semisimplification as a certain space of weight 2 on "$\Gamma_0(p)$ in level N", at least for k divisible by $p + 1$. Whence another way of comparing this space with that of modular forms (mod p). To be honest, I am too unfamiliar with Eichler's theory (especially with the levels π and N used here) to be able to state the correspondence precisely. But this should not be difficult for the experts (Gross, Ribet, Marie-France).

(26) *p-adic analogues*. Instead of considering the locally constant functions on $D_{\mathbb{A}}^\times / D_{\mathbb{Q}}^\times$, with values in $\mathbb{C}$, it would be more amusing to consider those *with p-adic values*, i.e. with values in $\overline{\mathbb{Q}}_p$. If we decompose $\mathbb{A}$ into $\mathbb{Q}_p \times \mathbb{A}'$, we would ask for these functions to be locally constant with respect to the variable in $D_{\mathbb{A}'}$ and to be continuous (or analytic, or more) with respect to the variable in D_p... Would there be p-adic Galois representations attached to such functions, presumably to eigenfunctions for the Hecke operators? Can we interpret the constructions of Hida (and Mazur) in such a way? I have no idea.

(27) *Generalizations*. We can extend this "day-dreaming" by asking which algebraic groups can replace $D^\times$ in all of the above. One thing is certain: one needs a condition of "compactness" at infinity.

...

J-P. Serre

PS—It is possible that all my k's must be replaced by $-k$'s, and other things of the same type; various conventions are possible here, and I have not yet made a choice.