Estimate of a constant related to the prime-counting function

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Abstract

Using new results on the bounds of the prime-counting function, we are able to improve on the bounds of a constant that appears in the asymptotic expansion of the sum of the reciprocal of the prime-counting function.

Contents

Chapter 1

Introduction

The prime-counting function, denoted by $\pi(x)$, counts the number of prime numbers up to a given number. A well-known result, referred to as the Prime Number Theorem, states that $\pi(x) \sim \frac{x}{\log x}$, or equivalently, $\pi(x) \sim \mathrm{li}(x)$ where $\mathrm{li}(x)$, the logarithmic integral function, is given by

$$\mathrm{li}(x) = \mathrm{PV} \int_0^x \frac{dt}{\log t} = \lim_{\varepsilon \to 0} \left(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t} \right)$$

The Prime Number Theorem was proven in 1896 independently by Hadamard [8] and de la Vallée Poussin [4]. They both used properties of the Riemann zeta function, $\zeta(s)$, which is the analytic continuation of $\sum_{n=1}^{\infty} n^{-s}$.

Related to $\pi(x)$ are two functions referred to as the Chebyshev functions; the first Chebyshev function, $\vartheta(x)$, and the second Chebyshev function, $\psi(x)$, are defined as

$$\vartheta(x) = \sum_{p \le x} \log p \qquad \psi(x) = \sum_{p^k \le x} \log p$$

One direct relation between $\pi(x)$ and $\vartheta(x)$ is the identity

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt$$

The Prime Number Theorem is equivalent to the Chebyshev functions being asymptotic to x.

One problem related to $\pi(x)$ that has been explored relatively recently is regarding the asymptotic behaviour of the sum of the reciprocal of $\pi(x)$. In 1980, de

Koninck and Ivić [11] showed that

$$\sum_{2\leq n\leq N} \frac{1}{\pi(n)} = \frac{1}{2}\log^2 N + O(\log N)$$

In 2000, Panaitopol [12] used an improved formula for $\pi(x)$ to show that

$$\sum_{2\leq n\leq N} \frac{1}{\pi(n)} = \frac{1}{2}\log^2 N - \log N - \log\log N + O(1)$$

Then, in 2002, Ivić [10] found that

$$\begin{aligned}\sum_{2\leq n\leq N} \frac{1}{\pi(n)} =& \frac{1}{2}\log^2 N - \log N - \log\log N + C \\ &+ \frac{k_2}{\log N} + \cdots + \frac{k_m}{(m-1)\log^{m-1} N} + O\left(\frac{1}{\log^m N}\right)\end{aligned}$$

where C is an absolute constant and $\{k_m\}_m$ is the integer sequence defined by the recurrence relation

$$k_1 = 1, \quad k_m + 1!k_{m-1} + \cdots + (n-1)!k_1 = n\cdot n!$$

In 2008, Hassani and Moshtagh [9] showed that, for $x \geq 2$,

$$\alpha(x) \leq \sum_{2\leq n\leq x} \frac{1}{\pi(n)} - \frac{1}{2}\log^2 x + \log x \leq \beta(x)$$

where $\alpha(x) = -1.51\log\log x + 0.8994$ and $\beta(x) = -0.79\log\log x + 6.4888$, and concluded that $C \approx 6.9$.

In 2016, Berkane and Dusart [2] further improved upon this. Letting

$$S(x) = \sum_{2\leq n\leq x} \frac{1}{\pi(n)} - \frac{1}{2}\log^2 x + \log x + \log\log x$$

they found that, for $x \geq 150721071$,

$$6.68400420 + \frac{6}{\sqrt[10]{\log^{11} x}} \leq S(x) \leq \frac{2}{\log x} + 6.78291066$$

with the upper bound holding also for $x \geq 25555987$, giving the bounds $6.6840 \leq C \leq 6.7830$.

Since then, improved bounds on $\pi(x)$ have been found by Axler [1]. His results rely on work by Büthe [3], whose bounds on $\vartheta(x)$ and $\psi(x)$ come from numerical computation and uses the zeroes of $\zeta(s)$ with imaginary part up to 10^{11}, and Dusart [5][6], who obtains bounds on $\vartheta(x)$ and $\psi(x)$ using various results on the location and density of the zeroes of $\zeta(s)$, including the numerical verification of the Riemann hypothesis for the first 10^{13} non-trivial zeroes by Gourdon [7].

Based on the method used by Berkane and Dusart, and applying the results found by Axler, we have managed to improve on the bounds of the constant C.

Theorem 1.1. *$S(x) \to C$ as $x \to \infty$ where $6.71433 < C < 6.74328$.*

This is a consequence of Theorem 4.1 proven in Chapter 4. The proof of the theorem starts by using bounds on $\pi(x)$ proven in Chapter 3, based on the work of Axler [1], to express $\sum_{2 \le n \le x} \frac{1}{\pi(n)}$ as the sum of quotients involving logarithms and the harmonic series. These are then compared to their integrals and Euler's constant respectively via results proven in Chapter 2. From there, the bound is rearranged into the form

$$\frac{1}{2}\log^2 x - \log x - \log\log x + C + R(x)$$

where C contains all terms independent of x. After calculating the value of C, it is then shown that $R(x)\log x$ and $R(x)\sqrt[10]{\log^{11} x}$ for the upper and lower bounds respectively are bounded by constants for large enough x. Theorem 1.1 then follows from the fact that $R(x) \to 0$ as $x \to \infty$ for both the upper and lower bounds.

Chapter 2

Preliminary Results

2.1 Bounds on sums of quotients involving logarithms

We define the Bernoulli numbers b_n by the exponential generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{b_n x^n}{n!}$$

In particular, this gives $b_0 = 1$, $b_1 = -\frac{1}{2}$, $b_2 = \frac{1}{6}$, $b_3 = 0$, and so on. We denote by b_n^+ the sequence of Bernoulli numbers with $b_1^+ = +\frac{1}{2}$. We also define the Bernoulli polynomials

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^k$$

In particular, $B_n(0) = b_n$ and $B_n(1) = b_n^+$. Additionally, these polynomials have the property that

$$\int_a^b B_n(x)dx = \frac{B_{n+1}(b) - B_{n+1}(a)}{n+1}$$

Next we state the Euler-MacLaurin summation formula.

Theorem 2.1 (Euler-MacLaurin). *If $a \leq N$ are integers and $f \in C^m([a, N])$, then*

$$\sum_{n=a+1}^{N} f(n) - \int_a^N f(t)dt = \sum_{k=1}^{m} \frac{b_k^+}{k!} \left(f^{(k-1)}(N) - f^{(k-1)}(a)\right) + R_m(a, N)$$

where $R_m(a, N)$ is given by

$$R_m(a, N) = \frac{(-1)^{m+1}}{m!} \int_a^N f^{(m)}(x) B_m(\{x\}) dx$$

where $\{x\}$ denotes the fractional part of x.

For fixed integers $m \geq 0$ and $a \geq 10$, we define the operator

$$\mathfrak{B}_m(f)(N) = f(a) + \sum_{k=0}^{m} \frac{b_{k+1}^{+}}{(k+1)!} \left(f^{(k)}(N) - f^{(k)}(a) \right)$$

We also define the functions

$$f_1(x) = \frac{\log x}{x}, \quad f_2(x) = \frac{1}{x \log x}, \quad f_3(x) = \frac{1}{x \log^2 x}$$

We will use the following lemma to get bounds of sums over these functions:

Lemma 2.1. *For integers $N \geq a \geq 10$ and $i \in \{1, 2, 3\}$, the following hold*

$$\mathfrak{B}_0(f_i)(N) + \frac{N-a}{12} f_i^{(2)}(N) \leq \sum_{n=a}^{N} f_i(n) - \int_a^N f_i(t)dt$$
$$\leq \mathfrak{B}_1(f_i)(N) - \frac{N-a}{720} f_i^{(4)}(N)$$

Proof. Starting with the lower bound, using the Euler-MacLaurin summation formula with $m = 1$, we get that

$$\sum_{n=a}^{N} f_i(n) - \int_a^N f_i(t)dt = \mathfrak{B}_0(f_i)(N) + \int_a^N f_i'(t) B_1(\{t\})\, dt$$

Focusing on the integral on the right-hand side, we get that

$$\begin{aligned}
\int_a^N f_i'(t) B_1(\{t\})\, dt &= \sum_{k=a}^{N-1} \int_k^{k+1} f_i'(t) B_1(t-k) dt \\
&= \frac{1}{2} \sum_{k=a}^{N-1} \left(\left[f_i'(t) \left(B_2(t-k) - \frac{1}{6} \right) \right]_k^{k+1} \right. \\
&\qquad \left. - \int_k^{k+1} f_i^{(2)}(t) \left(B_2(t-k) - \frac{1}{6} \right) dt \right) \\
&= \frac{1}{2} \sum_{k=a}^{N-1} \left(f_i'(k+1) \left(B_2(1) - \frac{1}{6} \right) - f_i'(k) \left(B_2(0) - \frac{1}{6} \right) \right) \\
&\qquad - \frac{1}{2} \int_a^N f_i^{(2)}(t) \left(B_2(\{t\}) - \frac{1}{6} \right) dt \\
&= \frac{1}{2} \int_a^N f_i^{(2)} \left(\{t\} - \{t\}^2 \right) dt
\end{aligned}$$

since $B_2(x) = x^2 - x + \frac{1}{6}$. By the mean value theorem for definite integrals, there exists $\xi \in (a, N)$ such that

$$\begin{aligned}\frac{1}{2}\int_a^N f_i^{(2)}\left(\{t\} - \{t\}^2\right) dt &= \frac{1}{2} f_i^{(2)}(\xi) \int_a^N \{t\} - \{t\}^2 dt \\ &= \frac{1}{2} f_i^{(2)}(\xi)(N-a)\int_0^1 t - t^2 dt \\ &= \frac{N-a}{2} f_i^{(2)}(\xi)\left[\frac{1}{2}t^2 - \frac{1}{3}t^3\right]_0^1 \\ &= \frac{N-a}{12} f_i^{(2)}(\xi)\end{aligned}$$

Since $f_i^{(2)}(x)$ is a decreasing function for $x \geq 10$ for each i, $f_i^{(2)}(\xi) \geq f_i^{(2)}(N)$ and thus we get the lower bound.

For the upper bound, we use the Euler-MacLaurin summation formula again with $m = 2$ to get

$$\sum_{n=a}^N f_i(n) - \int_a^N f_i(t)dt = \mathfrak{B}_1(f_i)(N) - \frac{1}{2}\int_a^N f_i^{(2)}(t) B_2\left(\{t\}\right) dt$$

Focusing on the integral on the right-hand side, we get that

$$\begin{aligned}\int_a^N f_i^{(2)}(t) B_2\left(\{t\}\right) dt &= \frac{1}{3}\sum_{k=a}^{N-1}\left(\left[f_i^{(2)}(t) B_3(t-k)\right]_k^{k+1} - \int_k^{k+1} f_i^{(3)}(t) B_3(t-k) dt\right) \\ &= -\frac{1}{3}\int_a^N f_i^{(3)}(t)\left(\{t\}^3 - \frac{3}{2}\{t\}^2 + \frac{1}{2}\{t\}\right) dt \\ &= \frac{1}{12}\int_a^N f_i^{(4)}(t)\left(\{t\}^4 - 2\{t\}^3 + \{t\}^2\right) dt\end{aligned}$$

By the mean value theorem for definite integrals, there exists $\xi \in (a, N)$ such that

$$\begin{aligned}\int_a^N f_i^{(4)}(t)\left(\{t\}^4 - 2\{t\}^3 + \{t\}^2\right) dt &= f_i^{(4)}(\xi)\int_a^N \{t\}^4 - 2\{t\}^3 + \{t\}^2 dt \\ &= (N-a) f_i^{(4)}(\xi)\int_0^1 t^4 - 2t^3 + t^2 dt \\ &= \frac{N-a}{30} f_i^{(4)}(\xi)\end{aligned}$$

Since $f_i^{(4)}(x)$ is a decreasing function for $x \geq 10$ for each i, $f_i^{(4)}(\xi) \geq f_i^{(4)}(N)$ and thus we get the upper bound. $\square$

2.2 Bounds on the partial sums of the harmonic series

Lemma 2.2. *For all $x \geq 1$, we have that*

$$\frac{1}{2(x+1)} \leq \sum_{1 \leq k \leq x} \frac{1}{k} - \log x - \gamma \leq \frac{1}{2x}$$

where $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right)$ *is Euler's constant.*

Proof. First we will show that each of the following functions is positive for $x \geq 1$

$$Q_1(x) = \frac{1}{2x} + \frac{1}{2(x+1)} - \log\left(1 + \frac{1}{x}\right)$$

$$Q_2(x) = \log\left(1 + \frac{1}{x}\right) - \frac{1}{x + \frac{1}{2}}$$

$$Q_3(x) = \frac{1}{x + \frac{1}{2}} - \frac{2x+5}{2(x+1)(x+2)}$$

For $Q_1(x)$, we first note that $Q_1(1) = \frac{3}{4} - \log 2 > 0$. Additionally, if we try to find the zeroes of $Q_1(x)$, we get that

$$\begin{aligned}
Q_1(x) = 0 &\iff \frac{1}{2x} + \frac{1}{2(x+1)} = \log\left(1 + \frac{1}{x}\right) \\
&\iff \exp\left(\frac{1}{2x} + \frac{1}{2(x+1)}\right) = 1 + \frac{1}{x} \\
&\iff \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{2x} + \frac{1}{2(x+1)}\right)^k = 1 + \frac{1}{x} \\
&\iff \frac{1}{8x^2(x+1)^2} + \sum_{k=3}^{\infty} \frac{1}{k!}\left(\frac{1}{2x} + \frac{1}{2(x+1)}\right)^k = 0
\end{aligned}$$

However, since all terms on the left-hand side of the last expression are strictly positive when $x \geq 1$, $Q_1(x) \neq 0$ for any $x \geq 1$. In addition, the singularities of $Q_1(x)$ are $x = 0$ and $x = -1$, neither of which are ≥ 1. Thus, since $Q_1(x)$ is continuous for $x \geq 1$, $Q_1(x) > 0$ for all $x \geq 1$.

For $Q_2(x)$, first note that $Q_2(x) \to 0$ as $x \to \infty$. Now we take the derivative of

$Q_2(x)$ and get

$$\begin{aligned}\frac{d}{dx}Q_2(x) &= \frac{-\frac{1}{x^2}}{1+\frac{1}{x}} + \frac{1}{\left(x+\frac{1}{2}\right)^2}\\ &= -\frac{1}{x^2+x} + \frac{1}{\left(x+\frac{1}{2}\right)^2}\\ &= \frac{-\left(x+\frac{1}{2}\right)^2 + x^2 + x}{x(x+1)\left(x+\frac{1}{2}\right)^2}\\ &= -\frac{1}{4x(x+1)\left(x+\frac{1}{2}\right)^2} < 0\end{aligned}$$

for $x \geq 1$. Finally we observe that $Q_2(1) = \log 2 - \frac{2}{3} > 0$. Thus we conclude that $Q_2(x) > 0$ for all $x \geq 1$.

For $Q_3(x)$, we first note that $Q_3(1) = \frac{1}{12} > 0$. Additionally, if we try to find the zeroes of $Q_3(x)$, we get that

$$\begin{aligned}Q_3(x) = 0 &\iff \frac{1}{x+\frac{1}{2}} = \frac{2x+5}{2x^2+6x+4}\\ &\iff (2x+5)\left(x+\frac{1}{2}\right) = 2x^2+6x+4\\ &\iff 2x^2+6x+\frac{5}{2} = 2x^2+6x+4\end{aligned}$$

Thus $Q_3(x) \neq 0$ for any x. In addition, all singularities of $Q_3(x)$ are in the range $x < 0$. Thus, since $Q_3(x)$ is continuous for $x \geq 1$, $Q_3(x) > 0$ for all $x \geq 1$.

For our next step, we define

$$\mathcal{L}(x) = \sum_{1\leq k\leq x}\frac{1}{k} - \log x - \frac{1}{2x}, \qquad \mathcal{U}(x) = \sum_{1\leq k\leq x}\frac{1}{k} - \log x - \frac{1}{2(x+1)}$$

We will use these to bound γ, giving the required result. For the lower bound, let $n \in \mathbb{N}$. Then

$$\begin{aligned}\mathcal{L}(n+1) &= \sum_{k=1}^{n+1}\frac{1}{k} - \log(n+1) - \frac{1}{2(n+1)}\\ &= \sum_{k=1}^{n}\frac{1}{k} - \log n - \frac{1}{2n} + \frac{1}{n+1} - \log\left(1+\frac{1}{n}\right) + \frac{1}{2n} - \frac{1}{2(n+1)}\\ &= \mathcal{L}(n) + Q_1(n) > \mathcal{L}(n)\end{aligned}$$

So the sequence $\{\mathcal{L}(n)\}_{n\in\mathbb{N}}$ is strictly increasing and limits to γ. Additionally, if $x \in [n, n+1)$, then

$$\frac{d}{dx}\mathcal{L}(x) = -\frac{1}{x} + \frac{1}{2x^2} < 0$$

and thus $\mathcal{L}(x)$ is strictly decreasing between each integer. Thus we get that

$$\mathcal{L}(x) \leq \sup_{x\geq 1} \mathcal{L}(x) \leq \sup_{n\geq 1} \mathcal{L}(n) = \gamma$$

For the upper bound, let $n \in \mathbb{N}$. Then

$$\begin{aligned}
\mathcal{U}(n+1) &= \sum_{k=1}^{n+1} \frac{1}{k} - \log(n+1) - \frac{1}{2(n+2)} \\
&= \mathcal{U}(n) + \frac{1}{n+1} - \log\left(1+\frac{1}{n}\right) + \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \\
&= \mathcal{U}(n) - \log\left(1+\frac{1}{n}\right) + \frac{2n+5}{2(n+1)(n+2)} \\
&= \mathcal{U}(n) - Q_2(n) - Q_3(n) < \mathcal{U}(n)
\end{aligned}$$

So the sequence $\{\mathcal{U}(n)\}_{n\in\mathbb{N}}$ is strictly decreasing and limits to γ. Additionally, if $x \in [n, n+1)$, then

$$\frac{d}{dx}\mathcal{U}(x) = -\frac{1}{x} + \frac{1}{2(x+1)^2} < 0$$

and thus $\mathcal{U}(x)$ is strictly decreasing between each integer. Thus we get that

$$\mathcal{U}(x) \geq \inf_{x\geq 1} \mathcal{U}(x) \geq \inf_{n\geq 1} \mathcal{U}(n) = \gamma$$

□

Chapter 3

Bounding the prime-counting function

3.1 Bounds on the first Chebyshev function

Lemma 3.1. *We have, for all $x \geq 19\,035\,709\,163$, that*

$$\vartheta(x) > x - \frac{0.15x}{\log^3 x}$$

and, for all $x > 1$, that

$$\vartheta(x) < x + \frac{0.15x}{\log^3 x}$$

Proof. We will start by showing that, for all $x \geq e^{35}$, that

$$|\vartheta(x) - x| < \frac{0.15x}{\log^3 x}$$

From Corollary 1.2 of [5], letting $R = 5.69693$, we have, for $x \geq 3$, that

$$|\vartheta(x) - x| < \frac{\sqrt{8}}{\sqrt{\pi\sqrt{R}}} x \left(\log x\right)^{\frac{1}{4}} e^{-\sqrt{\frac{\log x}{R}}}$$

Now define the function

$$g(x) = \left(\log x\right)^{\frac{13}{4}} e^{-\sqrt{\frac{\log x}{R}}}$$

Differentiating, we get that

$$g'(x) = \frac{13}{4x} \left(\log x\right)^{\frac{9}{4}} e^{-\sqrt{\frac{\log x}{R}}} - \frac{1}{2x\sqrt{R}} \left(\log x\right)^{\frac{11}{4}} e^{-\sqrt{\frac{\log x}{R}}}$$

We note that $g'(x) \geq 0$ for $x \geq e^{\frac{169R}{4}}$, so $g(x)$ is monotonically decreasing for $x \geq e^{\frac{169R}{4}}$. Thus, we get, for all $x \geq e^{5000}$, that

$$|\vartheta(x) - x| < \frac{\sqrt{8}}{\sqrt{\pi\sqrt{R}}} g(e^{5000}) \frac{x}{\log^3 x} < \frac{0.148x}{\log^3 x}$$

Next, from Corollary 4.5 from [6], we have, for $x > 0$, that

$$\psi(x) - \vartheta(x) < \left(1 + 1.47 \times 10^{-7}\right)\sqrt{x} + 1.78x^{\frac{1}{3}}$$

This also holds in absolute value since $\vartheta(x) < \psi(x)$. Additionally, from Proposition 3.2 in [6], we have, for $x \geq e^{b_i}$, that

$$|\psi(x) - x| < \varepsilon_i x$$

where for each b_i the corresponding value of ε_i is given in Table 1 of [6]. If we assume, in addition, that $x \leq e^{b_{i+1}}$, then we get that

$$\begin{aligned}
|\vartheta(x) - x| &\leq |\vartheta(x) - \psi(x)| + |\psi(x) - x| \\
&< (1 + 1.47 \times 10^{-7})\sqrt{x} + 1.78x^{\frac{1}{3}} + \varepsilon_i x \\
&= \left(\frac{(1 + 1.47 \times 10^{-7})\log^3 x}{\sqrt{x}} + \frac{1.78\log^3 x}{\sqrt[3]{x^2}} + \varepsilon_i \log^3 x\right)\frac{x}{\log^3 x} \\
&\leq \left(\frac{(1 + 1.47 \times 10^{-7})b_{i+1}^3}{\sqrt{e^{b_i}}} + \frac{1.78b_{i+1}}{\sqrt[3]{e^{2b_i}}} + \varepsilon_i b_{i+1}^3\right)\frac{x}{\log^3 x}
\end{aligned}$$

By going from $b_8 = 35$ to $b_{37} = 4500$ and substituting the corresponding values of ε_i, we find that the expression in brackets above is less than 0.15 for all values of $i \in \{8, \ldots, 37\}$. Thus we conclude that $|\vartheta(x) - x| < \frac{0.15x}{\log^3 x}$ for all $x \geq e^{35}$.

Next we will show that $\vartheta(x) < x + \frac{0.15x}{\log^3 x}$ for $1 < x < e^{35}$. From Theorem 2 in [3], we have that $\vartheta(x) < x - 0.05\sqrt{x}$ for $1 \leq x \leq 10^{19}$. Since $e^{35} < 10^{19}$, the result immediately follows.

Finally, we will show that $\vartheta(x) > x - \frac{0.15x}{\log^3 x}$ for $19\,035\,709\,163 \leq x < e^{35}$. Again from Theorem 2 in [3], we have that $\vartheta(x) \geq x - 1.95\sqrt{x}$ for $1423 \leq x \leq 10^{19}$. Now, define the function

$$g(x) = \frac{0.15\sqrt{x}}{\log^3 x}$$

Taking the derivative, we get that

$$g'(x) = \frac{0.075\log x - 0.45}{\sqrt{x}\log^4 x}$$

which is strictly positive for $x > e^6$. Additionally, we note that $g(34\,485\,879\,392) > 1.95$. Thus we conclude that $1.95 < \frac{0.15\sqrt{x}}{\log^3 x}$ for $x \geq 34\,485\,879\,392$. Thus we conclude that $\vartheta(x) > x - \frac{0.15x}{\log^3 x}$ for $34\,485\,879\,392 \leq x < e^{35}$.

Next, equation (6.2) on page 2004 of [3] tells us that, for $100 \leq x \leq 5 \times 10^{10}$,

$$-0.8 \leq \frac{x - \psi(x)}{\sqrt{x}} \leq 0.81$$

Thus, from Lemma 1 of [3], we get that, for $10\,000 \leq x \leq 5 \times 10^{10}$,

$$\vartheta(x) \geq x - 1.81\sqrt{x} - 0.8x^{\frac{1}{4}} - 1.03883\left(x^{\frac{1}{3}} + x^{\frac{1}{5}} + 2x^{\frac{1}{13}} \log x\right)$$

Now let

$$h_1(x) = \frac{100}{4\,563}x^{\frac{2}{15}} - \frac{64}{21\,125}$$

We observe that $h_1(x) > 0$ for all $x \geq 1$. Next let

$$h_2(x) = \frac{10}{117}x^{\frac{10}{39}} - \frac{8}{325}x^{\frac{8}{65}} - \frac{2}{13}$$

We note that $h_2(23) > 0$ and $h_2'(x) = \frac{h_1(x)}{x^{57/65}}$, and thus $h_2(x) > 0$ for $x \geq 23$. Now let

$$h_3(x) = \frac{1}{3}x^{\frac{10}{39}} - \frac{1}{5}x^{\frac{8}{65}} - \frac{2}{13}\log x - 2$$

We note that $h_3(19\,449) > 0$ and $h_3'(x) = \frac{h_2(x)}{x}$, and thus $h_3(x) > 0$ for $x \geq 19\,449$. Finally, we let

$$h_4(x) = x^{\frac{1}{3}} - x^{\frac{1}{5}} - 2x^{\frac{1}{13}} \log x$$

We note that $h_4(783\,674) > 0$ and $h_4'(x) = \frac{h_3(x)}{x^{12/13}}$, and thus $h_4(x) > 0$ for $x \geq 783\,674$. Thus, for $783\,674 \leq x \leq 5 \times 10^{10}$,

$$\vartheta(x) \geq x - 1.81\sqrt{x} - 0.8x^{\frac{1}{4}} - 2 \times 1.03883x^{\frac{1}{3}}$$

Now let

$$g_1(z) = 432z^3 - 540z^2 + 288z - 60$$

We observe that g_1 has only one real root at $z \approx 0.45593$ and that $g_1(1) = 120 > 0$, and thus $g_1(z) > 0$ for all $z \geq 1$. Next let

$$g_2(w) = \frac{w^7}{160\log^3 w} - \frac{3w^7}{640\log^4 w} + \frac{w^7}{960\log^5 w} - 10.86w$$

We note that $g_2(5) > 0$ and that

$$\frac{d}{dw}\left(\frac{g_2(w)}{w}\right) = \frac{w^5 g_1(\log w)}{11\,520 \log^6 w}$$

and thus $g_2(w) > 0$ for $w \geq 5$. Next let

$$g_3(w) = \frac{w^8}{1\,280\log^3 w} - \frac{w^8}{3\,840\log^4 w} - 5.43w^2 - 2.07766$$

We note that $g_3(7) > 0$ and $g_3'(w) = g_2(w)$, and thus $g_3(w) > 0$ for $w \geq 7$. Finally, we let

$$g_4(w) = \frac{w^9}{11\,520\log^3 w} - 1.81w^3 + 2.07766w + 0.8$$

We note that $g_4(7.46497465501) > 0$ and $g_4'(w) = g_3(w)$, and thus $g_4(w) > 0$ for $w \geq 7.46497466501$. If we then consider $w^3 g_4(w)$ and make the substitution $x = w^{12}$, then we get that

$$\frac{0.15x}{\log^3 x} \geq 1.81\sqrt{x} + 0.8x^{\frac{1}{4}} + 2.07766x^{\frac{1}{3}}$$

for $x \geq 29\,946\,085\,320$. Thus we conclude that $\vartheta(x) > x - \frac{0.15x}{\log^3 x}$ for $29\,946\,085\,320 \leq x \leq 34\,485\,879\,392$.

Finally, for $19\,035\,709\,163 \leq x < 29\,946\,085\,320$, set $f(x) = x(1 - \frac{0.15}{\log^3 x})$. Setting $h(z) = z^4 - 0.15z + 0.45$, we observe that $h(z)$ has no real roots and $h(0) = 0.45 > 0$ and thus $h(z) > 0$ for all real z. Since $f'(x) = \frac{h(\log x)}{\log^4 x}$, $f(x)$ is strictly increasing for $x > 0$. Next, letting p_n denote the n^{th} prime number, we note that, for $p_n \leq x < p_{n+1}$, $\vartheta(x) = \vartheta(p_n)$ while $f(x) < f(p_{n+1})$ since f is strictly increasing. Thus we only need to check that $\vartheta(p_n) > f(p_{n+1})$ for $\pi(19\,035\,709\,163) \leq n \leq \pi(29\,946\,085\,320)$, which we do by computer. □

3.2 Bounds on $\pi(x)$

Lemma 3.2. *We have, for all $x \geq 10\,031\,975\,087$, that*

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.69}{\log^2 x}}$$

and, for all $x \geq 38\,099\,531$, that

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{2.85}{\log^2 x}}$$

Proof. First we define the function

$$J_{\eta,x_0}(x) = \pi(x_0) - \frac{\vartheta(x_0)}{\log x_0} + \frac{x}{\log x} + \frac{\eta x}{\log^4 x} + \int_{x_0}^{x} \left(\frac{1}{\log^2 t} + \frac{\eta}{\log^5 t} \right) dt$$

If we assume that $|\vartheta(x) - x| < \frac{\eta x}{\log^3 x}$ for $x \geq x_0$, then using the identity

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt$$

we get, for all $x \geq x_0$, that

$$J_{-\eta,x_0}(x) < \pi(x) < J_{\eta,x_0}(x)$$

We know from Lemma 3.1, for $x \geq 19\,035\,709\,163$, that

$$|\vartheta(x) - x| < \frac{0.15x}{\log^3 x}$$

and thus if $x_1 \geq 19\,035\,709\,163$, then for all $x \geq x_1$ we get that

$$J_{-0.15,x_1}(x) < \pi(x) < J_{0.15,x_1}(x)$$

To prove the upper bound of $\pi(x)$, let $x_1 = 7 \times 10^{12}$ and define the functions

$$r(x) = \log x - 1 - \frac{1}{\log x} - \frac{3.69}{\log^2 x}, \quad f(x) = \frac{x}{r(x)}, \quad g(x) = f(x) - J_{0.15,x_1}(x)$$

We note that $\vartheta(x_1) = 6\,999\,996\,936\,360.165729$ (from [6] Table 2) and $\pi(x_1) = 245\,277\,688\,804$, and thus we calculate $g(x_1) \approx 60\,189 > 0$. Next, we set

$$h_1(z) = 0.54z^7 - 15.01z^6 - 8.13z^5 - 13.26z^4 - 3.68z^3 + 2.66z^2 + 1.27z + 6.12$$

The largest real root of h_1 is $z \approx 28.358$, and since the leading coefficient of h_1 is positive, we conclude that $h_1(z) > 0$ for $z > 28.36$. Calculating the derivative of $g(x)$, we get that

$$g'(x) > \frac{h_1(\log x)}{r^2(x) \log^9 x}$$

and thus $g'(x) > 0$ for $x > e^{28.36} \approx 2.07 \times 10^{12}$. Thus $g(x) > 0$ for $x \geq x_1$, and therefore $\pi(x) < f(x)$ for $x \geq 7 \times 10^{12}$.

Next, setting $x_2 = 5.05 \times 10^{10}$, we observe that $f(x_2) > \mathrm{li}(x_2)$. Additionally, setting

$$h_2(z) = 0.69z^3 - 15.76z^2 - 7.38z - 13.6161$$

we note that h_2 has only one real root at $z \approx 22.3352$, and since the leading coefficient of h_2 is positive, we conclude that $h_2(z) > 0$ for $z > 23.34$. Now we take the derivative of $f(x) - \mathrm{li}(x)$ to get

$$f'(x) - \mathrm{li}'(x) = \frac{h_2(\log x)}{r^2(x)\log^5 x}$$

and thus $f'(x) - \mathrm{li}'(x) > 0$ for $x > e^{22.34} \approx 1.37 \times 10^{10}$. Thus $f(x) > \mathrm{li}(x)$ for $x \geq x_2$. Using Theorem 2 of [3], we have that $\pi(x) < \mathrm{li}(x)$ for $2 \leq x \leq 10^{19}$, and therefore $\pi(x) < f(x)$ for $g \geq 5.05 \times 10^{10}$.

For $10\,031\,975\,087 \leq x \leq 5.05 \times 10^{10}$, first we let

$$h_3(z) = z^4 - 2z^3 - z^2 - 4.69 - 7.38$$

Since h_3 has only one real root at $z \approx 3.07501$ and the leading coefficient of h_3 is positive, we conclude that $h_3(z) > 0$ for $z > 3.0751$. Now we take the derivative of $f(x)$ to get that

$$f'(x) = \frac{h_3(\log x)}{r^2(x)\log^3 x}$$

and thus $f'(x) > 0$ for $x > e^{3.0751} \approx 21.65$, and hence $f(x)$ is strictly increasing for $x \geq 22$. Therefore we only need to check that $\pi(x) < f(x)$ for prime values of x, and we check this by computer.

For the lower bound, let $x_1 = 10^{10}$ and define the functions

$$r(x) = \log x - 1 - \frac{1}{\log x} - \frac{2.85}{\log^2 x}, \quad f(x) = \frac{x}{r(x)}, \quad g(x) = J_{-0.15,x_1}(x) - f(x)$$

We note that $\vartheta(x_1) = 9\,999\,939\,830.657757$ (from [6] Table 2) and $\pi(x_1) = 455\,052\,511$, and thus we calculate $g(x_1) \approx 29\,583 > 0$. Next we set

$$h_4(z) = 13.15z^6 + 4.95z^5 + 8.2275z^4 - 2.67z^3 + 2.16z^2 + 1.34663z + 3.65513$$

Since h_4 has no real roots and $h_4(0) = 3.65513 > 0$, $h_4(z) > 0$ for all real z. Calculating the derivative of $g(x)$, we get that

$$g'(x) > \frac{h_4(\log x)}{r^2(x)\log^9 x}$$

and thus $g'(x) > 0$ for all $x > 0$. Thus $g(x) > 0$ for $x \geq 10^{10}$, and therefore $\pi(x) > f(x)$ for $x \geq 19\,035\,709\,163$.

For $38\,099\,331 \le x \le 19\,035\,709\,163$, first we let

$$h_5(z) = z^4 - 2z^3 - z^2 - 3.85z - 5.7$$

Since h_5 has only one real root at $z \approx 2.98278$ and the leading coefficient of h_5 is positive, we conclude that $h_5(z) > 0$ for $z > 2.9828$. Now we take the derivative of $f(x)$ to get that

$$f'(x) = \frac{h_5(\log x)}{r^2(x)\log^3 x}$$

and thus $f'(x) > 0$ for $x > e^{2.9828} \approx 19.74$, and hence $f(x)$ is strictly increasing for $x \ge 20$. Now, for $p_n \le x < p_{n+1}$, $\pi(x) = \pi(p_n)$ while $f(x) < f(p_{n+1})$ since f is strictly increasing for $x \ge 20$. Thus we only need to check that $\pi(p_n) > f(p_{n+1})$ for $\pi(38\,099\,331) \le n \le \pi(19\,035\,709\,163)$, which we do by computer. $\square$

Chapter 4

Estimating the constant

Let $\sigma(x) = \sum_{1<n\le x} \frac{1}{\pi(n)}$.

Theorem 4.1. *For all $x \ge 4\cdot 10^{12}$, we have that*

$$6.714330921 + \frac{5.16}{\sqrt[10]{\log^{11} x}} \le \sigma(x) - \frac{1}{2}\log^2 x + \log x + \log\log x \le 6.74327915 + \frac{2.86}{\log x}$$

4.1 The upper bound

If we take $a \ge 38\,099\,531$, then Lemma 3.2 gives us that

$$\sigma(x) \le \sigma(a-1) + \sum_{a\le n\le x}\left(f_1(n) - \frac{1}{n} - f_2(n) - 2.85 f_3(n)\right)$$

Using the lemmas in Chapter 2, we then get that

$$\sigma(x) \le \frac{1}{2}\log^2 x - \log x - \log\log x + C_{\mathrm{up}}(a) + R_{\mathrm{up}}(x,a)$$

where

$$C_{\mathrm{up}}(a) = \sigma(a-1) - \frac{1}{2}\log^2 a + \log\log a - \frac{2.85}{\log a} - \gamma + \sum_{n\le a-1}\frac{1}{n}$$

and

$$\begin{aligned} R_{\mathrm{up}}(x,a) = {} & \mathfrak{B}_1(f_1)(x) - \mathfrak{B}_0(f_2)(x) - 2.85\mathfrak{B}_0(f_3)(x) + \frac{2.85}{\log x} \\ & - \frac{x-a}{720} f_1^{(4)}(x) - \frac{x-a}{12} f_2^{(2)}(x) - 2.85\frac{x-a}{12} f_3^{(2)}(x) - \frac{1}{2(x+1)} \end{aligned}$$

If we include in $C_{\text{up}}(a)$ all terms independent of x in $R_{\text{up}}(x,a)$, we get that

$$\begin{aligned}C_{\text{up}}(a) = \sigma(a-1) - \frac{1}{2}\log^2 a + \log\log a - \frac{2.85}{\log a} + \frac{\log a}{2a} \\ - \frac{1}{12a^2} + \frac{\log a}{12a^2} - \frac{1}{2a\log a} - \frac{2.85}{2a\log^2 a} - \gamma + \sum_{n\le a-1}\frac{1}{n}\end{aligned}$$

Setting $a = 4\cdot 10^{12}$, this gives us that $C_{\text{up}}(4\cdot 10^{12}) \approx 6.74327915$, and

$$\begin{aligned}R_{\text{up}}(x, 4\cdot 10^{12}) = & \frac{\log x}{2x} + \frac{1}{12x^2} - \frac{\log x}{12x^2} - \frac{1}{2x\log x} - \frac{2.85}{2x\log^2 x} + \frac{2.85}{\log x} - \frac{1}{2x+2} \\ & - \frac{(x-4\cdot 10^{12})(24\log x - 50)}{720x^5} \\ & - \frac{(x-4\cdot 10^{12})(2\log^2 x + 3\log x + 2)}{12x^3\log^3 x} \\ & - 2.85\frac{(x-4\cdot 10^{12})(2\log^2 x + 6\log x + 6)}{12x^3\log^4 x}\end{aligned}$$

Multiplying this by $\log x$ and only taking terms that are positive for $x \ge 4\cdot 10^{12}$, we get a function

$$F(x) = \frac{\log^2}{2x} + \frac{\log x}{12x^2} + 2.85$$

Differentiating, we get that

$$F'(x) = \frac{1}{12x^3}\left(12x\log x - 6x\log^2 x + 1 - 2\log x\right)$$

Differentiating only the part inside the brackets, we get that

$$\frac{d}{dx}\left(12x^3F'(x)\right) = 12 - 6\log^2 x - \frac{2}{x}$$

which is clearly negative for all $x \ge 5$, and since $F'(8) < 0$, $F'(x) < 0$ for all $x \ge 8$. Additionally, since $F(3\,276) < 2.86$, $F(x) < 2.86$ for all $x \ge 3\,276$. Thus $R_{\text{up}}(x, 4\cdot 10^{12}) \times \log x < 2.86$ for all $x \ge 4\cdot 10^{12}$. This gives us the upper bound

$$\sigma(x) \le \frac{1}{2}\log^2 x - \log x - \log\log x + \frac{2.86}{\log x} + 6.74327915$$

4.2 The lower bound

Lemma 3.2 and the lemmas in Chapter 2 give us that for $a \ge 10\,031\,975\,087$

$$\begin{aligned}\sigma(x) \ge\ & \sigma(a-1) + \sum_{a\le n\le x}\left(f_1(n) - \frac{1}{n} - f_2(n) - 3.69f_3(n)\right) \\ \ge\ & \frac{1}{2}\log^2 x - \log x - \log\log x + C_{\text{lw}}(a) + R_{\text{lw}}(x,a)\end{aligned}$$

where

$$C_{\mathrm{lw}}(a) = \sigma(a-1) - \frac{1}{2}\log^2 a + \log\log a - \frac{3.69}{\log a} - \gamma + \sum_{n\le a-1}\frac{1}{n}$$

and

$$\begin{aligned} R_{\mathrm{lw}}(x,a) = {} & \mathfrak{B}_0(f_1)(x) - \mathfrak{B}_1(f_2)(x) - 3.69\mathfrak{B}_1(f_3)(x) + \frac{3.69}{\log x} \\ & + \frac{x-a}{12} f_1^{(2)}(x) - \frac{x-a}{720} f_2^{(4)}(x) - 3.69\frac{x-a}{720} f_3^{(4)}(x) - \frac{1}{2x} \end{aligned}$$

Including in $C_{\mathrm{lw}}(a)$ all the terms independent of x in $R_{\mathrm{lw}}(x,a)$ gives us

$$\begin{aligned} C_{\mathrm{lw}}(a) = {} & \sigma(a-1) - \frac{1}{2}\log^2 a + \log\log a - \frac{3.69}{\log a} + \frac{\log a}{2a} \\ & - \frac{1}{2a\log a} - \frac{1}{12a^2\log a} - \frac{4.69}{12a^2\log^2 a} - \frac{3.69}{6a^2\log^3 a} - \gamma + \sum_{n\le a-1}\frac{1}{n} \end{aligned}$$

Setting $a = 4\cdot 10^{12}$ gives us that $C_{\mathrm{lw}}(4\cdot 10^{12}) \approx 6.714330921$, and

$$\begin{aligned} & R_{\mathrm{lw}}(x, 4\cdot 10^{12}) \\ & = \frac{\log x}{2x} - \frac{1}{2x\log x} + \frac{1}{12x^2\log x} + \frac{4.69}{12x^2\log^2 x} - \frac{3.69}{2x\log^2 x} \\ & + \frac{3.69}{6x^2\log^3 x} + \frac{3.69}{\log x} - \frac{1}{2x} + \frac{(x-4\cdot 10^{12})(2\log x - 3)}{12x^3} \\ & + \frac{(x-4\cdot 10^{12})(24\log^4 x + 50\log^3 x + 70\log^2 x + 60\log x + 24)}{720x^5\log^5 x} \\ & + 3.69\frac{(x-4\cdot 10^{12})(24\log^4 x + 100\log^3 x + 210\log^2 x + 240\log x + 120)}{720x^5\log^6 x} \end{aligned}$$

Multiplying this by $\sqrt[10]{\log^{11} x}$ and taking only the 7$^{\text{th}}$ term and the terms that are negative for $x \ge 4\cdot 10^{12}$, we get the function

$$\begin{aligned} G(x) &= 3.69\sqrt[10]{\log x} - \frac{\sqrt[10]{\log x}}{2x} - \frac{3.69}{2x\sqrt[10]{\log^9 x}} - \frac{\sqrt[10]{\log^{11} x}}{2x} \\ &= \frac{1}{2x\sqrt[10]{\log^9 x}}\left(7.38x\log x - \log x - 3.69 - \log^2 x\right) \end{aligned}$$

Differentiating only the part inside the brackets, we get

$$\begin{aligned} \frac{d}{dx}\left(2x\sqrt[10]{\log^9 x}G(x)\right) &= 7.38\log x + 7.38 - \frac{1}{x} - \frac{2\log x}{x} \\ &= \frac{1}{x}\left((7.38x - 2)\log x + 7.38x - 1\right) \end{aligned}$$

which is clearly positive when $x \geq 1$. Additionally, since $G(4 \cdot 10^{12}) > 5.16$, we have that $G(x) > 5.16$ for all $x \geq 4 \cdot 10^{12}$. Thus $R_{\text{lw}}(x, 4 \cdot 10^{12}) \times \sqrt[10]{\log^{11} x} > 5.16$ for all $x \geq 4 \cdot 10^{12}$. This gives us the lower bound

$$\sigma(x) \geq \frac{1}{2}\log^2 x - \log x - \log\log x + \frac{5.16}{\sqrt[10]{\log^{11} x}} + 6.714330921$$

4.3 Improving the estimate

Without improvements to the techniques used, improvement on the bounds for the constant can come from one of two places: either from improved bounds on $\pi(x)$ or by choosing a larger value of a.

If we write the bounds of $\pi(x)$ as

$$\frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{b}{\log^2 x}} < \pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{c}{\log^2 x}}$$

then, in the expressions for $C_{\text{up}}(a)$ and $C_{\text{lw}}(a)$, the most significant terms that directly involve b and c are $-\frac{b}{\log x}$ and $-\frac{c}{\log x}$ respectively. Thus, if we fix $a = 4 \cdot 10^{12}$, to improve either bound by, say, 0.01, we would need to either increase b or reduce c by $0.01 \log a \approx 0.29$. Given the current values of $b = 2.85$ and $c = 3.69$, this would mean increasing b to 3.14 or decreasing c to 3.40.

Looking at the difference between $C_{\text{up}}(a)$ and $C_{\text{lw}}(a)$, we get that

$$C_{\text{up}}(a) - C_{\text{lw}}(a) = \frac{c-b}{\log a} - \frac{1}{12a^2} + \frac{\log a}{12a^2} + \frac{1}{12a^2 \log a} - \frac{b}{2a\log^2 a} + \frac{c+1}{12a^2\log^2 a} + \frac{c}{6a^2\log^3 a}$$

which clearly goes to zero as a goes to infinity. Additionally, if we only look at the dominant term $\frac{c-b}{\log a}$, to get a difference of at least d for a given b and c, we would require that $a \geq e^{(c-b)/d}$. So, with the current b and c which have a difference of 0.84, to reduce the difference by 0.01 would require $a \approx 1.8 \cdot 10^{19}$, and to reduce it by 0.02 would require $a \approx 5.9 \cdot 10^{40}$.

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