

Defining Newforms in Characteristic p

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ABSTRACT

The theory of newforms, due to Atkin and Lehner [AL70], provides a powerful method for decomposing spaces of modular forms. However, many problems occur when trying to generalise this theory to characteristic p . Recently, Deo and Medvedovsky [DM19] have suggested a way around these problems by using purely algebraic notions to define newforms. In this thesis, we detailedly describe the methods of Deo and Medvedovsky and generalise their results where possible.

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INTRODUCTION

Modular forms are an indispensable tool in number theory. In recent years, they have garnered widespread attention due to their role in Wiles' proof of Fermat's Last Theorem ([Wil95], [TW95]) and the subsequent proof of the modularity conjecture [Bre+01]. This thesis is concerned with defining an important class of modular forms known as *newforms*.

For our purposes, a *modular form* is a function on the complex upper half plane satisfying certain holomorphy and transformation conditions. These conditions allow us to write each modular form f as a Fourier series

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi iz}.$$

Of particular interest are the values of the coefficients a_n , which encode valuable arithmetic information. A special class of modular forms we will study are *cuspidal forms*, which have $a_0 = 0$ in their Fourier expansions.

Associated to each non-zero modular form is a weight k and a level N . The vector space of all modular forms (resp. cuspidal forms) of weight k and level N is typically denoted $\mathcal{M}_k(N)$ (resp. $\mathcal{S}_k(N)$). In this thesis, we are interested in a decomposition of $\mathcal{S}_k(N)$ into "old" and "new" subspaces:

$$\mathcal{S}_k(N) = \mathcal{S}_k(N)^{\text{old}} \oplus \mathcal{S}_k(N)^{\text{new}}. \quad (1)$$

The space of *oldforms*, $\mathcal{S}_k(N)^{\text{old}}$, consists of cuspidal forms that arise from lower levels strictly dividing N . On the other hand, the space of *newforms*, $\mathcal{S}_k(N)^{\text{new}}$, is the orthogonal complement of $\mathcal{S}_k(N)^{\text{old}}$ with respect to a suitable inner product. The theory of oldforms and newforms was first developed by Atkin and Lehner [AL70] and later extended by Li [Li75].

One can also consider modular forms with coefficients over an arbitrary commutative ring B . To do this, we let $\mathcal{M}_k(N, \mathbb{Z})$ denote the space of modular forms with Fourier coefficients in \mathbb{Z} and then define $\mathcal{M}_k(N, B) := \mathcal{M}_k(N, \mathbb{Z}) \otimes_{\mathbb{Z}} B$. We also define $\mathcal{S}_k(N, B)$ analogously for cuspidal forms. If B has characteristic zero, the theory of newforms for $\mathcal{S}_k(N, B)$ naturally generalises the classical theory of Atkin and Lehner. However, when B has characteristic

p , congruences between oldforms and newforms result in many undesirable properties. For instance, we no longer have a decomposition as in (1).

Recently, Deo and Medvedovsky [DM19] have provided suggestions on how to deal with this behaviour in characteristic p . In particular, they suggest redefining newforms with robust, purely algebraic definitions. In [DM19], two such algebraic definitions are explored. These definitions are related to *Hecke operators* and are inspired by classical observations of Serre [Ser73] and Atkin and Lehner [AL70].

The goal of this thesis is to provide an extensive analysis of Deo and Medvedovsky's algebraic definitions of newforms. In particular, we fill in the details of proofs in [DM19] and suggest ways to extend their theory where possible. An outline of the thesis is as follows.

- In Chapter 1 we discuss the basic theory of modular forms and establish the notation that will be used throughout the thesis.
- In Chapter 2 we summarise the classical theory of newforms due to Atkin and Lehner.
- In Chapter 3 we introduce the algebraic definitions of newforms due to Deo and Medvedovsky. Moreover, we show that they agree with Atkin and Lehner's definitions in the classical setting.
- In Chapter 4 we define newforms over an arbitrary commutative ring B . We then describe the properties of these forms when B has characteristic zero.
- In Chapter 5 we explore the theory of newforms in characteristic p and analyse Deo and Medvedovsky's definitions in this setting.
- In Chapter 6 we introduce the notion of modular forms with character and suggest ways to generalise Deo and Medvedovsky's definitions to this setting. The results in this chapter constitute the most original contribution of this thesis.

We will assume that the reader is familiar with basic definitions and results from complex analysis and abstract algebra. The later chapters also use a large amount of commutative algebra. The relevant background material for these later chapters can be found in Appendix A.

CHAPTER 1

MODULAR FORMS

We begin by providing some preliminary definitions and examples of modular forms. In particular, we will define modular forms in terms of their "level", allowing the notion of newforms to arise naturally. For more details one can consult chapters 1.1 and 1.2 of Diamond and Shurman's book [DS05].

1.1 Modular Forms of Level 1

Let

$$\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$$

denote the complex upper half plane and

$$\text{SL}_2(\mathbb{Z}) = \{\gamma \in M_{2 \times 2}(\mathbb{Z}) : \det(\gamma) = 1\}$$

be the set of 2×2 matrices with integer entries and determinant 1.

When studying the geometry of \mathbb{H} it is common to consider the following action of $\text{SL}_2(\mathbb{Z})$ on \mathbb{H} :

$$\gamma \cdot z = \frac{az + b}{cz + d}, \quad \text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}). \quad (1.1)$$

One can check that this is a well-defined action in the sense that $\gamma \cdot z \in \mathbb{H}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot z = z$ and $(\gamma\gamma') \cdot z = \gamma \cdot (\gamma' \cdot z)$ for all $z \in \mathbb{H}$ and $\gamma, \gamma' \in \text{SL}_2(\mathbb{Z})$.

Modular forms are holomorphic functions on \mathbb{H} that satisfy certain symmetries with respect to the action (1.1). Such functions exhibit interesting arithmetic properties and are defined as follows.

Definition 1.1. Let k be a nonnegative integer. A (level 1) *modular form* of weight k is a function $f: \mathbb{H} \rightarrow \mathbb{C}$ satisfying the following three conditions:

1. f is holomorphic on \mathbb{H} ,
2. $f(\gamma \cdot z) = (cz + d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$,

3. f is holomorphic at infinity. That is, the limit $\lim_{\text{Im}(z) \rightarrow \infty} f(z)$ exists.

The set of all modular forms of weight k forms a vector space of functions, denoted $\mathcal{M}_k(\text{SL}_2(\mathbb{Z}))$.

From here onwards, we shall refer to the second condition as the *modularity condition*. To simplify notation let

$$j(\gamma, z) := cz + d$$

be the *factor of automorphy* for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$. For any $\gamma \in \text{SL}_2(\mathbb{Z})$ we then define the *slash operator* $|_k \gamma$ of weight k on functions $f : \mathbb{H} \rightarrow \mathbb{C}$ by

$$(f|_k \gamma)(z) := j(\gamma, z)^{-k} f(\gamma \cdot z). \quad (1.2)$$

The modularity condition then simply states that $f|_k \gamma = f$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$. The slash operator is also an action in the sense that $f|_k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = f$ and $f|_k \gamma \gamma' = f|_k \gamma |_k \gamma'$ for all functions $f : \mathbb{H} \rightarrow \mathbb{C}$.

Now, substituting $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ into the modularity condition tells us that modular forms are \mathbb{Z} -periodic, i.e. $f(z + 1) = f(z)$. As a result, all modular forms f have a Fourier series (or " q -expansion")

$$f(q) = \sum_{n \in \mathbb{Z}} a_n q^n, \quad \text{where } q = e^{2\pi iz}. \quad (1.3)$$

The fact that $\lim_{\text{Im}(z) \rightarrow \infty} f(z)$ exists means that $a_n = 0$ for all $n < 0$.

Definition 1.2. A *cuspidal form* is a modular form f such that $\lim_{\text{Im}(z) \rightarrow \infty} f(z) = 0$. Equivalently, a cuspidal form has $a_0 = 0$ in its q -expansion (1.3).

We now look at some examples.

Example 1.3.

- All constant functions on \mathbb{H} are modular forms of weight 0. The function $f(z) = 0$ is a modular form (and cuspidal form) for every weight $k \geq 0$.
- The *Eisenstein series*

$$G_k(z) = \sum_{c,d \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(cz + d)^k}$$

is a modular form of weight k for all even $k \geq 4$. Its q -expansion is given by

$$G_k(q) = 2\zeta(k) \left(1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \right)$$

where ζ is the Riemann zeta function, B_k is the k^{th} Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. Scaling by a factor of $1/(2\zeta(k))$ gives the simpler series

$$E_k(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n. \quad (1.4)$$

Throughout the rest of this thesis, we will refer to any scaling of $G_k(q)$ as an Eisenstein series.

- The *modular discriminant*

$$\Delta(z) = \frac{1}{1728} (E_4(z)^3 - E_6(z)^2)$$

is a cusp form of weight 12 and has the q -expansion

$$\Delta(q) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \dots$$

In the next chapter we shall see how the theory of newforms reveals nontrivial relationships between these Fourier coefficients. There are still many open problems regarding $\Delta(q)$. For instance, it is unknown whether the n^{th} Fourier coefficient of $\Delta(q)$ is nonzero for all $n \geq 1$.

- The only modular form of odd weight is $f(z) = 0$. To see this, suppose that f is a modular form of odd weight k . Substituting $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ into the modularity condition for f then gives $f(z) = (-1)^k f(z) = -f(z)$, which forces $f(z) = 0$.

1.2 Modular Forms of Level N

Studying the level 1 modular forms defined in the previous section eventually becomes quite restrictive. In fact, one can show that all modular forms of level 1 can be expressed as polynomials in the Eisenstein series $E_4(z)$ and $E_6(z)$ [Kil08, Proposition 3.6].

By replacing $SL_2(\mathbb{Z})$ in the modularity condition with some subgroup $\Gamma \leq SL_2(\mathbb{Z})$, we are left with a larger set of functions to study. The subgroups that interest us the most are those based on systems of congruences.

Definition 1.4. Let N be a positive integer. The *principal congruence subgroup of level N* is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

where the matrix congruence is interpreted entrywise.

Definition 1.5. A *congruence subgroup* is any subgroup $\Gamma \leq SL_2(\mathbb{Z})$ such that $\Gamma(N) \subseteq \Gamma$ for some positive integer N . The smallest such N is called the *level* of Γ .

For a given level $N \in \mathbb{Z}_{>0}$, the most widely studied congruence subgroups are

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}, \text{ and} \quad (1.5)$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\} \quad (1.6)$$

where $*$ means unspecified. Note that $\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq SL_2(\mathbb{Z})$.

We now alter our original definition of modular forms (Definition 1.1) to the following.

Definition 1.6. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and k be an integer. A modular form of weight k with respect to Γ is a function $f: \mathbb{H} \rightarrow \mathbb{C}$ satisfying the following three conditions:

1. f is holomorphic on \mathbb{H} ,
2. $f|_k \gamma = f$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,
3. $f|_k \alpha$ is holomorphic at infinity for all $\alpha \in SL_2(\mathbb{Z})$.

The first two conditions in Definition 1.6 are natural generalisations of the first two conditions in our original definition of modular forms (Definition 1.1). The third condition is a less obvious generalisation but is required for the space of modular forms to be finite dimensional. In the case where $\Gamma = SL_2(\mathbb{Z})$, note that the third condition agrees with Definition

1.1 since we always have $f|_k\alpha = f$ by the modularity condition.

If f is a modular form with respect to a congruence subgroup Γ of level N , we say that f is a *modular form of level N* . Since $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$, this justifies why the modular forms from Section 1.1 were called level 1 modular forms.

Another subtlety in defining modular forms with respect to a congruence subgroup Γ is the transfer over to Fourier series. Since Γ may not contain $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it does not necessarily follow that a modular form f with respect to Γ is \mathbb{Z} -periodic. However, we still have $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$ for some minimal $h \in \mathbb{Z}_{>0}$ giving $f(z+h) = f(z)$ and thus a Fourier series of the form

$$f(z) = \sum_{n \in \mathbb{Z}} a_n q_h^n, \quad \text{where } q_h = e^{2\pi iz/h}. \quad (1.7)$$

However, this is not something to worry about in the case of $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$ since both of these congruence subgroups contain $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The definition of a cusp form also carries over from the level 1 case in a similar way.

Definition 1.7. A modular form f (of weight k with respect to Γ) is a cusp form if $a(0) = 0$ in the q -expansion (1.7) of $f|_k\alpha$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

As in the case of level 1, the set of modular forms of weight k with respect to Γ forms a vector space $\mathcal{M}_k(\Gamma)$. The set of cusp forms also forms a vector subspace denoted by $\mathcal{S}_k(\Gamma)$. These spaces are finite dimensional and their dimensions can be computed explicitly [DS05, Chapter 3]. It is also sometimes useful to consider all weights simultaneously as in the following definition.

Definition 1.8. The *algebra of modular forms* is given by $\mathcal{M}(\Gamma) = \sum_{k=0}^{\infty} \mathcal{M}_k(\Gamma)$. Similarly the *algebra of cusp forms* is $\mathcal{S}(\Gamma) = \sum_{k=0}^{\infty} \mathcal{S}_k(\Gamma)$.

In particular, any $f \in \mathcal{M}(\Gamma)$ is a finite linear combination $f = f_1 + \cdots + f_n$ where each f_i occurs in a fixed weight. Note that $\mathcal{M}(\Gamma)$ is closed under multiplication since if f is a modular form of weight k and g is a modular form of weight k' , then fg is a modular form of weight $k+k'$.

We now finish off with some examples of level N modular forms.

Example 1.9.

- By generalising the Eisenstein Fourier series in (1.4) to $k = 2$ we obtain

$$E_2(q) = 1 - \frac{4}{B_2} \sum_{n=1}^{\infty} \sigma(n)q^n.$$

Note that this is **not** a (level 1) modular form. In particular, the modularity condition is not satisfied. However the function

$$E_2^N(z) := E_2(z) - NE_2(Nz)$$

is a modular form of weight 2 for $\Gamma_0(N)$.

- The *Jacobi theta function* is given by

$$\theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2}.$$

This function plays a crucial role in analytic number theory and is used to find the number of ways to express an integer as a sum of squares. In terms of modular forms, if k is an even integer then θ^k is a modular form of weight $k/2$ for $\Gamma_1(4)$.

- For some congruence subgroup Γ and odd weight k , if $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma$ then it is possible to have $\mathcal{M}_k(\Gamma) \neq \{0\}$. For instance, the function θ^2 from the previous example is in $\mathcal{M}_1(\Gamma_1(4))$.
- Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Since a modular form for Γ must satisfy fewer symmetries than a modular form for $\mathrm{SL}_2(\mathbb{Z})$ we have $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathcal{M}_k(\Gamma)$. That is, any modular form with respect to $\mathrm{SL}_2(\mathbb{Z})$ is also a modular form with respect to Γ .

To see where the idea of newforms arises, we consider $\Gamma_0(M)$ and $\Gamma_0(N)$ with $M|N$. From the definition of Γ_0 we have $\Gamma_0(N) \subseteq \Gamma_0(M)$ and thus $\mathcal{M}_k(\Gamma_0(M)) \subseteq \mathcal{M}_k(\Gamma_0(N))$. And so for fixed N , we can take any divisor $M|N$ and embed $\mathcal{M}_k(\Gamma_0(M))$ in $\mathcal{M}_k(\Gamma_0(N))$. Later we shall see that there are also other ways to embed $\mathcal{M}_k(\Gamma_0(M))$ in $\mathcal{M}_k(\Gamma_0(N))$.

The space of *oldforms* consists of forms in $\mathcal{M}_k(N)$ arising from lower levels. The corresponding space of *newforms* then consists of forms that do not arise from lower levels. These ideas will be made precise in the next chapter.

CHAPTER 2

CLASSICAL THEORY OF NEWFORMS

In this chapter we summarise the original theory of oldforms and newforms first established by Atkin and Lehner [AL70]. Following Atkin and Lehner, we will primarily concern ourselves with the congruence subgroup $\Gamma_0(N)$ defined in the previous chapter. As a consequence, we simplify our notation, writing $\mathcal{M}_k(N)$ to mean $\mathcal{M}_k(\Gamma_0(N))$ (or $\mathcal{S}_k(N)$ for $\mathcal{S}_k(\Gamma_0(N))$ in the case of cusp forms).

The following theory can also be extended to other classes of modular forms (see [Li75] and [DS05, Chapter 5]). However to prevent any overcomplications, we will refrain from discussing such generalisations until Chapter 6.

For the rest of this chapter, we assume that $k \in \mathbb{Z}_{\geq 0}$ and $N \in \mathbb{Z}_{>0}$ are fixed integers.

2.1 Atkin-Lehner and Hecke Operators

In order to study newforms, we need to define a collection of important linear operators on $\mathcal{M}_k(N)$. We begin by noting that the slash operator can be extended to matrices in

$$\mathrm{GL}_2(\mathbb{Q})^+ = \{\gamma \in M_{2 \times 2}(\mathbb{Q}) : \det(\gamma) > 0\}.$$

Namely, for a function $f: \mathbb{H} \rightarrow \mathbb{C}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^+$ we let

$$(f|_k \gamma)(z) = (\det \gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma \cdot z),$$

where $j(\gamma, z) = cz + d$ and $\gamma \cdot z = \frac{az+b}{cz+d}$ as in the $\mathrm{SL}_2(\mathbb{Z})$ case. The factor of $(\det \gamma)^{k/2}$ is a convention chosen so that scalar matrices act as the identity (provided k is even). The Atkin-Lehner operator is then defined as follows.

Definition 2.1. Let ℓ be a prime that divides N exactly once and $\gamma_\ell \in \mathrm{GL}_2(\mathbb{Q})^+$ be any matrix of the form $\begin{pmatrix} \ell & a \\ N & b \end{pmatrix}$ where a and b are integers chosen so that $\ell b - a(N/\ell) = 1$. The *Atkin-Lehner operator* w_ℓ is the map that sends $f \in \mathcal{M}_k(N)$ to $f|_k \gamma_\ell$.

Remarks.

- (i) The conditions on a and b imply that $\det(\gamma_\ell) = \ell$.
- (ii) w_ℓ does not depend on the choice of γ_ℓ . That is, if we let $\gamma_\ell = \begin{pmatrix} \ell & a \\ N & \ell b \end{pmatrix}$ and $\gamma'_\ell = \begin{pmatrix} \ell & a' \\ N & \ell b' \end{pmatrix}$ then

$$\gamma_\ell(\gamma'_\ell)^{-1} = \begin{pmatrix} \ell & a \\ N & \ell b \end{pmatrix} \frac{1}{\ell} \begin{pmatrix} \ell b' & -a' \\ -N & \ell \end{pmatrix} = \begin{pmatrix} \ell b' - a(N/\ell) & -a' + a \\ Nb' - Nb & -a'(N/\ell) + \ell b \end{pmatrix} \in \Gamma_0(N),$$

so that for $f \in \mathcal{M}_k(N)$, we have $w_\ell(f) = f|_k\gamma_\ell = f|_k\gamma'_\ell$ as required.

Next we define the Hecke operators. These operators can be explicitly described in terms of their Fourier coefficients. In what follows, we write $a_n(f)$ for the n^{th} Fourier coefficient of a modular form f . That is,

$$f(q) = \sum_{n=0}^{\infty} a_n(f)q^n.$$

Definition 2.2. Let p be a prime number. For a modular form f , the Hecke operator T_p acting on f is given by the q -expansion with n^{th} coefficient

$$\begin{aligned} a_n(T_p f) &= a_{np}(f), & \text{if } p \mid N, \\ a_n(T_p f) &= a_{pn}(f) + p^{k-1}a_{n/p}(f), & \text{if } p \nmid N, \end{aligned}$$

where $a_{n/p}(f) = 0$ if p does not divide n . For a prime power p^s with $s \geq 2$ we let T_1 act as the identity and then inductively define

$$T_{p^s} = T_p T_{p^{s-1}} - p^{k-1} T_{p^{s-2}}.$$

Finally, to define T_m for any positive integer m we let $T_{mm'} = T_m T_{m'}$ whenever $\gcd(m, m') = 1$.

For $m \geq 0$, we will also write U_m for the formal \mathbb{C} -linear operator $U_m: \mathbb{C}[[q]] \rightarrow \mathbb{C}[[q]]$ given by $a_n(U_m f) = a_{mn}(f)$. If $m \mid N$ then U_m coincides with T_m and we will generally write U_m in place of T_m in such situations.

The following proposition lists some algebraic properties of these operators. Proofs of the following results can be found in sections 2 and 3 of [AL70].

Proposition 2.3. *Let p and p' be distinct primes not dividing N and let ℓ be a prime dividing N exactly α times. Let f be a modular form of weight k for $\Gamma_0(N)$. Then*

(a) $T_p(f)$ and $U_\ell(f)$ are in $\mathcal{M}_k(N)$.

(b) If $\alpha = 1$ then $w_\ell(f) \in \mathcal{M}_k(N)$. Moreover, w_ℓ is an involution, i.e. $w_\ell^2(f) = f$.

(c) $T_p T_{p'}(f) = T_{p'} T_p(f)$,
 $T_p U_\ell(f) = U_\ell T_p(f)$ and if $\alpha = 1$,
 $T_p w_\ell(f) = w_\ell T_p(f)$.

(d) If $\alpha = 1$ then $U_\ell(f) + \ell^{\frac{k}{2}-1} w_\ell(f) \in \mathcal{M}_k(N/\ell)$.

(e) If $\alpha > 1$, then $U_\ell(f) \in \mathcal{M}_k(N/\ell)$.

Remark. The above proposition also holds if we restrict to cusp forms.

2.2 Definition of Newforms

Let M be a positive integer dividing N and $e \mid (N/M)$. We consider the map

$$\begin{aligned} i_e: \mathcal{M}_k(M) &\rightarrow \mathcal{M}_k(N) \\ f &\mapsto (z \mapsto f(ez)). \end{aligned} \tag{2.1}$$

Proposition 2.4. *The map i_e is well defined.*

Proof. We need to show that $g(z) := f(ez)$ is in $\mathcal{M}_k(N)$. The holomorphicity of f carries over to g so it suffices to check that g satisfies the modularity condition $g|_k \gamma = g$ for any $\gamma \in \Gamma_0(N)$. So, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary matrix in $\Gamma_0(N)$. We have

$$\begin{aligned} (g|_k \gamma)(z) &= (cz + d)^{-k} g(\gamma \cdot z) \\ &= (cz + d)^{-k} g\left(\frac{az + b}{cz + d}\right) \\ &= (cz + d)^{-k} f\left(e \frac{az + b}{cz + d}\right) \\ &= (cz + d)^{-k} f\left(\frac{a(ez) + be}{\frac{c}{e}(ez) + d}\right) \\ &= (cz + d)^{-k} \left(\frac{c}{e}(ez) + d\right)^k f(ez) \quad (*) \\ &= g(z), \end{aligned}$$

as required. Note that (*) follows since $\begin{pmatrix} a & be \\ c/e & d \end{pmatrix} \in \Gamma_0(M)$. In particular, $N \mid c$ and $e \mid (N/M)$ implies that $(c/e) \mid M$. \square

Remark. The above arguments also hold if we restrict i_e to a map over cusp forms.

The map i_e thus gives an embedding of $\mathcal{M}_k(M)$ into $\mathcal{M}_k(N)$. Using this embedding, we make the following definition.

Definition 2.5. The space of *oldforms* $\mathcal{M}_k(N)^{\text{old}}$ is the \mathbb{C} -linear subspace of $\mathcal{M}_k(N)$ spanned by the maps $i_e: \mathcal{M}_k(M) \rightarrow \mathcal{M}_k(N)$ for all M strictly dividing N and $e \mid (N/M)$. More explicitly,

$$\mathcal{M}_k(N)^{\text{old}} = \sum_{\substack{M \mid N \\ M \neq N \\ e \mid (N/M)}} i_e(\mathcal{M}_k(M)).$$

Now that we have a formal notion of "old", we can describe what it means to be "new". Following Atkin and Lehner [AL70] we restrict our attention to cusp forms. In this direction let

$$\mathcal{S}_k(N)^{\text{old}} = \sum_{\substack{M \mid N \\ M \neq N \\ e \mid (N/M)}} i_e(\mathcal{S}_k(M)).$$

We now consider the *Petersson inner product* $\langle \cdot, \cdot \rangle_N: \mathcal{S}_k(N) \times \mathcal{S}_k(N) \rightarrow \mathbb{C}$, which is given by

$$\langle f, g \rangle_N := \int_{z \in D_N} f(z) \overline{g(z)} y^k \frac{dx dy}{y^2}, \quad (2.2)$$

where $z = x + iy$ and D_N is a fundamental domain for the action of $\Gamma_0(N)$ on the upper half plane \mathbb{H} . One can check that this is a well-defined inner product on $\mathcal{S}_k(N)$ and the integral (2.2) does not depend on the fundamental domain chosen [DS05, Chapter 5.4]. Moreover, for a prime $p \nmid N$ the Hecke operators T_p are Hermitian with respect to the Petersson inner product [AL70, Lemma 13]. In other words,

$$\langle T_p f, g \rangle_N = \langle f, T_p g \rangle_N. \quad (2.3)$$

Definition 2.6. The space of *newforms* in $\mathcal{S}_k(N)$ is the orthogonal complement of $\mathcal{S}_k(N)^{\text{old}}$ with respect to the Petersson inner product. That is,

$$\mathcal{S}_k(N)^{\text{new}} = \{f \in \mathcal{S}_k(N) : \langle f, g \rangle_N = 0 \text{ for all } g \in \mathcal{S}_k(N)_{\text{old}}\}.$$

Since $\mathcal{S}_k(N)$ is finite dimensional, it then follows that

$$\mathcal{S}_k(N) = \mathcal{S}_k(N)^{\text{old}} \oplus \mathcal{S}_k(N)^{\text{new}}.$$

In particular, $\mathcal{S}_k(N)^{\text{old}} \cap \mathcal{S}_k(N)^{\text{new}} = \{0\}$. To help identify oldforms and newforms we will make use of the following theorem (adapted from [AL70]).

Theorem 2.7. *Let $f(q) = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_k(N)$ and suppose that $a_n = 0$ whenever $\gcd(n, N) = 1$. Then f is an oldform.*

Proof. See [AL70, Theorem 1] for the original proof or [Car99] for a more recent proof involving representation theory. \square

2.3 Properties of Newforms

We now prove some statements regarding oldforms and newforms. Namely, we are interested in how newforms interact with the operators defined in Section 2.1. For the most part we will follow Section 4 of [AL70] whilst adopting more modern notation.

We begin by utilising the Hermitian property of T_p for primes $p \nmid N$ (see (2.3)). In particular, we have the following result.

Lemma 2.8. *The space of oldforms $\mathcal{S}_k(N)^{\text{old}}$ and newforms $\mathcal{S}_k(N)^{\text{new}}$ are preserved under the action of the Hecke operators T_p for $p \nmid N$.*

Proof. For oldforms, the statement follows since T_p commutes with the i_e embedding maps. That is, if $f = \sum_{n=1}^{\infty} a_n(f)q^n$ is a cusp form then

$$T_p(i_e f) = i_e(T_p f) = \sum_{n=1}^{\infty} [a(pn) + p^{2k-1}a(n/p)]q^{ne}.$$

For newforms, consider an arbitrary $f \in \mathcal{S}_k(N)^{\text{new}}$. Then for any $p \nmid N$ and $g \in \mathcal{S}_k(N)^{\text{old}}$ we have

$$\langle T_p(f), g \rangle_N = \langle f, T_p(g) \rangle_N = 0,$$

since $T_p(g) \in \mathcal{S}_k(N)^{\text{old}}$. As a consequence, $T_p(f) \in \mathcal{S}_k(N)^{\text{new}}$ as required. \square

In light of this result, we now analyse how the Hecke operators behave on the space of newforms. From here onwards, we say $f \in \mathcal{S}_k(N)$ is an *eigenform* if f is an eigenfunction of

all the Hecke operators T_p with $p \nmid N$. This means that an eigenform is also an eigenfunction for any of the Hecke operators T_m with $\gcd(m, N) = 1$.

Lemma 2.9. *There exists a basis of eigenforms for $\mathcal{S}_k(N)^{\text{new}}$.*

Proof. Since the T_p operators are Hermitian (for $p \nmid N$) and commute with one another (Proposition 2.3), the result follows by a version of the spectral theorem (see, for example [Gan59, Theorem 11 of Chapter IX in Volume 1]). \square

Remark. Lemma 2.9 also holds for $\mathcal{S}_k(N)$ or $\mathcal{S}_k(N)^{\text{old}}$ by the same reasoning.

Lemma 2.10. *For any eigenform $f(q) = \sum_{n=1}^{\infty} a_n(f)q^n \in \mathcal{S}_k(N)^{\text{new}}$, we have $a_1(f) \neq 0$.*

Proof. Suppose for a contradiction that $a_1(f) = 0$. Let m be a positive integer such that $\gcd(m, N) = 1$. Since f is an eigenform there exists $\lambda_m \in \mathbb{C}$ such that $T_m f = \lambda_m f$. Hence,

$$a_m(f) = a_1(T_m(f)) = \lambda_m a_1(f) = 0.$$

Applying Theorem 2.7 then gives $f \in \mathcal{S}_k(N)^{\text{old}}$ and thus $f = 0$ since $\mathcal{S}_k(N)^{\text{new}}$ and $\mathcal{S}_k(N)^{\text{old}}$ intersect trivially. This contradicts f being an eigenform so we must have $a_1(f) \neq 0$ as required. \square

The previous lemma implies that we can normalise the eigenforms $f \in \mathcal{S}_k(N)^{\text{new}}$ to have $a_1(f) = 1$.

Definition 2.11. A *primitive form* (of weight k for $\Gamma_0(N)$) is an eigenform $f \in \mathcal{S}_k(N)^{\text{new}}$ with $a_1(f) = 1$.

Let f be a primitive form. Writing $\lambda_p(f)$ for the T_p eigenvalue of f (so that $T_p f = \lambda_p(f)f$) gives

$$a_{pn}(f) + p^{k-1}a_{n/p}(f) = \lambda_p(f)a_n(f). \quad (2.4)$$

Thus, by setting $n = 1$,

$$\lambda_p(f) = a_p(f). \quad (2.5)$$

This demonstrates how elegantly newforms behave under the action of the Hecke operators. In fact, substituting (2.5) into (2.4) gives

$$a_{np}(f) - a_n(f)a_p(f) + p^{k-1}a_{n/p}(f) = 0 \quad (2.6)$$

which we can use to inductively compute a_m when $\gcd(m, N) = 1$, provided that we know the Hecke eigenvalues $\lambda_p(f) = a_p(f)$ for each prime $p \nmid N$. We now define an equivalence relation to further analyse this behaviour.

Definition 2.12. Let f and g be eigenforms in $\mathcal{S}_k(N)$. We write $f \sim g$ if f and g have the same T_p eigenvalues for all $p \nmid N$.

Lemma 2.13. *If f_1 and f_2 are primitive forms in $\mathcal{S}_k(N)^{\text{new}}$ with $f_1 \sim f_2$ then $f_1 = f_2$.*

Proof. Since f_1 is equivalent to f_2 , (2.6) tells us that $a_m(f_1) = a_m(f_2)$ whenever $\gcd(m, N) = 1$. Now, $f_1 - f_2 \in \mathcal{S}_k(N)^{\text{new}}$ and applying Theorem 2.7 gives $f_1 - f_2 \in \mathcal{S}_k(N)^{\text{old}}$. As $\mathcal{S}_k(N)^{\text{new}} \cap \mathcal{S}_k(N)^{\text{old}} = \{0\}$ it follows that $f_1 = f_2$ as required. \square

Lemma 2.14. *We have*

$$\mathcal{S}_k(N) = \sum_{M|N} \sum_{e|(N/M)} i_e(\mathcal{S}_k(M)^{\text{new}}),$$

so that any eigenform $f \in \mathcal{S}_k(N)$ is equivalent (in the sense of Definition 2.12) to a new eigenform $h \in \mathcal{S}_k(M)^{\text{new}}$ where M is a divisor of N .

Proof. We proceed by induction on the factors of N . In level 1 there are no lower levels so $\mathcal{S}_k(1) = \mathcal{S}_k(1)^{\text{new}}$. Now assume that the result holds for all proper divisors of N . Let $f \in \mathcal{S}_k(N)$, and write $f = f_{\text{new}} + f_{\text{old}}$ where $f_{\text{new}} \in \mathcal{S}_k(N)^{\text{new}}$ and $f_{\text{old}} \in \mathcal{S}_k(N)^{\text{old}}$. By definition f_{old} is a linear combination of forms $i_e(g)$ with $g \in \mathcal{S}_k(M)$ for some M strictly dividing N and $e \mid (N/M)$. But then, by the induction hypothesis, any such g can be written as the sum of forms $i_d(h)$ where $h \in \mathcal{S}_k(M')^{\text{new}}$ with $M' \mid M \mid N$ and $d \mid (M/M')$.

For the statement about equivalent forms first recall that each space $\mathcal{S}_k(M)^{\text{new}}$ has a basis of eigenforms. And so, noting that the T_p and i_e operators commute (for $p \nmid N$) we have that any eigenform in $\mathcal{S}_k(N)$ must be equivalent to an eigenform in $\mathcal{S}_k(M)^{\text{new}}$ for some $M \mid N$. \square

Lemma 2.15. *Let $f \in \mathcal{S}_k(N)^{\text{old}}$ and $g \in \mathcal{S}_k(N)^{\text{new}}$ be eigenforms. Then f and g belong to different equivalence classes.*

Proof. Suppose for a contradiction that $f \sim g$. By Lemma 2.14, $f \sim h$ (and thus $g \sim h$) for some new eigenform $h \in \mathcal{S}_k(M)^{\text{new}}$ where $M \mid N$. Since multiplication of g or h by a scalar does not affect their T_p eigenvalues we may assume $a_1(g) = a_1(h) = 1$. Applying the argument in Lemma 2.13 we then have $g - h \in \mathcal{S}_k(N)^{\text{old}}$. Now $h = i_1(h) \in \mathcal{S}_k(N)^{\text{old}}$ so it follows that $g \in \mathcal{S}_k(N)^{\text{old}}$. But as g is a newform this implies that $g = 0$, a contradiction. \square

We now move onto the main theorem (adapted from [AL70, Theorem 3]), which completely describes newforms in terms of the Atkin-Lehner and Hecke operators. To prove the theorem, we will make repeated use of the previous lemma along with the results listed in Proposition 2.3.

Theorem 2.16. *Let $f \in \mathcal{S}_k(N)^{\text{new}}$ be a primitive form, p be any prime with $p \nmid N$, and ℓ be any prime dividing N exactly α times. If $f(q) = q + \sum_{n=2}^{\infty} a_n q^n$ we then have*

$$(a) \quad T_p(f) = a_p f,$$

$$(b) \quad U_\ell(f) = a_\ell f \text{ and}$$

$$(c) \quad \text{if } \alpha = 1 \text{ then } w_\ell(f) = \varepsilon_\ell f, \text{ where } \varepsilon_\ell = \pm 1.$$

Further, if $\alpha \geq 2$, then $a_\ell = 0$. While if $\alpha = 1$, we have $a_\ell = -\ell^{\frac{k}{2}-1} \varepsilon_\ell$. In terms of Fourier coefficients, (a) and (b) become

$$(a)' \quad a_{np} - a_n a_p + p^{k-1} a_{n/p} = 0,$$

$$(b)' \quad a_{n\ell} - a_n a_\ell = 0.$$

Proof. First we recall that the U_ℓ and T_p operators commute so that $U_\ell(f) \sim f$. Now, if $\alpha \geq 2$ we have $U_\ell(f) \in \mathcal{S}_k(N/\ell)$ by Proposition 2.3 and thus $U_\ell(f) \in \mathcal{S}_k(N)^{\text{old}}$. So by Lemma 2.15, $U_\ell f \equiv 0$. Moreover, w_ℓ commutes with T_p so that $w_\ell(f) \sim f$. However, if $w_\ell(f) \in \mathcal{S}_k(N)^{\text{old}}$ then $w_\ell(f) = 0$ meaning that $f = w_\ell^2(f) = 0$, a contradiction. So we must have $w_\ell(f) \in \mathcal{S}_k(N)^{\text{new}}$ by Lemma 2.15. Because $w_\ell(f) \sim f$, Lemma 2.13 then implies that $w_\ell f$ is a scalar multiple of f , i.e. $w_\ell f = \varepsilon_\ell f$. We know that $\varepsilon_\ell = \pm 1$ since $w_\ell^2(f) = f$.

Now if instead $\alpha = 1$, Proposition 2.3 gives $f \sim U_\ell(f) + \ell^{\frac{k}{2}-1} w_\ell(f) \in \mathcal{S}_k(N)^{\text{old}}$. Hence by Lemma 2.15, $U_\ell(f) + \ell^{\frac{k}{2}-1} w_\ell(f) = 0$. That is, $U_\ell f = -\ell^{\frac{k}{2}-1} \varepsilon_\ell f$ as required. The translation to Fourier series follows directly from the definition of the Hecke operators. \square

Example 2.17. We consider the modular discriminant

$$\Delta(q) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \dots$$

from Section 1.1. Recall that $\Delta(q)$ is a cusp form of weight 12 and level 1. Since $\Delta(q)$ is a level 1 cusp form it cannot arise from any lower levels and is thus a newform. As

a consequence, its Fourier coefficients satisfy the relations described in Theorem 2.16. In particular, if $\tau(n)$ denotes the n^{th} Fourier coefficient of $\Delta(q)$ then:

$$\begin{aligned}\tau(mn) &= \tau(m)\tau(n) \text{ if } \gcd(m, n) = 1, \text{ and} \\ \tau(p^{r+1}) &= \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}) \text{ for } p \text{ prime and } r > 0.\end{aligned}$$

These relations were first conjectured by Ramanujan [Ram16].

Example 2.18. The space $\mathcal{S}_2(45)^{\text{new}}$ is spanned by the primitive form⁽ⁱ⁾

$$f = q + q^2 - q^4 - q^5 - 3q^8 + O(q^{10}).$$

In this example, $N = 45$, and 3 divides 45 twice whereas 5 divides 45 once. Hence by Theorem 2.16, we expect $U_3(f) = 0$ and $U_5(f) = \pm 1$, which certainly agrees with the q -expansion above.

These examples demonstrate how studying newforms can reveal nontrivial data about the coefficients of modular forms. There are many other useful reasons to study newforms, such as their use in classifying cusp forms and their relation to Dirichlet L -functions [DS05, §5.9]. How to generalise this theory to other types of modular forms (besides cusp forms on $\Gamma_0(N)$) is thus a very natural and widely studied question.

⁽ⁱ⁾This was computed using the software system SageMath.

CHAPTER 3

NEWFORMS IN SQUAREFREE LEVEL

In this chapter we look at the equivalent ways of defining newforms suggested by Deo and Medvedovsky [DM19]. To do this, we will restrict our attention to the space of “ ℓ -newforms” and focus on the case where N is a squarefree integer.

The definitions in this chapter favour more algebraic notions, avoiding the use of the Petersson inner product. Following [DM19] we will explore two such algebraic notions. The first involves classifying newforms by their U_ℓ -eigenvalue. The second notion relates newforms to the kernel of a “trace” operator on $\mathcal{M}_k(N)$, inspired by an observation of Serre [Ser73, §3.1]. By defining newforms in such a way, further generalisations become easier to define and work with.

3.1 The Space of ℓ -newforms

Let $N > 1$ be a fixed integer. Recall that the space of weight k -oldforms for $\Gamma_0(N)$ is defined as

$$\mathcal{M}_k(N)^{\text{old}} = \sum_{\substack{M|N \\ M \neq N \\ e|(N/M)}} i_e(\mathcal{M}_k(M)).$$

Now, each M strictly dividing N is also a divisor of N/ℓ for some prime $\ell \mid N$. Hence we only need to consider forms arising from the levels N/ℓ for each $\ell \mid N$. In other words,

$$\mathcal{M}_k(N)^{\text{old}} = \sum_{\substack{\ell \text{ prime} \\ \ell \mid N}} \mathcal{M}_k(N)^{\ell\text{-old}}, \tag{3.1}$$

where

$$\mathcal{M}_k(N)^{\ell\text{-old}} = \mathcal{M}_k(N/\ell) + i_\ell(\mathcal{M}_k(N/\ell)).$$

Here we slightly abuse notation, writing $\mathcal{M}_k(N/\ell)$ when we actually mean $i_1(\mathcal{M}_k(N/\ell))$. For a detailed proof of the equality in (3.1), see Proposition B.1 in Appendix B.

Now, similarly let $\mathcal{S}_k(N)^{\ell\text{-old}} = \mathcal{S}_k(N/\ell) + i_\ell(\mathcal{S}_k(N/\ell))$ and define $\mathcal{S}_k(N)^{\ell\text{-new}}$ to be the

orthogonal complement of $\mathcal{S}_k(N)^{\ell\text{-old}}$. We then have $\mathcal{S}_k(N) = \mathcal{S}_k(N)^{\ell\text{-old}} \oplus \mathcal{S}_k(N)^{\ell\text{-new}}$ and

$$\mathcal{S}_k(N)^{\text{new}} = \bigcap_{\substack{\ell \text{ prime} \\ \ell | N}} \mathcal{S}_k(N)^{\ell\text{-new}}.$$

It therefore suffices to study $\mathcal{S}_k(N)^{\ell\text{-old}}$ and $\mathcal{S}_k(N)^{\ell\text{-new}}$ for each prime $\ell \mid N$ in order to determine $\mathcal{S}_k(N)^{\text{old}}$ and $\mathcal{S}_k(N)^{\text{new}}$ respectively. As with $\mathcal{S}_k(N)^{\text{old}}$ and $\mathcal{S}_k(N)^{\text{new}}$, these ℓ -old and ℓ -new subspaces have bases of eigenforms. The following result of Atkin-Lehner can then be used to give us further information about $\mathcal{S}_k(N)^{\ell\text{-old}}$ and $\mathcal{S}_k(N)^{\ell\text{-new}}$.

Theorem 3.1. *Let $f \in \mathcal{S}_k(M_1)^{\text{new}}$ and $g \in \mathcal{S}_k(M_2)^{\text{new}}$ be primitive forms with $a_p(f) = a_p(g)$ for infinitely many primes p . Then $M_1 = M_2$ and $f = g$.*

Proof. See [AL70, Theorem 4]. □

Corollary 3.2. *Let ℓ be a prime dividing N exactly α times. Then,*

$$\mathcal{S}_k(N)^{\ell\text{-new}} = \sum_{\substack{M \\ \ell^\alpha | M | N}} \sum_{e | (N/M)} i_e(\mathcal{S}_k(M)^{\text{new}}).$$

Proof. Recall from Lemma 2.14 that

$$\mathcal{S}_k(N) = \sum_{M | N} \sum_{e | (N/M)} i_e(\mathcal{S}_k(M)^{\text{new}}). \quad (3.2)$$

And then, since $\mathcal{S}_k(N)^{\ell\text{-old}} = \mathcal{S}_k(N/\ell) + i_\ell(\mathcal{S}_k(N/\ell))$, any ℓ -old form arises from a newform at a level M with $\ell^\alpha \nmid M$. In other words,

$$\mathcal{S}_k(N)^{\ell\text{-old}} = \sum_{\substack{M \\ \ell^\alpha \nmid M | N}} \sum_{e | (N/M)} i_e(\mathcal{S}_k(M)^{\text{new}}). \quad (3.3)$$

Comparing this with (3.2) gives

$$\mathcal{S}_k(N)^{\ell\text{-new}} \subseteq \sum_{\substack{M \\ \ell^\alpha | M | N}} \sum_{e | (N/M)} i_e(\mathcal{S}_k(M)^{\text{new}}).$$

For the reverse inclusion, suppose that M is a divisor of N such that $\ell \mid M$. By Theorem 3.1 and (3.3) any eigenform in $i_e(\mathcal{S}_k(M)^{\text{new}})$ cannot be contained in $\mathcal{S}_k(N)^{\ell\text{-old}}$ and must therefore be in $\mathcal{S}_k(N)^{\ell\text{-new}}$ as required. □

Corollary 3.3. *Let $f \in \mathcal{S}_k(N)^{\ell\text{-old}}$ and $g \in \mathcal{S}_k(N)^{\ell\text{-new}}$ be eigenforms. Then f and g belong to different equivalence classes. That is, there exists some prime $p \nmid N$ such that $T_p(f) \neq T_p(g)$.*

Proof. As shown in the proof of the previous corollary, ℓ -new and ℓ -old forms arise from newforms at different levels. Hence the eigenforms f and g must belong to different equivalence classes by Theorem 3.1. \square

Corollary 3.4. *Let $f \in \mathcal{S}_k(N)^{\ell\text{-new}}$ be an eigenform where ℓ is a prime dividing N exactly α times. If $\alpha = 1$ then $w_\ell(f) = \varepsilon_\ell f$ and $U_\ell(f) = -\ell^{\frac{k}{2}-1} \varepsilon_\ell f$ for some $\varepsilon_\ell = \pm 1$. Otherwise if $\alpha \geq 2$ then $U_\ell(f) = 0$*

Proof. Using Corollary 3.3 repeat the argument from the proof of Theorem 2.16. \square

3.2 The Atkin-Lehner Operator Revisited

For the rest of this chapter, we assume that ℓ is a fixed prime dividing N exactly once. Note that if N is squarefree then this is always true for every prime dividing N . Now, since ℓ divides N exactly once we can always define the Atkin-Lehner operator w_ℓ . Following Deo and Medvedovsky [DM19, Section 3.4] we will scale w_ℓ by defining $W_\ell = \ell^{k/2} w_\ell$. This scaling of w_ℓ will be particularly important when we generalise to characteristic p in Chapter 5. Note that since w_ℓ is an involution $W_\ell^2 = \ell^k$.

We now show how to conveniently define ℓ -oldforms in terms of W_ℓ .

Proposition 3.5. *If $f \in \mathcal{M}_k(N/\ell) \subseteq \mathcal{M}_k(N)$ then $W_\ell f = \ell^k i_\ell f$.*

Proof. We have

$$\begin{aligned}
(W_\ell f)(z) &= \ell^{k/2} \left(f \Big|_k \begin{pmatrix} \ell & a \\ N & \ell b \end{pmatrix} \right) (z) \\
&= \frac{\ell^k}{(Nz + \ell b)^k} f \left(\frac{\ell z + a}{Nz + \ell b} \right) \\
&= \frac{\ell^k}{(Nz + \ell b)^k} f \left(\frac{(\ell z) + a}{\frac{N}{\ell}(\ell z) + \ell b} \right) \\
&= \frac{\ell^k}{(Nz + \ell b)^k} (Nz + \ell b)^k f(\ell z) \quad \left(\text{Since } \begin{pmatrix} 1 & a \\ N/\ell & \ell b \end{pmatrix} \in \Gamma_0(N/\ell) \right)
\end{aligned}$$

$$= \ell^k(i_\ell f)(z),$$

as required. □

Hence W_ℓ is just a rescaling of i_ℓ on $\mathcal{M}_k(N/\ell)$. As a result

$$\mathcal{M}_k(N)^{\ell\text{-old}} = \mathcal{M}_k(N/\ell) + W_\ell(\mathcal{M}_k(N/\ell)).$$

Proposition 3.6. *If $f \in \mathcal{M}_k(N/\ell) \cap W_\ell(\mathcal{M}_k(N/\ell))$, then f is constant.*

Proof. Let $f \in \mathcal{M}_k(N/\ell) \cap W_\ell(\mathcal{M}_k(N/\ell))$ so that $f = W_\ell(g)$ for some $g \in \mathcal{M}_k(N/\ell)$. We have

$$\ell^k g = W_\ell^2(g) = W_\ell(f) = \ell^k i_\ell(f)$$

and thus

$$f = W_\ell(g) = \ell^k i_\ell(g) = \ell^k i_\ell^2(f). \quad (3.4)$$

If $f = \sum_{n=0}^{\infty} a_n q^n$ then (3.4) becomes

$$\sum_{n=0}^{\infty} a_n q^n = \ell^k \sum_{n=0}^{\infty} a_n q^{n\ell^2}. \quad (3.5)$$

Now, suppose for a contradiction that f is not constant. Let $n > 0$ be the least integer with $a_n \neq 0$. Then the left-hand side of (3.5) has a q^n term whereas the right-hand side of (3.5) does not. This is impossible so f must be constant as required. □

In Appendix B (Proposition B.2) we see that the algebra $\mathcal{M}(N) = \sum_{k=0}^{\infty} \mathcal{M}_k(N)$ is in fact a direct sum $\mathcal{M}(N) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(N)$. In particular, constant-valued modular forms only occur in weight 0. Combining this with Proposition 3.6 gives the following result.

Corollary 3.7. *Let $k > 0$. Then $\mathcal{M}_k(N/\ell) \cap W_\ell(\mathcal{M}_k(N/\ell)) = \{0\}$ and as a result,*

$$\mathcal{M}_k(N)^{\ell\text{-old}} = \mathcal{M}_k(N/\ell) \oplus W_\ell(\mathcal{M}_k(N/\ell)).$$

3.3 First Algebraic Notion: U_ℓ Eigenvalue

We have shown (Corollary 3.4) that $\mathcal{S}_k(N)^{\ell\text{-new}}$ has a basis of eigenforms $\{f_1, f_2, \dots, f_n\}$ such that

$$U_\ell(f_i) = \pm \ell^{\frac{k}{2}-1} f_i.$$

In light of this property, we define the following operator.

Definition 3.8. The \mathcal{D}_ℓ operator is the linear map $\mathcal{D}_\ell: \mathcal{M}_k(N) \rightarrow \mathcal{M}_k(N)$ defined by

$$\mathcal{D}_\ell = \ell^2 U_\ell^2 - \ell^k,$$

so that

$$\ker(\mathcal{D}_\ell) = \{f \in \mathcal{M}_k(N) : U_\ell^2(f) = \ell^{k-2} f\}.$$

We claim that when \mathcal{D}_ℓ is restricted to cusp forms, this kernel is precisely the ℓ -newforms.

Theorem 3.9. *We have*

$$\mathcal{S}_k(N)^{\ell\text{-new}} = \ker(\mathcal{D}_\ell|_{\mathcal{S}_k(N)}).$$

To prove this theorem, we will need some preliminary lemmas.

Lemma 3.10. *If $g \in \mathcal{S}_k(N/\ell)$ is an eigenform of weight k then $|a_\ell(g)| < (\ell + 1)\ell^{\frac{k-2}{2}}$, where $a_\ell(g)$ is the T_ℓ eigenvalue of g .*

Proof. This relies on the Weil bound [Del74, Theorem 8.2] which states that $|a_\ell(g)| \leq 2\ell^{\frac{k-1}{2}}$. It thus suffices to show that $2\ell^{\frac{k-1}{2}} < (\ell + 1)\ell^{\frac{k-2}{2}}$. Squaring both sides of this inequality and dividing by ℓ^{k-2} , we see that this is equivalent to $4\ell < (\ell + 1)^2$, which is always true for $\ell > 1$. \square

Remark. The Weil bound is only stated for primitive forms in [Del74]. However, any eigenform $g \in \mathcal{S}_k(N/\ell)$ has the same T_ℓ -eigenvalue as some primitive form in a lower level (Lemma 2.14).

Lemma 3.11. *For $k > 0$, let $g \in \mathcal{M}_k(N/\ell)$ be an eigenform and consider the two dimensional subspace, $V_{\ell,g}$, spanned by g and $W_\ell g$. The characteristic polynomial of U_ℓ^2 on $V_{\ell,g}$ is $P_{\ell,g}(X) = X^2 - (a_\ell^2(g) - 2\ell^{k-1})X + \ell^{2k-2}$, where $a_\ell(g)$ is the T_ℓ -eigenvalue of g .*

Proof. First note that g and $W_\ell g$ are linearly independent by Corollary 3.7. We need to find out how U_ℓ acts on g and $W_\ell(g)$. By the definition of the Hecke operators of $\mathcal{M}_k(N/\ell)$,

$$T_\ell = U_\ell + \ell^{-1}W_\ell, \tag{3.6}$$

noting that $(W_\ell f)(q) = \ell^k f(q^\ell)$ by Proposition 3.5. Evaluating (3.6) at g and rearranging then gives

$$U_\ell(g) = a_\ell(g)g - \ell^{-1}W_\ell(g).$$

On the other hand, $U_\ell(i_\ell g) = g$ and thus $U_\ell(W_\ell(g)) = \ell^k g$. Putting everything together, the transformation matrix of U_ℓ with respect to the basis $\{g, W_\ell(g)\}$ is

$$U_\ell = \begin{pmatrix} a_\ell(g) & \ell^k \\ -\ell^{-1} & 0 \end{pmatrix}.$$

The transformation matrix for U_ℓ^2 is then

$$U_\ell^2 = \begin{pmatrix} a_\ell^2(g) - \ell^{k-1} & \ell^k a_\ell(g) \\ -\ell^{-1} a_\ell(g) & -\ell^{k-1} \end{pmatrix}$$

which has the desired characteristic polynomial. \square

We are now ready to prove Theorem 3.9.

Proof of Theorem 3.9. From Corollary 3.4 it follows that any $f \in \mathcal{S}_k(N)^{\ell\text{-new}}$ satisfies $U_\ell^2(f) = \ell^{k-2}f$ so that $\mathcal{S}_k(N)^{\ell\text{-new}} \subseteq \ker(\mathcal{D}_\ell|_{\mathcal{S}_k(N)})$. It thus suffices to show that no ℓ -old forms have a U_ℓ^2 eigenvalue of ℓ^{k-2} .

Note that there are no nonzero cusp forms of weight 0 [DS05, Theorem 3.5.1] so we may assume $k > 0$. Now, $\mathcal{S}_k(N/\ell)$ has a basis of eigenforms so that $\mathcal{S}_k(N)^{\ell\text{-old}}$ is the direct sum of the two-dimensional subspaces $V_{\ell,g}$ described in Lemma 3.11. These spaces are U_ℓ -invariant so we deal with them separately. By Lemma 3.11, the characteristic polynomial of U_ℓ^2 on $V_{\ell,g}$ is

$$P_{\ell,g}(X) = X^2 - (a_\ell^2(g) - 2\ell^{k-1})X + \ell^{2k-2}.$$

Therefore, if λ_1 and λ_2 denote the U_ℓ^2 eigenvalues on $V_{\ell,g}$,

$$\lambda_1 + \lambda_2 = a_\ell^2(g) - 2\ell^{k-1} \text{ and } \lambda_1 \lambda_2 = \ell^{2k-2}.$$

Suppose for a contradiction that one of these eigenvalues is ℓ^{k-2} . Then the other eigenvalue must be ℓ^k and hence

$$a_\ell^2(g) = \ell^{k-2} + 2\ell^{k-1} + \ell^k = \ell^{k-2}(\ell + 1)^2, \tag{3.7}$$

which contradicts the Weil bound (Lemma 3.10). \square

3.4 Second Algebraic Notion: Kernel of Trace Operator

For our next algebraic definition of newforms we look at how to project $\mathcal{S}_k(N)$ onto $\mathcal{S}_k(N/\ell)$. Fortunately there is a natural way to do this using the following operator.

Definition 3.12. For any weight $k \geq 0$, the *trace operator* from level N to level N/ℓ is given by

$$\begin{aligned} \mathrm{Tr}_\ell: \mathcal{M}_k(N) &\rightarrow \mathcal{M}_k(N/\ell) \\ f &\mapsto \sum_{\gamma \in \Gamma_0(N) \backslash \Gamma_0(N/\ell)} f|_k \gamma. \end{aligned}$$

See Proposition B.3 in the appendix as for why this operator is well defined.

By explicitly finding coset representatives for $\Gamma_0(N) \backslash \Gamma_0(N/\ell)$, one obtains the following formula for Tr_ℓ [MO03, Lemma 2.2]:

$$\mathrm{Tr}_\ell(f) = f + \ell^{1-k/2} U_\ell \omega_\ell f.$$

In terms of the scaled W_ℓ operator; this is:

$$\mathrm{Tr}_\ell(f) = f + \ell^{1-k} U_\ell W_\ell f. \tag{3.8}$$

If $f \in \mathcal{M}_k(N/\ell)$ we can simplify this expression further.

Lemma 3.13. *Let $f \in \mathcal{M}_k(N/\ell)$. We have*

$$(a) \quad \mathrm{Tr}_\ell(f) = (\ell + 1)f,$$

$$(b) \quad \mathrm{Tr}_\ell(W_\ell f) = \ell T_\ell(f).$$

Proof. For part (a), we note that $U_\ell(i_\ell(f)) = f$ and thus

$$\mathrm{Tr}_\ell(f) = f + \ell^{1-k} U_\ell W_\ell f = f + \ell^{1-k} U_\ell \ell^k i_\ell f = f + \ell f = (\ell + 1)f.$$

For part (b), we first recall that $W_\ell^2 = \ell^k$. As a consequence,

$$\mathrm{Tr}_\ell(W_\ell f) = W_\ell f + \ell^{1-k} U_\ell W_\ell^2 f = W_\ell f + \ell U_\ell f.$$

The result then follows since $\ell T_\ell = \ell U_\ell + \ell^k i_\ell = \ell U_\ell + W_\ell$. □

Using these results we can now define newforms in terms of the trace operator. This characterisation was first noted by Serre [Ser73, §3.1(c) remarque (3)] in the specific case when $N = \ell$.

Theorem 3.14. *We have*

$$\mathcal{S}_k(N)^{\ell\text{-new}} = \ker(\mathrm{Tr}_\ell|_{\mathcal{S}_k(N)}) \cap \ker(\mathrm{Tr}_\ell W_\ell|_{\mathcal{S}_k(N)}).$$

Proof. Using (3.8) and Corollary 3.4 we see that $\mathrm{Tr}_\ell(f) = \mathrm{Tr}_\ell(W_\ell f) = 0$ for any ℓ -new form $f \in \mathcal{S}_k(N)^{\ell\text{-new}}$. Hence $\mathcal{S}_k(N)^{\ell\text{-new}} \subseteq \ker(\mathrm{Tr}_\ell) \cap \ker(\mathrm{Tr}_\ell W_\ell)$.

We now need to show that for any eigenform $f \in \mathcal{S}_k(N)^{\mathrm{old}}$, $f \notin \ker(\mathrm{Tr}_\ell) \cap \ker(\mathrm{Tr}_\ell W_\ell)$. As in our proof of Theorem 3.9, it will suffice to consider $k > 0$ and f contained in the space $V_{\ell,g}$ spanned by g and $W_\ell(g)$ for some eigenform $g \in \mathcal{S}_k(N/\ell)$. By Lemma 3.13 we can represent Tr_ℓ and $\mathrm{Tr}_\ell W_\ell$ as matrices acting on the ordered basis $\{g, W_\ell(g)\}$:

$$\mathrm{Tr}_\ell = \begin{pmatrix} \ell + 1 & \ell a_\ell(g) \\ 0 & 0 \end{pmatrix}, \quad \mathrm{Tr}_\ell W_\ell = \begin{pmatrix} \ell a_\ell(g) & (\ell + 1)\ell^k \\ 0 & 0 \end{pmatrix},$$

where $a_\ell(g)$ is the T_ℓ eigenvalue of g . The kernels of these matrices only intersect nontrivially when $a_\ell(g)^2 = (\ell + 1)^2 \ell^{k-2}$. Thus $f \notin \ker(\mathrm{Tr}_\ell) \cap \ker(\mathrm{Tr}_\ell W_\ell)$ otherwise we would contradict the Weil bound (Lemma 3.10). \square

3.5 Extension to all Modular Forms

We now have two equivalent definitions of newforms on $\mathcal{S}_k(N)$ in terms of the operators

$$\mathcal{D}_\ell(f) = \ell^2 U_\ell^2 f - \ell^k f \text{ and } \mathrm{Tr}_\ell(f) = f + \ell^{1-k} U_\ell W_\ell f.$$

These operators are well-defined on the entire space of modular forms $\mathcal{M}_k(N)$. Hence we can use them to define notions of newforms on spaces besides $\mathcal{S}_k(N)$.

Definition 3.15. Let \mathcal{C} be a Hecke-invariant subspace of $\mathcal{M}_k(N)$. Define

$$\mathcal{C}^{U_\ell\text{-new}} := \ker(\mathcal{D}_\ell|_{\mathcal{C}}), \text{ and}$$

$$\mathcal{C}^{\mathrm{Tr}_\ell\text{-new}} := \ker(\mathrm{Tr}_\ell|_{\mathcal{C}}) \cap \ker(\mathrm{Tr}_\ell W_\ell|_{\mathcal{C}}).$$

Here, Hecke-invariant means that \mathcal{C} is closed under the action of the Hecke operators. Our previous results in this chapter tell us that

$$\mathcal{S}_k(N)^{\ell\text{-new}} = \mathcal{S}_k(N)^{U_{\ell\text{-new}}} = \mathcal{S}_k(N)^{\text{Tr}_{\ell\text{-new}}}$$

and we now consider whether a similar statement holds more generally for $\mathcal{M}_k(N)$.

First we note that $\mathcal{M}_k(N)$ decomposes as

$$\mathcal{M}_k(N) = \mathcal{S}_k(N) \oplus \mathcal{E}_k(N)$$

where $\mathcal{E}_k(N)$ is a Hecke-invariant subspace of $\mathcal{M}_k(N)$ called the *Eisenstein space* [DS05, Chapter 4 and 5.11]. As our theory of newforms on $\mathcal{S}_k(N)$ is already established, we restrict our attention to $\mathcal{E}_k(N)$.

Recall that in $\mathcal{S}_k(N)$, the ℓ -new subspace consists of eigenforms that are not equivalent to any ℓ -old eigenforms (Corollary 3.3). Since the Hecke operators are well-defined on the Eisenstein space, the notions of eigenforms and equivalent eigenforms also make sense in $\mathcal{E}_k(N)$. We thus make the following definition.

Definition 3.16. The space of ℓ -new Eisenstein forms $\mathcal{E}_k(N)^{\ell\text{-new}}$ is the span of all eigenforms in $\mathcal{E}_k(N)$ that are not equivalent to any eigenforms in $\mathcal{E}_k(N)^{\ell\text{-old}} = \mathcal{E}_k(N/\ell) + W_{\ell}(\mathcal{E}_k(N/\ell))$.

In support of this definition, we note that $\mathcal{E}_k(N)$ has a basis of eigenforms and every newform in $\mathcal{E}_k(N)^{\text{new}} = \cap_{\ell|N} \mathcal{E}_k(N)^{\ell\text{-new}}$ is a simultaneous eigenfunction for all the Hecke operators T_m with $m \geq 1$ [DS05, Proposition 5.2.3].

We now see how $\mathcal{E}_k(N)^{\ell\text{-new}}$ is related to $\mathcal{E}_k(N)^{U_{\ell\text{-new}}}$ and $\mathcal{E}_k(N)^{\text{Tr}_{\ell\text{-new}}}$.

Theorem 3.17. *Suppose $k \neq 2$. We then have*

$$\mathcal{E}_k(N)^{\ell\text{-new}} = \mathcal{E}_k(N)^{U_{\ell\text{-new}}} = \mathcal{E}_k(N)^{\text{Tr}_{\ell\text{-new}}} = \{0\}.$$

Proof. First suppose that $k = 0$. Then the only modular forms are constant functions [DS05, Theorem 3.5.1]. Hence $\mathcal{E}_0(N) = \mathcal{E}_0(1)$ so that there are no nonzero ℓ -new forms in $\mathcal{E}_0(N)$. Now let $f(z) = c$ be some nonzero, constant-valued function on \mathbb{H} . We then have

$$\mathcal{D}_{\ell}(f(z)) = \ell^2 U_{\ell}^2 c - \ell^0 c = (\ell^2 - 1)c \neq 0, \text{ and}$$

$$\mathrm{Tr}_\ell(W_\ell(f(z))) = \mathrm{Tr}_\ell(f(z)) = c + \ell^{1-0}U_\ell W_\ell c = c + \ell U_\ell \ell^0 i_\ell c = (1 + \ell)c \neq 0.$$

Hence $\mathcal{E}_0(N)^{\ell\text{-new}} = \mathcal{E}_0(N)^{U_\ell\text{-new}} = \mathcal{E}_0(N)^{\mathrm{Tr}_\ell\text{-new}} = \{0\}$.

Now assume $k \neq 0, 2$. In Appendix B (Proposition B.4) we show that $\mathcal{E}_k(N) = \mathcal{E}_k(N)^{\ell\text{-old}}$ and thus $\mathcal{E}_k(N)^{\ell\text{-new}} = \{0\}$. As in the proof of Theorems 3.9 and 3.14, any ℓ -old form in $\mathcal{M}_k(N)^{U_\ell\text{-new}}$ or $\mathcal{M}_k(N)^{\mathrm{Tr}_\ell\text{-new}}$ arises from an eigenform $g \in \mathcal{M}_k(N/\ell)$ with its T_ℓ -eigenvalue satisfying $a_\ell^2(g) = (\ell + 1)^2 \ell^{k-2}$. However, no eigenform in $\mathcal{E}_k(N/\ell)$ has such an eigenvalue [DS05, Proposition 5.2.3]. Hence

$$\mathcal{E}_k(N)^{\ell\text{-new}} = \mathcal{E}_k(N)^{U_\ell\text{-new}} = \mathcal{E}_k(N)^{\mathrm{Tr}_\ell\text{-new}} = \{0\},$$

as required. □

When $k = 2$ however, these notions of "newness" no longer coincide. This is illustrated in the following example.

Example 3.18. Suppose that N is not prime and let $g = E_2^{N/\ell}(z)$ be as defined in Example 1.9. Note that $g \in \mathcal{E}_2(N/\ell)$ with $T_\ell(g) = (\ell + 1)g$ [DS05, Proposition 5.2.3]. Now let $h = \ell g - W_\ell(g)$, noting that $h \notin \mathcal{E}_2(N)^{\ell\text{-new}}$. However, using Lemma 3.13 we have

$$\mathrm{Tr}_\ell(h) = \ell \mathrm{Tr}_\ell(g) - \mathrm{Tr}_\ell(W_\ell(g)) = (\ell + 1)g - \ell(\ell + 1)g = 0, \text{ and}$$

$$\mathrm{Tr}_\ell(W_\ell(h)) = \ell \mathrm{Tr}_\ell(W_\ell(g)) - \ell^2 \mathrm{Tr}_\ell(g) = \ell^2(\ell + 1)g - \ell^2(\ell + 1)g = 0.$$

Hence $h \in \mathcal{E}_2(N)^{\mathrm{Tr}_\ell\text{-new}}$ and one can similarly show that $h \in \mathcal{E}_2(N)^{U_\ell\text{-new}}$.

Question 3.19. Although the above example shows that $\mathcal{E}_2(N)^{\mathrm{Tr}_\ell\text{-new}} \neq \mathcal{E}_2(N)^{\ell\text{-new}}$ (whenver N is not prime) is it still true that $\mathcal{E}_2(N)^{\mathrm{Tr}_\ell\text{-new}} = \mathcal{E}_2(N)^{U_\ell\text{-new}}$?

This problematic behaviour when $k = 2$ is overlooked by Deo and Medvedovsky [DM19, Proposition 6.1 and Proposition 6.3]. However, it does not affect their subsequent results.

For a more detailed analysis of Eisenstein newforms and their relation to the trace operator see [Wei77, Chapter 3].

CHAPTER 4

NEWFORMS IN CHARACTERISTIC ZERO

We now consider a generalisation of modular forms, allowing q -expansions to have coefficients over some commutative ring B . The aim of this chapter is to show that when B has characteristic zero, the theory of newforms mimics that discussed in Chapters 2 and 3.

The definitions in this chapter are adapted from [DM19, Sections 2, 5 and 6]. From here onwards, we will assume the reader is familiar with the basic theory of flat modules. A summary can be found in Appendix A.1.

4.1 Modular Forms over a Commutative Ring B

Let $N > 1$ be a fixed positive integer. Let $\mathcal{M}_k(N, \mathbb{Z})$ (resp. $\mathcal{S}_k(N, \mathbb{Z})$) denote the set of modular forms (resp. cusp forms) of weight k for $\Gamma_0(N)$ with integral Fourier coefficients. For any commutative ring B we then define

$$\begin{aligned}\mathcal{M}_k(N, B) &:= \mathcal{M}_k(N, \mathbb{Z}) \otimes_{\mathbb{Z}} B, \text{ and} \\ \mathcal{S}_k(N, B) &:= \mathcal{S}_k(N, \mathbb{Z}) \otimes_{\mathbb{Z}} B,\end{aligned}$$

where $\mathcal{M}_k(N, \mathbb{Z})$, $\mathcal{S}_k(N, \mathbb{Z})$ and B are viewed as \mathbb{Z} -modules⁽ⁱ⁾. Using the isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} B \cong B$, we can treat $\mathcal{M}_k(N, B)$ (and similarly $\mathcal{S}_k(N, B)$) as a subset of $B[[q]]$. Namely,

$$\mathcal{M}_k(N, B) = \text{span}_B \left\{ \sum_{n=0}^{\infty} \tilde{a}_n q^n \in B[[q]] : \sum_{n=0}^{\infty} a_n q^n \in \mathcal{M}_k(N, \mathbb{Z}) \right\},$$

where \tilde{a}_n is the image of a_n under the unique ring homomorphism $\mathbb{Z} \rightarrow B$. Note that $\mathcal{M}_k(N, \mathbb{C})$ agrees with the classical definition of modular forms since $\mathcal{M}_k(N)$ has a basis in $\mathcal{M}_k(N, \mathbb{Z})$ [DI95, Corollary 12.3.12]. The same is true for $\mathcal{S}_k(N, \mathbb{C})$.

Now, let $\mathcal{M}(N, B) = \sum_{k=0}^{\infty} \mathcal{M}_k(N, B)$ denote the algebra of modular forms (for $\Gamma_0(N)$)

⁽ⁱ⁾There is also a geometric definition of modular forms with coefficients in B that was first introduced by Katz [Kat73]. However for most of the cases that we are interested in, Katz's definition agrees with our more elementary definitions (see [DI95, Theorem 12.3.7]). Certainly, $\mathcal{M}_k(N, B) \subseteq \mathcal{M}_k(N, B)^{\text{Katz}}$, where $\mathcal{M}_k(N, B)^{\text{Katz}}$ is the space of Katz's geometric modular forms.

over B). Similarly let $\mathcal{S}(N, B) = \sum_{k=0}^{\infty} \mathcal{S}_k(N, B)$ be the algebra of cusp forms. These algebras are much easier to work with when B is a domain of characteristic zero.

Proposition 4.1. *Let B be a domain of characteristic zero. Then*

$$\mathcal{M}(N, B) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(N, B).$$

An analogous result holds for $\mathcal{S}(N, B)$.

Proof. For $B = \mathbb{C}$, and thus $B = \mathbb{Z}$, this is proven in Appendix B (Proposition B.2). Another way of saying this is that the map

$$\bigoplus_{k=0}^{\infty} \mathcal{M}_k(N, \mathbb{Z}) \rightarrow \mathcal{M}(N, \mathbb{Z}) \tag{4.1}$$

is injective. Now suppose that B is an arbitrary domain of characteristic zero. Then B is flat as a \mathbb{Z} -module (Corollary A.4). Tensoring (4.1) by B therefore preserves injectivity, giving the desired result. \square

Next we define the Atkin-Lehner and Hecke operators over B . Recall that for p prime and $f = \sum_{n=0}^{\infty} a_n(f)q^n \in \mathcal{M}_k(N, \mathbb{C})$, the Hecke operators are defined by

$$\begin{aligned} a_n(T_p f) &= a_n(U_p f) = a_{np}(f), & \text{if } p \mid N, \\ a_n(T_p f) &= a_{pn}(f) + p^{k-1} a_{n/p}(f), & \text{if } p \nmid N. \end{aligned}$$

Suppose now that $f \in \mathcal{M}_k(N, \mathbb{Z})$ and $k > 0$. Then by the above definitions we also have $T_p(f) \in \mathcal{M}_k(N, \mathbb{Z})$. As a result, the Hecke operators are defined on $\mathcal{M}_k(N, B)$ as follows.

Definition 4.2. Let B be a commutative ring, p be a prime and $k > 0$. The Hecke operator T_p is defined on simple tensors in $\mathcal{M}_k(N, B)$ by

$$T_p(f \otimes_{\mathbb{Z}} x) := T_p(f) \otimes_{\mathbb{Z}} x,$$

where $f \in \mathcal{M}_k(N, \mathbb{Z})$ and $x \in B$.

Remark. If p is invertible in B or $p \mid N$ then the above definition naturally extends to $k = 0$. In addition, suppose that B is a domain of characteristic zero. Then T_p is well-defined on the algebra $\mathcal{M}(N, B)$ by Proposition 4.1.

Now let ℓ be a prime dividing N exactly once. A result due to B. Conrad [Pra09, Theorem A.1] tells us that the Atkin-Lehner operator W_ℓ is $\mathbb{Z}[1/\ell]$ -integral. In other words, if f is in $\mathcal{M}_k(N, \mathbb{Z}[1/\ell])$ then so is $W_\ell(f)$. Thus, if ℓ is invertible in B we can naturally define W_ℓ on $\mathcal{M}_k(N, B)$ as follows.

Definition 4.3. Let B be a $\mathbb{Z}[1/\ell]$ -algebra (so that ℓ is invertible in B). The Atkin-Lehner operator W_ℓ is defined on $\mathcal{M}_k(N, B)$ by

$$W_\ell(f \otimes_{\mathbb{Z}[1/\ell]} x) := W_\ell(f) \otimes_{\mathbb{Z}[1/\ell]} x$$

where $f \in \mathcal{M}_k(N, \mathbb{Z}[1/\ell])$ and $x \in B$. If B is a $\mathbb{Z}[1/\ell]$ -domain of characteristic zero then this definition extends to the algebra $\mathcal{M}(N, B)$.

Remark. The above definition makes sense since

$$\mathcal{M}_k(N, B) = \mathcal{M}_k(N, \mathbb{Z}) \otimes_{\mathbb{Z}} B = \mathcal{M}_k(N, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/\ell] \otimes_{\mathbb{Z}[1/\ell]} B = \mathcal{M}_k(N, \mathbb{Z}[1/\ell]) \otimes_{\mathbb{Z}[1/\ell]} B,$$

by the associativity of the tensor product.

These definitions indicate that a theory of oldforms and newforms on $\mathcal{M}_k(N, B)$ will not have a very interesting structure unless we assume some conditions on B . In this chapter, there are three main cases we will consider (in level of decreasing generality):

- (1) The case where B is an arbitrary commutative ring.
- (2) The case where B is a domain of characteristic zero.
- (3) The case where B is a characteristic zero $\mathbb{Z}[1/\ell]$ -domain, for some prime ℓ dividing N exactly once.

Notably, many common rings are $\mathbb{Z}[1/\ell]$ -domains such as the p -adic integers \mathbb{Z}_p (provided $p \neq \ell$) as well as any field such as \mathbb{Q} , \mathbb{R} , \mathbb{C} or \mathbb{Q}_p .

4.2 Oldforms over a Commutative Ring B

Let ℓ be a prime dividing N . Recall that the space of ℓ -old forms (of weight k) is given by

$$\mathcal{M}_k(N)^{\ell\text{-old}} = \mathcal{M}_k(N/\ell) + i_\ell(\mathcal{M}_k(N/\ell)),$$

or if ℓ divides N exactly once then

$$\mathcal{M}_k(N)^{\ell\text{-old}} = \mathcal{M}_k(N/\ell) + W_\ell(\mathcal{M}_k(N/\ell)).$$

For ℓ -old forms over any commutative ring B , we let $\mathcal{M}_k(N, \mathbb{Z})^{\ell\text{-old}} := \mathcal{M}_k(N)^{\ell\text{-old}} \cap \mathbb{Z}[[q]]$ and then define

$$\mathcal{M}_k(N, B)^{\ell\text{-old}} := \mathcal{M}_k(N, \mathbb{Z})^{\ell\text{-old}} \otimes_{\mathbb{Z}} B.$$

Similarly, we define $\mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-old}} = \mathcal{S}_k(N)^{\ell\text{-old}} \cap \mathbb{Z}[[q]]$ and $\mathcal{S}_k(N, B)^{\ell\text{-old}} = \mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-old}} \otimes_{\mathbb{Z}} B$. Note that $\mathcal{M}_k(N/\ell)$ and $\mathcal{S}_k(N/\ell)$ have bases in $\mathbb{Z}[[q]]$ and i_ℓ is \mathbb{Z} -integral. Hence the above definitions imply that $\mathcal{M}_k(N, \mathbb{C})^{\ell\text{-old}} = \mathcal{M}_k(N)^{\ell\text{-old}}$ and $\mathcal{S}_k(N, \mathbb{C})^{\ell\text{-old}} = \mathcal{S}_k(N)^{\ell\text{-old}}$.

From these definitions it follows that $\mathcal{M}_k(N/\ell, B) + i_\ell(\mathcal{M}_k(N/\ell, B)) \subseteq \mathcal{M}_k(N, B)^{\ell\text{-old}}$. On the other hand, it is not always true that $\mathcal{M}_k(N, B)^{\ell\text{-old}} \subseteq \mathcal{M}_k(N/\ell, B) + i_\ell(\mathcal{M}_k(N/\ell, B))$. This is illustrated in the following example.

Example 4.4. Let $p \geq 5$ and $E'_{p-1}(z) := -\frac{B_{p-1}}{2(p-1)}E_{p-1}(z)$ be the Eisenstein series of weight $p-1$ scaled so that $a_1 = 1$. In particular,

$$E'_{p-1}(q) = -\frac{B_{p-1}}{2(p-1)} + \sum_{n=1}^{\infty} \sigma_{p-2}(n)q^n \in \mathcal{M}_{p-1}(1, \mathbb{C}). \quad (4.2)$$

Now let

$$f(z) = E'_{p-1}(q) - E'_{p-1}(q^\ell) \quad (4.3)$$

noting that (4.3) is the only way to express f as the sum of a form in $\mathcal{M}_{p-1}(1, \mathbb{C})$ and a form in $i_\ell(\mathcal{M}_{p-1}(1, \mathbb{C}))$ (see Corollary 3.7). Since the constant terms cancel in (4.3) we have $f \in \mathcal{M}_k(N, \mathbb{Z})^{\ell\text{-old}}$ and thus $\bar{f} \in \mathcal{M}_k(N, \mathbb{F}_p)^{\ell\text{-old}}$ where \bar{f} is the image of f in $\mathbb{F}_p[[q]]$. However, \bar{f} is not in $\mathcal{M}_k(N/\ell, \mathbb{F}_p) + i_\ell(\mathcal{M}_k(N/\ell, \mathbb{F}_p))$. To see this, we note that the denominator of B_{p-1} in (4.2) is divisible by p (Proposition B.5), but p is not invertible in \mathbb{F}_p .

If B is a $\mathbb{Z}[1/\ell]$ -algebra then this problem no longer occurs.

Proposition 4.5. *If B is a $\mathbb{Z}[1/\ell]$ -algebra, then*

$$\mathcal{S}_k(N, B)^{\ell\text{-old}} = \mathcal{S}_k(N/\ell, B) \oplus W_\ell(\mathcal{S}_k(N/\ell, B)).$$

To prove this result, we first need the following lemma.

Lemma 4.6. *Let B be a $\mathbb{Z}[1/\ell]$ -algebra. If $f \in \mathcal{M}_k(N/\ell, B) \cap W_\ell \mathcal{M}_k(N/\ell, B)$, then f is constant.*

Proof. Generalise the argument from Proposition 3.6. □

Proof of Proposition 4.5. By the Lemma 4.6 we have $S_k(N/\ell, B) \cap W_\ell(S_k(N/\ell, B)) = \{0\}$ since nonzero cusp forms cannot be constant. Hence it suffices to show that

$$\mathcal{S}_k(N, B)^{\ell\text{-old}} = \mathcal{S}_k(N/\ell, B) + W_\ell(\mathcal{S}_k(N/\ell, B)). \quad (4.4)$$

Now, note that (4.4) is true when $B = \mathbb{C}$. And then, since $\mathcal{S}_k(N, \mathbb{C})$ has a basis in $\mathcal{S}_k(N, \mathbb{Q})$, (4.4) also holds when $B = \mathbb{Q}$ by a dimension counting argument.

Next we consider the case when $B = \mathbb{Z}[1/\ell]$. Directly from the definition of oldforms over B (see Section 4.2) we see that $\mathcal{S}_k(N/\ell, \mathbb{Z}[1/\ell]) + W_\ell(\mathcal{S}_k(N/\ell, \mathbb{Z}[1/\ell])) \subseteq \mathcal{S}_k(N, \mathbb{Z}[1/\ell])^{\ell\text{-old}}$. The reverse inclusion is not as clear.

Let $f \in \mathcal{S}_k(N, \mathbb{Z}[1/\ell])^{\ell\text{-old}}$. Since $\mathbb{Z}[1/\ell] \subseteq \mathbb{Q}$ we have $\mathcal{S}_k(N, \mathbb{Q}) = \mathcal{S}_k(N, \mathbb{Z}[1/\ell]) \otimes_{\mathbb{Z}[1/\ell]} \mathbb{Q}$ and thus

$$f = \frac{1}{M}(g + W_\ell h),$$

where $M \in \mathbb{Z}$ and $g, h \in \mathcal{S}_k(N/\ell, \mathbb{Z}[1/\ell])$. Because f has coefficients in $\mathbb{Z}[1/\ell]$ we must then have $g \equiv -W_\ell(h) \pmod{M\mathbb{Z}[1/\ell]}$ ⁽ⁱⁱ⁾. Now, $\mathbb{Z}[1/\ell]/M\mathbb{Z}[1/\ell]$ is also a $\mathbb{Z}[1/\ell]$ -algebra so by Lemma 4.6

$$W_\ell(h) \equiv g \equiv 0 \pmod{M\mathbb{Z}[1/\ell]}.$$

Hence both $\frac{1}{M}g$ and $\frac{1}{M}W_\ell(h)$ have coefficients in $\mathbb{Z}[1/\ell]$ and thus f is an element of $\mathcal{S}_k(N/\ell, \mathbb{Z}[1/\ell]) + W_\ell(\mathcal{S}_k(N/\ell, \mathbb{Z}[1/\ell]))$ as required.

Finally, we note that if B is any $\mathbb{Z}[1/\ell]$ -algebra then

$$\begin{aligned} \mathcal{S}_k(N, B)^{\ell\text{-old}} &\cong \mathcal{S}_k(N/\ell, \mathbb{Z}[1/\ell])^{\ell\text{-old}} \otimes_{\mathbb{Z}[1/\ell]} B \\ &\cong (\mathcal{S}_k(N/\ell, \mathbb{Z}[1/\ell]) \oplus W_\ell(\mathcal{S}_k(N/\ell, \mathbb{Z}[1/\ell]))) \otimes_{\mathbb{Z}[1/\ell]} B \\ &= \mathcal{S}_k(N/\ell, B) \oplus W_\ell(\mathcal{S}_k(N/\ell, B)), \end{aligned}$$

as required. □

⁽ⁱⁱ⁾That is, $a_n(g) \equiv a_n(-W_\ell(h))$ for all $n \geq 0$.

4.3 Newforms over a Commutative Ring B

For any commutative ring B , we define

$$\mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-new}} := \mathcal{S}_k(N)^{\ell\text{-new}} \cap \mathbb{Z}[[q]], \text{ and}$$

$$\mathcal{S}_k(N, B)^{\ell\text{-new}} := \mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-new}} \otimes_{\mathbb{Z}} B.$$

Also let

$$\mathcal{S}(N, B)^{\ell\text{-new}} = \sum_{k=0}^{\infty} \mathcal{S}_k(N, B)^{\ell\text{-new}}.$$

Then, to see that $\mathcal{S}_k(N, \mathbb{C})^{\ell\text{-new}}$ agrees with the classical definition, we prove that $\mathcal{S}_k(N)^{\ell\text{-new}}$ has a basis in $\mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-new}}$.

Proposition 4.7. *The space $\mathcal{S}_k(N)^{\ell\text{-new}}$ has a basis in $\mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-new}}$.*

Proof. First note that for all $M \geq 1$, $\mathcal{S}_k(M)^{\text{new}}$ has a basis in $\mathbb{Z}[[q]]$. To see this, one can use the fact that Galois conjugates of newforms are new [DI95, Corollary 12.4.5] and then apply the argument in [DS05, Corollary 6.5.6].

Now, by Proposition 3.2 we have the decomposition

$$\mathcal{S}_k(N)^{\ell\text{-new}} = \sum_{\substack{M \\ \ell^\alpha | M | N}} \sum_{e|(N/M)} i_e(\mathcal{S}_k(M)^{\text{new}}), \quad (4.5)$$

where α is the number of times that ℓ divides N . Since each $\mathcal{S}_k(M)^{\text{new}}$ has a basis over \mathbb{Z} and i_e is \mathbb{Z} -integral, it thus follows that $\mathcal{S}_k(N)^{\ell\text{-new}}$ has a basis in $\mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-new}}$ as required. \square

If $k \neq 2$, the above argument also applies to $\mathcal{M}_k(N)^{\ell\text{-new}} = \mathcal{S}_k(N)^{\ell\text{-new}} \oplus \mathcal{E}_k(N)^{\ell\text{-new}}$ as defined in Section 3.5. In particular, one can use the explicit basis of eigenforms described in [DS05, Proposition 5.2.3] to produce a decomposition analogous to (4.5) for $\mathcal{E}_k(N)^{\ell\text{-new}}$ and thus $\mathcal{M}_k(N)^{\ell\text{-new}}$ ⁽ⁱⁱⁱ⁾. In light of this, we also define

$$\mathcal{M}_k(N, \mathbb{Z})^{\ell\text{-new}} := \mathcal{M}_k(N)^{\ell\text{-new}} \cap \mathbb{Z}[[q]], \text{ and}$$

$$\mathcal{M}_k(N, B)^{\ell\text{-new}} := \mathcal{M}_k(N, \mathbb{Z})^{\ell\text{-new}} \otimes_{\mathbb{Z}} B.$$

⁽ⁱⁱⁱ⁾See also [Wei77, Proposition 21].

4.4 Span of Oldforms and Newforms in Characteristic Zero

Recall the classical decomposition of $\mathcal{S}_k(N, \mathbb{C})$ into ℓ -old and ℓ -new forms:

$$\mathcal{S}_k(N, \mathbb{C})^{\ell\text{-old}} \oplus \mathcal{S}_k(N, \mathbb{C})^{\ell\text{-new}} = \mathcal{S}_k(N, \mathbb{C}). \quad (4.6)$$

A natural question to ask is whether this decomposition holds if we replace \mathbb{C} with an arbitrary commutative ring B . Unfortunately, this does not turn out to be the case in general. For instance, in characteristic p there are congruences between ℓ -old and ℓ -new forms so that we cannot even use a direct sum as in (4.6). However, at least when B is a domain of characteristic zero, the space of ℓ -old and ℓ -new forms intersect trivially.

Proposition 4.8. *Let B be a domain of characteristic zero. Then for any weight k ,*

$$\mathcal{S}_k(N, B)^{\ell\text{-old}} \cap \mathcal{S}_k(N, B)^{\ell\text{-new}} = \{0\}.$$

Proof. First note that the proposition holds for $B = \mathbb{C}$ and thus $B = \mathbb{Z}$. Equivalently, the map

$$\mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-new}} \oplus \mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-old}} \rightarrow \mathcal{S}_k(N, \mathbb{Z}) \quad (4.7)$$

is injective. Now suppose that B is an arbitrary domain of characteristic zero. As B is flat over \mathbb{Z} (Corollary A.4), tensoring (4.7) with B preserves injectivity thus giving the desired result. \square

In addition, if B is a field then the ℓ -new and ℓ -old forms span $\mathcal{S}_k(N, B)$.

Proposition 4.9. *Let B be a field of characteristic zero. Then for any weight k ,*

$$\mathcal{S}_k(N, B)^{\ell\text{-old}} \oplus \mathcal{S}_k(N, B)^{\ell\text{-new}} = \mathcal{S}_k(N, B).$$

Proof. Since $\mathcal{S}_k(N, B)^{\ell\text{-old}}$ and $\mathcal{S}_k(N, B)^{\ell\text{-new}}$ intersect trivially (Proposition 4.8), it suffices to show that

$$\dim_B(\mathcal{S}_k(N, B)^{\ell\text{-old}}) + \dim_B(\mathcal{S}_k(N, B)^{\ell\text{-new}}) = \dim_B(\mathcal{S}_k(N, B)). \quad (4.8)$$

First note that this holds when $B = \mathbb{C}$ since $\mathcal{S}_k(N, \mathbb{C})^{\ell\text{-new}}$ is the orthogonal complement of $\mathcal{S}_k(N, \mathbb{C})^{\ell\text{-old}}$ with respect to the Petersson inner product. Now, $\mathcal{S}_k(N, \mathbb{C})^{\ell\text{-old}}$, $\mathcal{S}_k(N, \mathbb{C})^{\ell\text{-new}}$ and $\mathcal{S}_k(N, \mathbb{C})$ all have bases over \mathbb{Z} and thus \mathbb{Q} . Hence (4.8) holds for $B = \mathbb{Q}$

as well.

We now take B to be an arbitrary field of characteristic 0. Note that \mathbb{Q} is a subfield of B . In particular, we have a unique injective ring homomorphism from \mathbb{Q} to B , sending $\frac{a}{b} \in \mathbb{Q}$ to $\varphi(a)\varphi(b)^{-1}$ where φ is the unique ring homomorphism from \mathbb{Z} to B . As a result,

$$\dim_B(\mathcal{S}_k(N, B)) = \dim_B(\mathcal{S}_k(N, \mathbb{Q}) \otimes_{\mathbb{Q}} B) = \dim_{\mathbb{Q}}(\mathcal{S}_k(N, \mathbb{Q}))$$

and similarly for $\mathcal{S}_k(N, B)^{\ell\text{-old}}$ and $\mathcal{S}_k(N, B)^{\ell\text{-new}}$. Thus (4.8) is true for all characteristic 0 fields as required. \square

The following example demonstrates what can happen if we remove the assumption that B is a field.

Example 4.10. The space $\mathcal{S}_4(5)$ is spanned by a single cuspform

$$f = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7 - 23q^9 + O(q^{10}).$$

Similarly, the space $\mathcal{S}_4(10)^{2\text{-new}}$ is spanned by a single cuspform

$$g = q + 2q^2 - 8q^3 + 4q^4 + 5q^5 - 16q^6 - 4q^7 + 8q^8 + 37q^9 + O(q^{10}).$$

Note that $f \in \mathcal{S}_4(10, \mathbb{Z})^{2\text{-old}}$ and $g \in \mathcal{S}_4(10, \mathbb{Z})^{2\text{-new}}$. Looking at the first few terms we also see that $a_n(f) \equiv a_n(g) \pmod{2}$ and by [Ste07, Corollary 9.20] this congruence holds for all $n \geq 0$. Hence, we also have

$$h := \frac{1}{2}(f - g) \in \mathcal{S}_4(10, \mathbb{Z}). \tag{4.9}$$

But $h \notin \mathcal{S}_4(10, \mathbb{Z})^{2\text{-old}} \oplus \mathcal{S}_4(10, \mathbb{Z})^{2\text{-new}}$. To see this, note that $h = \frac{1}{2}f - \frac{1}{2}g$ is the only way to write h as the sum of a forms in $\mathcal{S}_4(10, \mathbb{C})^{2\text{-old}}$ and $\mathcal{S}_4(10, \mathbb{C})^{2\text{-new}}$ by Proposition 4.9. However, $\frac{1}{2}f$ and $\frac{1}{2}g$ are not in $\mathcal{S}_4(10, \mathbb{Z})$.

4.5 U_ℓ -new and Tr_ℓ -new in Characteristic Zero

We now look at how the notions of U_ℓ -new and Tr_ℓ -new from Chapter 3 carry over in this more general setting. In what follows, we will always assume that ℓ is a fixed prime dividing N .

Definition 4.11. Let B be any commutative ring and \mathcal{C} be a Hecke-invariant submodule of $\mathcal{M}_k(N, B)$. Define

$$\mathcal{C}^{U_\ell\text{-new}} := \ker(\mathcal{D}_\ell|_{\mathcal{C}}).$$

If B is a domain of characteristic zero then we can take \mathcal{C} to be any Hecke-invariant submodule of $\mathcal{M}(N, B) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(N, B)$.

Theorem 4.12. *Let B be a domain of characteristic zero. We have*

$$\mathcal{S}(N, B)^{\ell\text{-new}} = \mathcal{S}(N, B)^{U_\ell\text{-new}}.$$

Proof. Since B is a domain of characteristic zero, $\mathcal{S}(N, B) = \bigoplus_{k=0}^{\infty} \mathcal{S}_k(N, B)$. And so, because \mathcal{D}_ℓ is weight preserving it suffices to prove

$$\mathcal{S}_k(N, B)^{\ell\text{-new}} = \mathcal{S}_k(N, B)^{U_\ell\text{-new}} \tag{4.10}$$

in a single weight k . By Theorem 3.9, (4.10) holds for $B = \mathbb{C}$ and thus also for $B = \mathbb{Z}$. And then, if B is any other domain of characteristic zero we have

$$\mathcal{S}_k(N, B)^{\ell\text{-new}} = \mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-new}} \otimes_{\mathbb{Z}} B = \mathcal{S}_k(N, \mathbb{Z})^{U_\ell\text{-new}} \otimes_{\mathbb{Z}} B = \ker(D_\ell|_{\mathcal{S}_k(N, \mathbb{Z})}) \otimes_{\mathbb{Z}} B. \tag{4.11}$$

However, since B is flat over \mathbb{Z} ,

$$\ker(D_\ell|_{\mathcal{S}_k(N, \mathbb{Z})}) \otimes_{\mathbb{Z}} B = \ker(\mathcal{D}_\ell|_{\mathcal{S}_k(N, \mathbb{Z}) \otimes_{\mathbb{Z}} 1_B}) = \ker(\mathcal{D}_\ell|_{\mathcal{S}_k(N, B)}) = \mathcal{S}_k(N, B)^{U_\ell\text{-new}}$$

(see Proposition A.5 in the Appendix). Combining this with (4.11) gives the desired result. \square

We now make a similar generalisation for Tr_ℓ -new forms.

Definition 4.13. Let B be a $\mathbb{Z}[1/\ell]$ -algebra and \mathcal{C} be a Hecke-invariant submodule of $\mathcal{M}_k(N, B)$. Define

$$\mathcal{C}^{\text{Tr}_\ell\text{-new}} := \ker(\text{Tr}_\ell|_{\mathcal{C}}) \cap \ker(\text{Tr}_\ell W_\ell|_{\mathcal{C}}).$$

If B is a $\mathbb{Z}[1/\ell]$ -domain of characteristic zero then we can take \mathcal{C} to be any Hecke-invariant submodule of $\mathcal{M}(N, B) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(N, B)$.

Remark. Here, we require that B is a $\mathbb{Z}[1/\ell]$ -algebra so that W_ℓ and Tr_ℓ are well-defined. This follows from our general definition of W_ℓ (Definition 4.3) and the formula $\text{Tr}_\ell(f) = f + \ell^{1-k} U_\ell W_\ell f$.

Theorem 4.14. *Let B be a $\mathbb{Z}[1/\ell]$ -domain of characteristic zero. We have*

$$\mathcal{S}(N, B)^{\ell\text{-new}} = \mathcal{S}(N, B)^{U_{\ell}\text{-new}} = \mathcal{S}(N, B)^{\text{Tr}_{\ell}\text{-new}}$$

and if $k \neq 2$

$$\mathcal{M}_k(N, B)^{\ell\text{-new}} = \mathcal{M}_k(N, B)^{U_{\ell}\text{-new}} = \mathcal{M}_k(N, B)^{\text{Tr}_{\ell}\text{-new}}.$$

Proof. In Chapter 3 we showed that all of these statements were true over \mathbb{C} . To generalise to any characteristic zero $\mathbb{Z}[1/\ell]$ -domain B , we repeat the arguments from the proof of Theorem 4.12 making use of Theorems 3.14 and 3.17. \square

Considering all primes dividing N gives the corresponding result for $\mathcal{S}(N, B)^{\text{new}} = \bigcap_{\ell|N} \mathcal{S}(N, B)^{\ell\text{-new}}$ and $\mathcal{M}_k(N, B)^{\text{new}} = \bigcap_{\ell|N} \mathcal{M}_k(N, B)^{\ell\text{-new}}$.

Corollary 4.15. *Let B be a $\mathbb{Z}[1/N]$ -domain of characteristic zero. If N is squarefree then*

$$\mathcal{S}(N, B)^{\text{new}} = \bigcap_{\ell|N} \mathcal{S}(N, B)^{U_{\ell}\text{-new}} = \bigcap_{\ell|N} \mathcal{S}(N, B)^{\text{Tr}_{\ell}\text{-new}}$$

and if $k \neq 2$,

$$\mathcal{M}_k(N, B)^{\text{new}} = \bigcap_{\ell|N} \mathcal{M}_k(N, B)^{U_{\ell}\text{-new}} = \bigcap_{\ell|N} \mathcal{M}_k(N, B)^{\text{Tr}_{\ell}\text{-new}}.$$

Proof. Follows directly from Theorem 4.14. Note that since N is squarefree, each prime ℓ divides N exactly once. Moreover, each prime $\ell \mid N$ is invertible in B with $\ell^{-1} = \frac{(N/\ell)}{N}$. \square

The theory of newforms in characteristic zero thus naturally generalises the classical theory over \mathbb{C} . In the next chapter we will perform a similar analysis in characteristic p .

CHAPTER 5

NEWFORMS IN CHARACTERISTIC p

In this chapter, we attempt to generalise our results of the previous chapter to characteristic p . However, due to congruences between modular forms, the theory of newforms in characteristic p is more complicated than in characteristic zero.

Again we assume that $N > 1$ is a fixed positive integer. Moreover, for any two modular forms $f, g \in \mathcal{M}_k(N, \mathbb{Z})$ we will always write $f \equiv g \pmod{p}$ to mean that $a_n(f) \equiv a_n(g) \pmod{p}$ for all $n \geq 0$.

The main results of this chapter are adapted from [DM19, Section 6.2].

5.1 Modular Forms in Characteristic p

In Chapter 4 we showed that if B is a domain of characteristic zero then

$$\mathcal{M}(N, B) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(N, B).$$

This is no longer true in characteristic p as a result of the following proposition.

Proposition 5.1. *Let $p \geq 5$ be a prime number. The Eisenstein series E_{p-1} can be scaled so that $E_{p-1} \in \mathcal{M}_k(N, \mathbb{Z})$. Moreover, we can choose this scaling to be such that*

$$E_{p-1}(q) \equiv 1 \pmod{p}.$$

If $p = 2$ or $p = 3$ then E_4 and E_6 can similarly be scaled so that

$$E_4(q) \equiv E_6(q) \equiv 1 \pmod{p}.$$

Proof. Recall that E_{p-1} is given by

$$E_{p-1}(q) = 1 - \frac{2(p-1)}{B_{p-1}} \sum_{n=1}^{\infty} \sigma_{p-2}(n) q^n,$$

where B_{p-1} is the $(p-1)^{\text{th}}$ Bernoulli number. Now, write $B_{p-1} = \frac{a}{b}$ where a and b are coprime integers. As a result

$$aE_{p-1}(q) = a - 2(p-1)b \sum_{n=1}^{\infty} \sigma_{p-2}(n)q^n \in \mathcal{M}_k(N, \mathbb{Z}).$$

However, $b \equiv 0 \pmod{p}$ by Proposition B.5 and thus

$$aE_{p-1}(q) \equiv a \pmod{p}.$$

Next we note that $\gcd(a, p) = 1$ since $\gcd(a, b) = 1$ and $b \equiv 0 \pmod{p}$. Therefore there exists $m \in \{1, \dots, p-1\}$ such that

$$maE_{p-1}(q) \equiv 1 \pmod{p},$$

which is the desired result. Without too much abuse of notation, we will always assume that E_{p-1} has been scaled as such so that we write

$$E_{p-1}(q) \equiv 1 \pmod{p}.$$

Finally, if $p = 2$ or $p = 3$ then we can perform a similar argument for E_4 and E_6 noting that $B_4 = -\frac{1}{30}$ and $B_6 = \frac{1}{42}$. □

From now on, we normalise E_{p-1} as in Prop 5.1?

Corollary 5.2. *Let p be a prime number. If $p \geq 5$ then $\mathcal{M}_k(N, \mathbb{F}_p) \subseteq \mathcal{M}_{k+(p-1)}(N, \mathbb{F}_p)$. On the other hand, if $p = 2$ or $p = 3$ then $\mathcal{M}_k(N, \mathbb{F}_p) \subseteq \mathcal{M}_{k+4}(N, \mathbb{F}_p)$. In particular, the sum $\sum_{k=0}^{\infty} \mathcal{M}_k(N, \mathbb{F}_p)$ is never direct.*

Proof. If $p \geq 5$ then for any $f \in \mathcal{M}_k(N)$ we have $f \equiv E_{p-1}f \pmod{p}$. Since $E_{p-1}f$ is in $\mathcal{M}_{k+(p-1)}(N)$ we attain the desired result. For $p = 2$ or $p = 3$ we can argue similarly using E_4 . □

These are essentially the only congruences we will have to consider in characteristic p .

Proposition 5.3. *Let $p \geq 5$ be a prime number and B be a domain of characteristic p . The kernel of the map*

$$\bigoplus_{k=0}^{\infty} \mathcal{M}_k(N, B) \rightarrow B[[q]]$$

is generated by $\overline{E_{p-1}} - \overline{1}$. Here, $\overline{E_{p-1}}$ and $\overline{1}$ are the images of $E_{p-1} \in \mathcal{M}_{p-1}(N, \mathbb{Z})$ and $1 \in \mathcal{M}_0(N, \mathbb{Z})$ in $B[[q]]$.

Remark. Although $E_{p-1} \equiv 1 \pmod{p}$ we view \overline{E}_{p-1} and $\overline{1}$ as distinct modular forms in $\bigoplus_{k=0}^{\infty} \mathcal{M}_k(N, B)$ since they arise from forms in $\mathcal{M}(N, \mathbb{Z})$ with different weights.

Proof of Proposition 5.3. For $B = \mathbb{F}_p$ this is shown in [Gor02, Theorem 5.4 in Chapter 4]. Now suppose that B is an arbitrary domain of characteristic p . Then B is a vector space over \mathbb{F}_p . Namely, this means that B is a free \mathbb{F}_p -module and thus flat over \mathbb{F}_p . Since flat modules preserve kernels (Proposition A.5) we obtain the desired result. \square

In light of this observation, we define

$$\mathcal{M}(N, B)^i := \bigcup_{k \equiv i \pmod{p-1}} \mathcal{M}_k(N, B),$$

so that when B is a domain of characteristic $p \geq 5$

$$\mathcal{M}(N, B) = \bigoplus_{i \in 2\mathbb{Z}/(p-1)\mathbb{Z}} \mathcal{M}(N, B)^i. \quad (5.1)$$

We also analogously define $\mathcal{S}(N, B)^i = \bigcup_{k \equiv i \pmod{p-1}} \mathcal{S}_k(N, B)$ for cusp forms. Note that when $p \geq 5$ each element of $\mathcal{M}(N, B)^i$ can be assumed to appear in a single weight k since $\mathcal{M}_k(N, B) \subseteq \mathcal{M}_{k+(p-1)}(N, B)$.

On the other hand, suppose that B has characteristic 2 or 3. In this case, $\mathcal{M}(N, B) = \mathcal{M}(N, B)^0$. Each element of $\mathcal{M}(N, B)^0$ can also be assumed to appear in a single weight by multiplying by suitable powers of \overline{E}_4 and \overline{E}_6 .

5.2 Atkin-Lehner, Hecke, \mathcal{D}_ℓ and Tr_ℓ Operators in Characteristic p

Let ℓ be a prime dividing N exactly once. In Chapter 4 we showed that if B is a $\mathbb{Z}[1/\ell]$ -algebra then the scaled Atkin-Lehner operator $W_\ell = \ell^{k/2} w_\ell$ is well-defined on $\mathcal{M}_k(N, B)$. In addition, $\mathcal{M}_k(N/\ell, B) \cap W_\ell(\mathcal{M}_k(N/\ell, B)) = B$ (Lemma 4.6). If B is also a domain (of any characteristic) then these results extend to the algebra $\mathcal{M}(N, B) = \sum_{k=0}^{\infty} \mathcal{M}_k(N, B)$.

Proposition 5.4. *Let B be a $\mathbb{Z}[1/\ell]$ -domain. Then:*

- (a) *The operator W_ℓ is a well-defined automorphism on the algebra $\mathcal{M}(N, B)$.*
- (b) *We have $\mathcal{M}(N/\ell, B) \cap W_\ell(\mathcal{M}(N/\ell, B)) = B$.*

Proof.

- (a) In characteristic zero $\mathcal{M}(N, B) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(N, B)$ so we can simply define W_ℓ on $\mathcal{M}(N, B)$ by extending its definition on each $\mathcal{M}_k(N, B)$ by linearity. Note that W_ℓ is an automorphism since it has an inverse $W_\ell^{-1} = \ell^{-k} W_\ell$.

Now suppose that B has characteristic p . For any $f \in \mathcal{M}(N, B)$ we have a unique decomposition $f = \sum_i f_i$ where $f_i \in \mathcal{M}(N, B)^i$ as in (5.1). Hence it suffices to show that W_ℓ is well-defined on each $\mathcal{M}(N, B)^i$. As discussed in Section 5.1, each form in $\mathcal{M}(N, B)^i$ can be assumed to appear in a single weight k . We need to show that W_ℓ does not depend on this weight.

So, let $f_i, f'_i \in \mathcal{M}(N, \mathbb{Z})^i$ be forms with the same mod B reduction⁽ⁱ⁾ and weights k and k' respectively. Without loss of generality, assume that $k' \geq k$ so that $k' - k = m(p-1)$ for some $m \geq 0$. If $p \geq 5$ then $E_{p-1}^m f_i$ has the same weight as f'_i and also has the same q -expansion mod B . Since W_ℓ is well-defined on $\mathcal{M}_{k'}(N, B)$ (i.e. in fixed weight) we then have

$$W_\ell(f'_i) \equiv W_\ell(E_{p-1}^m f_i) \equiv W_\ell(E_{p-1})^m W_\ell(f_i) \equiv (\ell^{p-1} E_{p-1}(\ell z))^m W_\ell(f_i) \equiv W_\ell(f_i),$$

where the congruences are taken mod B . Here $\ell^{(p-1)} \equiv 1$ by Fermat's little theorem⁽ⁱⁱ⁾. If $p = 2$ or $p = 3$ then we can perform an analogous argument using powers of E_4 and E_6 . Therefore, W_ℓ is well-defined on each $\mathcal{M}(N, B)^i$ and thus $\mathcal{M}(N, B)$ as required.

- (b) Again, if B has characteristic zero then the result follows since $\mathcal{M}(N, B) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(N, B)$. Now suppose that B has characteristic p . Let f and g be forms in $\mathcal{M}(N, B)$ such that $f = W_\ell(g)$. We write $f = \sum_i f_i, g = \sum_i g_i$ where $f_i, g_i \in \mathcal{M}(N, B)^i$. By multiplying by suitable powers of Eisenstein series we may also assume that each f_i and g_i have the same fixed weight k_i . Now, since $f = W_\ell(g)$, we have $\sum_i f_i = \sum_i W_\ell(g_i)$. However, W_ℓ is weight-preserving so linear independence implies that $f_i = W_\ell(g_i)$. As a result, each f_i is constant (Lemma 4.6) so that $f = \sum_i f_i \in B$ as required. \square

In light of these results, we define for any $\mathbb{Z}[1/\ell]$ -domain B

$$\mathcal{S}(N, B)^{\ell\text{-old}} = \mathcal{S}(N/\ell, B) \oplus W_\ell(\mathcal{S}(N/\ell, B)).$$

⁽ⁱ⁾That is, $\varphi(a_n(f_i)) = \varphi(a_n(f'_i))$ where φ is the unique ring homomorphism $\varphi: \mathbb{Z} \rightarrow B$.

⁽ⁱⁱ⁾Since ℓ is invertible in B we must have $(\ell, p) = 1$.

This agrees with the standard definition of ℓ -old forms in fixed weight (cf. Proposition 4.5).

We also emphasise the importance of using the scaled Atkin-Lehner operator W_ℓ . In particular, the following example shows that the operator $w_\ell = \ell^{-k/2}W_\ell$ is **not** always well-defined on $\mathcal{M}(N, B)$.

Example 5.5. Let $B = \mathbb{F}_5$ so that $\bar{1} \in \mathcal{M}_0(N, B)$ and $\bar{E}_4 \in \mathcal{M}_4(N, B)$ have the same q -expansion. We then have $w_3(1) = 1$ and $w_3(E_4) = 3^2 E_4(\gamma_\ell \cdot z) \equiv -1 \pmod{5}$, so that $w_3(\bar{1}) \neq w_3(\bar{E}_4)$.

The main idea here is that $\ell^k \equiv \ell^{k'} \pmod{p}$ if $k \equiv k' \pmod{p-1}$ but we might have $\ell^{k/2} \not\equiv \ell^{k'/2}$. Taking this into consideration, we define the following operator.

Definition 5.6. Let B be a $\mathbb{Z}[1/\ell]$ -algebra. The *weight-separating operator* $S_\ell: \mathcal{M}_k(N, B) \rightarrow \mathcal{M}_k(N, B)$ is given by $S_\ell(f) = \ell^k f$ with inverse $S_\ell^{-1}(f) = \ell^{-k} f$. If B is also a domain then we can define S_ℓ on $\mathcal{M}(N, B)$ by extending this definition by linearity.

Now, let B be a $\mathbb{Z}[1/\ell]$ -domain and $f \in \mathcal{M}_k(N, B)$. In terms of the weight-separating operator, we have

$$\mathcal{D}_\ell(f) = \ell^2 U_\ell^2 f - S_\ell f, \text{ and } \text{Tr}_\ell(f) = f + \ell S_\ell^{-1} U_\ell W_\ell f.$$

As a result, these operators can be extended to $\mathcal{M}(N, B)$ and we can consider notions of U_ℓ -new and Tr_ℓ -new forms in characteristic p . Similarly, the operator T_ℓ is well-defined on $\mathcal{M}(N/\ell, B)$, where it is given by

$$T_\ell = U_\ell + \ell S_\ell W_\ell f.$$

5.3 Newforms for $\mathbb{Z}[1/\ell]$ -domains in Characteristic p

refer to where the definitions are?

Again let ℓ be a prime dividing N exactly once. We now consider how the notions of ℓ -new, U_ℓ -new and Tr_ℓ -new forms carry over into characteristic p . For the rest of this section, we will always assume that B is a $\mathbb{Z}[1/\ell]$ -domain of characteristic p unless otherwise stated.

We begin by giving an example showing that in general, all three notions of newness disagree on $\mathcal{M}_k(N, B)$. This is in contrast to the case of characteristic zero in which all three notions coincide provided $k \neq 2$ (Theorem 4.14).

Example 5.7. Let $p \geq 3$ and $f = \bar{1} \in \mathcal{M}_0(N, \mathbb{F}_p)$. Note that $f \notin \mathcal{M}_0(N, \mathbb{F}_p)^{\ell\text{-new}}$ since f is the mod p reduction of $1 \in \mathcal{M}_0(1) \subseteq \mathcal{M}_0(N)^{\ell\text{-old}}$. However,

$$\begin{aligned}\mathcal{D}_\ell(f) &= \ell^2 U_\ell^2(1) - \ell^0 = \ell^2 - 1 \text{ and,} \\ \text{Tr}_\ell(f) &= 1 + \ell^{1-0} U_\ell W_\ell(f) = 1 + \ell.\end{aligned}$$

Hence if $\ell \equiv -1 \pmod{p}$ then $f \in \mathcal{M}(N, \mathbb{F}_p)^{U_\ell\text{-new}}$ and $f \in \mathcal{M}(N, \mathbb{F}_p)^{\text{Tr}_\ell\text{-new}}$. On the other hand, if $\ell \equiv 1 \pmod{p}$ then $f \in \mathcal{M}(N, \mathbb{F}_p)^{U_\ell\text{-new}}$ but $f \notin \mathcal{M}(N, \mathbb{F}_p)^{\text{Tr}_\ell\text{-new}}$. This example also generalises to higher weights by using the mod p reduction of Eisenstein series.

By restricting to cusp forms, we can overcome much of this problematic behaviour. In this direction, we prove a result (adapted from [DM19, Proposition 6.4]) regarding the ℓ -old forms that appear in $\mathcal{S}(N, B)^{U_\ell\text{-new}}$ and $\mathcal{S}(N, B)^{\text{Tr}_\ell\text{-new}}$.

Proposition 5.8. *Let B be a $\mathbb{Z}[1/\ell]$ -algebra and $f, g \in \mathcal{S}_k(N/\ell, B)$ for some fixed weight k . The following are equivalent*

- (1) $f + W_\ell(g) \in \ker(\mathcal{D}_\ell)$.
- (2) $f + W_\ell(g) \in \ker(\text{Tr}_\ell) \cap \ker(\text{Tr}_\ell W_\ell)$.
- (3) $\ell T_\ell f = -(\ell + 1)S_\ell g$ and $\ell T_\ell g = -(\ell + 1)f$.

In other words,

$$\begin{aligned}(\mathcal{S}_k(N, B)^{\ell\text{-old}})^{U_\ell\text{-new}} &= (\mathcal{S}_k(N, B)^{\ell\text{-old}})^{\text{Tr}_\ell\text{-new}} \\ &= \{f + W_\ell(g) : \ell T_\ell f = -(\ell + 1)S_\ell g \text{ and } \ell T_\ell g = -(\ell + 1)f\}\end{aligned}$$

If B is also a domain then the above results extend to the algebra $\mathcal{S}(N, B)$.

Proof. First we show (1) \Leftrightarrow (3). Recall that on $\mathcal{M}(N/\ell, B)$ we have $W_\ell = S_\ell U_\ell$, $U_\ell W_\ell = S_\ell$ and $\ell U_\ell = \ell T_\ell - W_\ell$. So,

$$\begin{aligned}\mathcal{D}_\ell(f) &= \ell^2 U_\ell^2 f - S_\ell f \\ &= \ell U_\ell(\ell T_\ell f - W_\ell f) - S_\ell f \\ &= \ell T_\ell(\ell T_\ell f) - W_\ell(\ell T_\ell f) - \ell U_\ell W_\ell f - S_\ell f \\ &= \ell^2 T_\ell^2 f - (\ell + 1)S_\ell f - W_\ell \ell T_\ell f.\end{aligned}$$

Whereas,

$$\begin{aligned}
\mathcal{D}_\ell(W_\ell g) &= \ell^2 U_\ell^2 W_\ell g - S_\ell W_\ell g \\
&= \ell^2 U_\ell S_\ell g - W_\ell S_\ell g \\
&= \ell^2 T_\ell S_\ell g - \ell W_\ell S_\ell g - W_\ell S_\ell g \\
&= \ell^2 T_\ell S_\ell g - (\ell + 1) W_\ell S_\ell g.
\end{aligned}$$

Hence, if $f + W_\ell(g) \in \ker(\mathcal{D}_\ell)$ then

$$\begin{aligned}
0 &= \mathcal{D}_\ell(f + W_\ell g) \\
&= (\ell^2 T_\ell^2 f - (\ell + 1) S_\ell f + \ell^2 T_\ell S_\ell g) - W_\ell(\ell T_\ell f + (\ell + 1) S_\ell g).
\end{aligned}$$

But by Proposition 4.5 we have $\mathcal{S}_k(N/\ell, B) \cap W_\ell \mathcal{S}_k(N/\ell, B) = \{0\}$. Thus $f + W_\ell(g) \in \ker(\mathcal{D}_\ell)$ if and only if

$$\ell^2 T_\ell^2 f - (\ell + 1) S_\ell f + \ell^2 T_\ell S_\ell g = 0 \text{ and } \ell T_\ell f + (\ell + 1) S_\ell g = 0.$$

The second equation tells us that $\ell T_\ell f = -(\ell + 1) S_\ell g$. Substituting this into the first equation and simplifying then gives

$$\ell T_\ell g = -(\ell + 1) f,$$

as required.

For (2) \Leftrightarrow (3) we recall that $\text{Tr}_\ell(f) = (\ell + 1)f$ and $\text{Tr}_\ell(W_\ell g) = \ell T_\ell(g)$ (see Lemma 3.13). And so,

$$\text{Tr}_\ell(f + W_\ell g) = 0 \Leftrightarrow (\ell + 1)f + \ell T_\ell g = 0 \Leftrightarrow \ell T_\ell g = -(\ell + 1)f.$$

On the other hand,

$$\begin{aligned}
0 &= \text{Tr}_\ell W_\ell(f + W_\ell g) = \text{Tr}_\ell(W_\ell f + S_\ell g) \\
&\Leftrightarrow \ell T_\ell f = -(\ell + 1) S_\ell g,
\end{aligned}$$

as required. If B is a domain then we can repeat the above argument for $f, g \in \mathcal{S}(N/\ell, B)$ by using Proposition 5.4. \square

Remark. In fixed weight k , this result can be expressed more symmetrically in terms of $w_\ell = \ell^{-k/2} W_\ell$. In particular, if $\lambda_k = -(\ell + 1)\ell^{\frac{k-2}{2}}$ then

$$\begin{aligned} (\mathcal{S}_k(N, B)^{\ell\text{-old}})^{U_\ell\text{-new}} &= (\mathcal{S}_k(N, B)^{\ell\text{-old}})^{\text{Tr}_\ell\text{-new}} \\ &= \{f + w_\ell(g) : T_\ell f = \lambda_k g \text{ and } T_\ell g = \lambda_k f\}. \end{aligned}$$

The constant λ_k also appears in the work of Ribet [Rib83] and Diamond [Dia91] on congruences between ℓ -old and ℓ -new forms. The connection between these results is explored further in [DM19, Section 7].

We now move onto the main theorem, comparing the notions of ℓ -new, U_ℓ -new and Tr_ℓ -new cusp forms in characteristic p .

Theorem 5.9. *Let B be a $\mathbb{Z}[1/\ell]$ -domain of characteristic p . We then have*

$$\mathcal{S}(N, B)^{\ell\text{-new}} \subseteq \mathcal{S}(N, B)^{U_\ell\text{-new}} = \mathcal{S}(N, B)^{\text{Tr}_\ell\text{-new}}.$$

Proof. Since B is a domain of characteristic p we have $\mathcal{S}(N, B) = \bigoplus_{i \in 2\mathbb{Z}/(p-1)\mathbb{Z}} \mathcal{S}(N, B)^i$ and each element of $\mathcal{S}(N, B)^i$ can be assumed to appear in a fixed weight. As a result, it will suffice to prove the theorem for some fixed weight k . In what follows, we will write X^B to indicate that an operator X is acting on $\mathcal{S}_k(N, B)$.

First we show that $\mathcal{S}_k(N, B)^{\ell\text{-new}} \subseteq \mathcal{S}_k(N, B)^{U_\ell\text{-new}}$. So, let $f \otimes b$ be a simple tensor in $\mathcal{S}_k(N, B)^{\ell\text{-new}} = \mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-new}} \otimes_{\mathbb{Z}} B$. We then have

$$\mathcal{D}_\ell^B(f \otimes b) = \mathcal{D}_\ell^{\mathbb{Z}}(f) \otimes b = 0$$

noting that $f \in \mathcal{S}_k(N, \mathbb{Z})^{\ell\text{-new}} = \ker(\mathcal{D}_\ell^{\mathbb{Z}})$. Hence $f \otimes b \in \ker(\mathcal{D}_\ell^B) = \mathcal{S}_k(N, B)^{U_\ell\text{-new}}$ so that in general, $\mathcal{S}_k(N, B)^{\ell\text{-new}} \subseteq \mathcal{S}_k(N, B)^{U_\ell\text{-new}}$ as required.

We now show that $\mathcal{S}_k(N, B)^{U_\ell\text{-new}} = \mathcal{S}_k(N, B)^{\text{Tr}_\ell\text{-new}}$ by mimicking the argument in [DM19, Theorem 1]. In what follows, we assume the reader is familiar with the basic properties of p -adic numbers. A summary can be found in Appendix A.2.

First note that it suffices to prove $\mathcal{S}_k(N, \mathbb{F}_p)^{U_\ell\text{-new}} = \mathcal{S}_k(N, \mathbb{F}_p)^{\text{Tr}_\ell\text{-new}}$. To see why this is sufficient, note that B is an \mathbb{F}_p -vector space and thus flat over \mathbb{F}_p . Therefore, tensoring with B will preserve the kernels of \mathcal{D}_ℓ , Tr_ℓ and $\text{Tr}_\ell W_\ell$ (Proposition A.5).

Now let $f \in \mathcal{S}_k(N, \mathbb{F}_p)^{U_\ell\text{-new}}$. Considering how \mathbb{F}_p is embedded in \mathbb{Z}_p , there exists $\tilde{f} \in \mathcal{S}_k(N, \mathbb{Z}_p)$

this is certainly not true because of the characteristic. what is true is that $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$

$$\tilde{f} \in \mathcal{S}_k(N, \mathbb{Z}_p) \subset \mathcal{S}_k(N, \mathbb{Q}_p)$$

such that f is the mod- p reduction of \tilde{f} . We take such an \tilde{f} and lift it to a modular form? over \mathbb{Q}_p . Since \mathbb{Q}_p is a field of characteristic zero, we then have $\tilde{f} = \tilde{f}_0^{\text{old}} + \tilde{f}_0^{\text{new}}$ where $\tilde{f}_0^{\text{old}} \in \mathcal{S}_k(N, \mathbb{Q}_p)^{\ell\text{-old}}$ and $\tilde{f}_0^{\text{new}} \in \mathcal{S}_k(N, \mathbb{Q}_p)^{\ell\text{-new}}$ (see Proposition 4.9). Let $b \in \mathbb{Z}_{\geq 0}$ be sufficiently large so that $p^b \tilde{f}_0^{\text{old}}$ and $p^b \tilde{f}_0^{\text{new}}$ are in $\mathcal{S}_k(N, \mathbb{Z}_p)$. That is,

$$\tilde{f} = p^{-b}(\tilde{f}^{\text{old}} + \tilde{f}^{\text{new}}),$$

where $\tilde{f}^{\text{old}} \in \mathcal{S}_k(N, \mathbb{Z}_p)^{\ell\text{-old}}$ and $\tilde{f}^{\text{new}} \in \mathcal{S}_k(N, \mathbb{Z}_p)^{\ell\text{-new}}$.

Since $f \in \ker(\mathcal{D}_\ell^{\mathbb{F}_p})$ it follows that $\mathcal{D}_\ell^{\mathbb{Z}_p}(\tilde{f})$ is in $p\mathbb{Z}_p[[q]]$. Next we note that $\mathcal{D}_\ell^{\mathbb{Z}_p}(\tilde{f}^{\text{new}}) = 0$ since $\mathcal{S}_k(N, \mathbb{Z}_p)^{\ell\text{-new}} = \mathcal{S}_k(N, \mathbb{Z}_p)^{U_{\ell\text{-new}}}$ (Theorem 4.12). As a result, $\mathcal{D}_\ell^{\mathbb{Z}_p}(\tilde{f}) = p^{-b}\mathcal{D}_\ell^{\mathbb{Z}_p}(\tilde{f}^{\text{old}})$. So, letting f^{old} be the reduction of \tilde{f}^{old} mod p^{b+1} we have $f^{\text{old}} \in \ker(\mathcal{D}_\ell^{\mathbb{Z}/p^{b+1}\mathbb{Z}})$. By Proposition 5.8 we then have $f^{\text{old}} \in \ker(\text{Tr}_\ell)^{\mathbb{Z}/p^{b+1}\mathbb{Z}} \cap \ker(\text{Tr}_\ell W_\ell)^{\mathbb{Z}/p^{b+1}\mathbb{Z}}$. Lifting back up to characteristic zero we see that both $\text{Tr}_\ell^{\mathbb{Z}_p}(\tilde{f}^{\text{old}})$ and $(\text{Tr}_\ell W_\ell)^{\mathbb{Z}_p}(\tilde{f}^{\text{old}})$ are in $p^{b+1}\mathbb{Z}_p[[q]]$.

Now, $(\text{Tr}_\ell)^{\mathbb{Z}_p}(\tilde{f}^{\text{new}}) = 0$ and hence both $\text{Tr}_\ell^{\mathbb{Z}_p}(\tilde{f})$ and $(\text{Tr}_\ell W_\ell)^{\mathbb{Z}_p}(\tilde{f})$ are in $p\mathbb{Z}_p[[q]]$. Therefore, $\text{Tr}_\ell^{\mathbb{F}_p}(f) = 0$ and $(\text{Tr}_\ell W_\ell)^{\mathbb{F}_p}(f) = 0$. That is, $f \in \ker(\text{Tr}_\ell)^{\mathbb{F}_p} \cap \ker(\text{Tr}_\ell W_\ell)^{\mathbb{F}_p}$. Thus $\ker(\mathcal{D}_\ell^{\mathbb{F}_p}) \subseteq \ker(\text{Tr}_\ell)^{\mathbb{F}_p} \cap \ker(\text{Tr}_\ell W_\ell)^{\mathbb{F}_p}$. Reversing all of the steps above gives the reverse containment. \square

Remarks.

- (i) In [DM19, Section 6.1], the space of ℓ -new forms $\mathcal{S}_k(N, B)^{\ell\text{-new}} := \mathcal{S}_k(N, \mathbb{Z}) \otimes_{\mathbb{Z}} B$ is only defined when B has characteristic zero. However, here we also considered the case where B has characteristic p for completeness.
- (ii) The relation $\mathcal{S}(N, B)^{\ell\text{-new}} \subseteq \mathcal{S}(N, B)^{U_{\ell\text{-new}}}$ holds when B is any domain of characteristic p (even if ℓ isn't invertible).

Question 5.10. Is the inclusion $\mathcal{S}(N, B)^{\ell\text{-new}} \subseteq \mathcal{S}(N, B)^{U_{\ell\text{-new}}}$ strict?

The author of this thesis admits to not knowing a comprehensive answer to this question. However, the following example shows a case where $\mathcal{S}_k(N, B)^{\ell\text{-new}} \subsetneq \mathcal{S}_k(N, B)^{U_{\ell\text{-new}}}$ in fixed weight k .

Example 5.11. The space $\mathcal{S}_2(11)$ is spanned by a single eigenform

$$f = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 + O(q^{10}).$$

Similarly, the space $\mathcal{S}_2(33)^{3\text{-new}}$ is spanned by a single eigenform

$$g = q + q^2 - q^3 - q^4 - 2q^5 - q^6 + 4q^7 - 3q^8 + q^9 + O(q^{10}).$$

Note that $f \in \mathcal{S}_2(33, \mathbb{Z})^{3\text{-old}}$ and $g \in \mathcal{S}_2(33, \mathbb{Z})^{3\text{-new}}$. Define $h := f - w_3(f) = f - 3i_3(f)$ so that

$$h = q - 2q^2 - 4q^3 + 2q^4 + q^5 + 8q^6 - 2q^7 + q^9 + O(q^{10}) \in \mathcal{S}_2(33, \mathbb{Z})^{3\text{-old}}.$$

Reducing mod 5 we then obtain

$$\bar{g} = q + q^2 + 4q^3 + 4q^4 + 3q^5 + 4q^6 + 4q^7 + 2q^8 + q^9 + O(q^{10}) \in \mathcal{S}_2(33, \mathbb{F}_5)^{3\text{-new}}, \text{ and}$$

$$\bar{h} = q + 3q^2 + q^3 + 2q^4 + q^5 + 3q^6 + 3q^7 + q^9 + O(q^{10}) \in \mathcal{S}_2(33, \mathbb{F}_5)^{3\text{-old}}.$$

By the remark following Proposition 5.8 we see that $\bar{h} \in \mathcal{S}_2(33, \mathbb{F}_5)^{U_3\text{-new}}$. However, $\bar{h} \notin \mathcal{S}_2(33, \mathbb{F}_5)^{3\text{-new}} = \text{Span}_{\mathbb{F}_5}\{\bar{g}\}$.

5.4 Summary/Conclusion

Example 5.11 from the previous section indicates that the notion of ℓ -new may differ from the algebraic notions of U_ℓ -new and Tr_ℓ -new in characteristic p . Because of this discrepancy, we now summarise the differences between each of these notions and their relative usefulness.

First we consider the original motivation for defining newforms over \mathbb{C} . In particular, the theory of oldforms and newforms allows us to classify cusp forms in $\mathcal{S}_k(N)$ for some weight k and level N . This classification is useful due to the desirable properties of newforms that we explored in Section 2.3. However, in characteristic p this classification breaks down. Namely, it is often the case that

$$\mathcal{S}_k(N, B)^{\ell\text{-old}} \cap \mathcal{S}_k(N, B)^{\ell\text{-new}} \neq \{0\}$$

when B has prime characteristic. See Example 4.10 for an instance of this behaviour in characteristic 2 or [DM19, Example 1] for an example in characteristic 7.

An alternative way to classify oldforms and newforms over \mathbb{C} is by their Hecke eigenvalues

(Lemma 2.15). However, the Hecke operators are not always diagonalisable in characteristic p making such a classification impossible (see [Joc82]). It is thus evident that ℓ -old and ℓ -new forms in characteristic p cannot be used to decompose the space $\mathcal{S}_k(N, B)$. Despite this problem, we may still hope to retain algebraic properties of newforms in characteristic p .

This is where the notions of U_ℓ -new and Tr_ℓ -new are useful. These notions are explicitly defined in terms of the Atkin-Lehner and Hecke operators and thus consist of forms that are well-behaved with respect to these operators. For instance, a U_ℓ -new form will always have a U_ℓ^2 eigenvalue of ℓ^{k-2} . Even in characteristic 0, defining newforms in this way is arguably more robust and elegant than using the Petersson inner product. Moreover, the intersection between ℓ -old and U_ℓ -new (or Tr_ℓ -new) forms can be described quite simply (Proposition 5.8).

An area of future research would be to study applications of U_ℓ -new and Tr_ℓ -new forms to other aspects of modular form theory. Importantly, Deo and Medvedovsky [DM19, Section 8] show that these notions of “newness” appear when looking at Monsky’s Hecke-stable filtration [Mon15], [Mon16].

Now, although the notions of U_ℓ -new and Tr_ℓ -new agree on cusp forms (Theorem 5.9), these notions do not agree on the space of all modular forms (Example 5.7). Moreover, we can only define Tr_ℓ -new forms over commutative rings where ℓ is invertible. So although similar, it is unclear which algebraic notion - U_ℓ -new or Tr_ℓ -new, provides a “better” definition of newness. In Chapter 6 we shall generalise these notions even further so that their different properties become more apparent.

CHAPTER 6

FURTHER GENERALISATIONS

In the last chapter we explored Deo and Medvedovsky's notions of U_ℓ -new and Tr_ℓ -new forms in characteristic p . These notions were only defined for modular forms for $\Gamma_0(N)$ but there are certainly other types of modular forms that we can consider. In this chapter we introduce the notion of modular forms with character and suggest definitions for U_ℓ -new and Tr_ℓ -new forms in this setting. Moreover, we consider ways of removing the requirement that N is squarefree (i.e. that each prime factor ℓ of N divides N exactly once).

Note that most of the proofs in this chapter will be quite short and instead describe how to generalise proofs from previous chapters. We will also frequently make use of *Dirichlet characters*. For information about these objects, see Appendix A.3.

As per usual we assume that N is a fixed positive integer.

6.1 Modular Forms with Character

Might be good to specify the coefficients here. \mathbb{C} ?

Definition 6.1. Let $k \geq 0$ and χ be a Dirichlet character mod N . A modular form with character χ (of weight k and level N) is an element of the vector space

$$\mathcal{M}_k^\chi(N) = \left\{ f \in \mathcal{M}_k(\Gamma_1(N)) : f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)f \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}.$$

The space of cusp forms with character χ , denoted $\mathcal{S}_k^\chi(N)$, is defined analogously.

In particular, note that $\mathcal{M}_k^{\chi_0}(N) = \mathcal{M}_k(N)$, where χ_0 is the trivial Dirichlet character mod N . We also have the following decomposition of $\mathcal{M}_k(\Gamma_1(N))$ [Ste07, Proposition 9.2]:

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{M}_k^\chi(N), \tag{6.1}$$

where the direct sum is over all Dirichlet characters mod N . Therefore, instead of directly studying the space $\mathcal{M}_k(\Gamma_1(N))$, we can instead study $\mathcal{M}_k^\chi(N)$ for each Dirichlet character χ mod N .

We can also define $\mathcal{S}_k^\chi(N)^{\text{new}}$ in a similar way to how we defined $\mathcal{S}_k(N)^{\text{new}}$ in Chapter 2. So, suppose that χ is a Dirichlet character mod N induced by a Dirichlet character χ' mod M . For each $e \mid (N/M)$ we have an embedding map $i_e: \mathcal{S}_k^{\chi'}(M) \hookrightarrow \mathcal{S}_k^\chi(N)$ that sends $f \in \mathcal{S}_k^{\chi'}(M)$ to the function $z \mapsto f(ez)$. We then use the images of these embedding maps (at each level from which χ is induced) to obtain $\mathcal{S}_k^\chi(N)^{\text{old}}$ and define $\mathcal{S}_k^\chi(N)^{\text{new}}$ to be the orthogonal complement of $\mathcal{S}_k^\chi(N)^{\text{old}}$ with respect to the Petersson inner product.

Similarly, for any prime ℓ dividing N , we define $\mathcal{S}_k^\chi(N)^{\ell\text{-old}}$ and $\mathcal{S}_k^\chi(N)^{\ell\text{-new}}$ as was done for $\mathcal{S}_k(N)$ in Section 3.1. Note that if χ is **not** induced by a character mod N/ℓ then there are no forms arising from level N/ℓ and thus $\mathcal{S}_k^\chi(N)^{\ell\text{-new}} = \mathcal{S}_k^\chi(N)$. For this reason, we will frequently restrict to the case where χ is induced by a Dirichlet character mod N/ℓ .

The properties of newforms for $\mathcal{S}_k^\chi(N)$ naturally generalise those for $\mathcal{S}_k(N)$ given by Atkin and Lehner. In particular, Li [Li75] provides analogues of the results from sections 2.3 and 3.1 for forms with character.

These definitions also generalise to modular forms for $\Gamma_1(N)$ and by (6.1) we have

$$\mathcal{M}_k(\Gamma_1(N))^{\ell\text{-new}} = \bigoplus_{\chi} \mathcal{M}_k^\chi(N)^{\ell\text{-new}}.$$

In fact, newforms for $\Gamma_1(N^2)$ can be used to describe newforms for the principal congruence subgroup $\Gamma(N)$. This is achieved by considering a group $\Gamma(N)^*$ that is conjugate to $\Gamma(N)$ and contains $\Gamma_1(N^2)$. See [Wei77, pp 20-21] for more details.

In Chapter 4 we discussed why $\mathcal{M}_k(N)$, $\mathcal{S}_k(N)$ and $\mathcal{S}_k(N)^{\ell\text{-new}}$ had bases in $\mathbb{Z}[[q]]$. These arguments also hold for modular forms for $\Gamma_1(N)$. Namely, the results we used from Diamond and Im's paper [DI95, Corollary 12.3.2 and Corollary 12.4.5] are stated for both $\Gamma_0(N)$ and $\Gamma_1(N)$. However, it is not always true that $\mathcal{M}_k^\chi(N)$ has a basis in $\mathbb{Z}[[q]]$. Instead, we have the following weaker result which will allow us to define $\mathcal{M}_k^\chi(N, B)$ for a restricted class of commutative rings B .

Proposition 6.2. *Let χ be a Dirichlet character mod N . The spaces $\mathcal{M}_k^\chi(N)$, $\mathcal{S}_k^\chi(N)$ and $\mathcal{S}_k^\chi(N)^{\ell\text{-new}}$ have bases consisting of forms with Fourier coefficients in $\mathbb{Z}[\chi]$. Here $\mathbb{Z}[\chi]$ denotes the ring generated by the image of $\chi: \mathbb{Z} \rightarrow \mathbb{C}$.*

To prove this result, we will need to introduce a new operator.

Definition 6.3. Let $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $\gamma = \begin{pmatrix} * & * \\ * & \bar{a} \end{pmatrix}$ be a matrix in $\Gamma_0(N)$ such that $\bar{a} \equiv a \pmod{N}$. The *diamond operator* $\langle a \rangle$ is the map that sends $f \in \mathcal{M}_k(\Gamma_1(N))$ to $f|_k\gamma$.

See [DS05, Section 5.2] as for why this operator is well-defined, along with some other basic results. Importantly, we note that if $f \in \mathcal{M}_k^\chi(N)$ then $\langle a \rangle f = \chi(a)f$.

Proof of Proposition 6.2. We only prove the result for $\mathcal{M}_k^\chi(N)$ but the proofs for $\mathcal{S}_k^\chi(N)$ and $\mathcal{S}_k^\chi(N)^{\ell\text{-new}}$ are identical.

Consider the map

$$\pi_\chi: \mathcal{M}_k(\Gamma_1(N)) \rightarrow \mathcal{M}_k^\chi(N), \quad \pi_\chi(f) = \frac{1}{\varphi(N)} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(a) \langle a \rangle f,$$

where $\varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$. We show that π_χ is a well-defined projection. So, let $f \in \mathcal{M}_k(\Gamma_1(N))$ and write $f = \sum_\psi f_\psi$ where the sum is over all Dirichlet characters $\psi \pmod{N}$ and $f_\psi \in \mathcal{M}_k^\psi(N)$ (refer to the decomposition in (6.1)). Then,

$$\begin{aligned} \pi_\chi(f) &= \frac{1}{\varphi(N)} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \left(\bar{\chi}(a) \langle a \rangle \sum_\psi f_\psi \right) \\ &= \frac{1}{\varphi(N)} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \left(\bar{\chi}(a) \sum_\psi \psi(a) f_\psi \right) \\ &= \frac{1}{\varphi(N)} \varphi(N) f_\psi \quad (*) \\ &= f_\psi, \end{aligned}$$

where $(*)$ follows from the orthogonality relations for Dirichlet characters (Proposition A.11).

Now, let \mathcal{B} be a basis for $\mathcal{M}_k(\Gamma_1(N))$ such that every $f \in \mathcal{B}$ has Fourier coefficients in \mathbb{Z} . Since the diamond operator is \mathbb{Z} -integral ([DI95, Proposition 12.3.11]) it follows that $\pi_\chi(\mathcal{B})$ is a spanning set for $\mathcal{M}_k^\chi(N)$ consisting of forms with Fourier coefficients in $\mathbb{Z}[\chi]$. Taking a linearly independent subset of $\pi_\chi(\mathcal{B})$ then gives the desired result. \square

In light of this result we now let $\mathcal{M}_k^\chi(N, \mathbb{Z}[\chi])$ be the set of all modular forms in $\mathcal{M}_k^\chi(N)$ whose Fourier coefficients are in $\mathbb{Z}[\chi]$. For any $\mathbb{Z}[\chi]$ -algebra B we then define

$$\mathcal{M}_k^\chi(N, B) := \mathcal{M}_k(N, \mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} B,$$

and analogously for $\mathcal{S}_k^\chi(N, B)$ and $\mathcal{S}_k^\chi(N, B)^{\ell\text{-new}}$.

Finally we consider the structure of $\mathcal{M}^\chi(N, B) = \sum_{k=0}^{\infty} \mathcal{M}_k^\chi(N, B)$. Note that $\mathcal{M}^\chi(N, B)$ isn't an algebra if χ is nontrivial. In particular, if f and g are modular forms with character χ then fg is a modular form with character χ^2 . Regardless, $\mathcal{M}^\chi(N, B)$ is still an interesting module to study.

Proposition 6.4. *Let χ be a Dirichlet character mod N and B be a $\mathbb{Z}[\chi]$ -domain of characteristic zero. Then $\mathcal{M}^\chi(N, B) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k^\chi(N, B)$.*

Proof. First note that the statement holds for $B = \mathbb{C}$ (see the remark after Proposition B.2 in Appendix B). Now, suppose that B is a general $\mathbb{Z}[\chi]$ -domain of characteristic zero. Since each element in the image of χ is a root of unity we have $\mathbb{Z}[\chi] = \mathbb{Z}[\zeta_n]$ where ζ_n is a primitive n^{th} root of unity for some $n \in \mathbb{Z}_{>0}$. And so, by Corollary A.4 we have that B is flat over $\mathbb{Z}[\chi]$. We can then repeat the argument used for $\mathcal{M}(N, B)$ in Chapter 4 (Proposition 4.1). \square

Proposition 6.5. *Let B be a $\mathbb{Z}[\chi]$ -domain of characteristic p . Then,*

$$\mathcal{M}^\chi(N, B) = \bigoplus_{i \in \mathbb{Z}/(p-1)\mathbb{Z}} \mathcal{M}^\chi(N, B)^i \quad (6.2)$$

where

$$\mathcal{M}^\chi(N, B)^i := \bigcup_{k \equiv i \pmod{p-1}} \mathcal{M}_k^\chi(N, B),$$

Proof. As with the case of $\mathcal{M}(N, B)$ in Chapter 5, the proposition follows from a result of Goren [Gor02, Theorem 5.4 in Chapter 4]. In [Gor02] this result is stated for forms for $\Gamma_1(N)$ and thus also forms with character. However, unlike the case for $\mathcal{M}(N, B)$ we may now have forms of odd weight. \square

It also turns out that each element of $\mathcal{M}^\chi(N, B)^i$ appears in a single weight. To see why this is true we need to define the notion of *even* and *odd* Dirichlet characters.

Definition 6.6. Let χ be a Dirichlet character mod N . If $\chi(-1) = 1$ we say that χ is *even* and if $\chi(-1) = -1$ we say that χ is *odd*.

Remark. Since $(\chi(-1))^2 = \chi((-1)^2) = \chi(1) = 1$ we always have $\chi(-1) = \pm 1$ as implied by the above definition.

Lemma 6.7. *Let χ be a Dirichlet character mod N and $f \in \mathcal{M}_k^\chi(N)$. If k and χ have different parity then $f = 0$.*

Proof. We only prove the lemma for k even and χ odd. The argument for k odd and χ is even is identical.

So, suppose that k is even and $\chi(-1) = -1$. By the definition of the slash operator we have

$$f|_k \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (-1)^k f = f.$$

However, since $f \in \mathcal{M}_k^\chi(N)$, we have

$$f|_k \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \chi(-1)f = -f.$$

Hence $f = -f$ so that $f = 0$ as required. \square

Proposition 6.8. *Let B be a domain of characteristic p and $i \in \mathbb{Z}/(p-1)\mathbb{Z}$. Then each element of $\mathcal{M}^\chi(N, B)^i$ appears in a single weight.*

Proof. We argue similarly to the case for trivial character in Chapter 5. So, suppose that for some $f \in \mathcal{M}^\chi(N, B)^i$ we have $f = f_1 + \cdots + f_n$ where each f_j is nonzero and appears in a different fixed weight. If $p \geq 5$ then we can multiply each f_j by a suitable power of E_{p-1} so that f appears in a single weight. Now assume that $p = 2$ or $p = 3$. By the previous lemma any two of the f_j 's differ in weight by a multiple of 2. And so, we can multiply each f_j by suitable powers of E_4 and E_6 so that f appears in a single weight. \square

6.2 Atkin-Lehner and Hecke operators on $\mathcal{M}_k^\chi(N, B)$

We now generalise the Atkin-Lehner and Hecke operators to allow us to extend the notions of U_ℓ -new and Tr_ℓ -new to modular forms with character. From here onwards, we always assume $N > 1$ and χ is a Dirichlet character mod N .

On $\mathcal{M}_k^\chi(N)$, the Atkin-Lehner operator is the same as our original definition (Definition 2.1). That is, if ℓ is a prime dividing N exactly once and $a, b \in \mathbb{Z}$ are such that $\ell b - a(N/\ell) = 1$, then $w_\ell(f) = f|_k \begin{pmatrix} \ell & a \\ N & \ell b \end{pmatrix}$ is certainly well-defined for any $f \in \mathcal{M}_k^\chi(N)$.

Proposition 6.9. *Let ℓ be a prime dividing N exactly once and assume that χ is induced by a Dirichlet character mod N/ℓ . Then for any $f \in \mathcal{M}_k^\chi(N)$ we also have $w_\ell(f) \in \mathcal{M}_k^\chi(N)$.*

Proof. Let a and b be integers satisfying $\ell b - a(N/\ell) = 1$ and let $\gamma_\ell = \begin{pmatrix} \ell & a \\ N & \ell b \end{pmatrix}$. Then, let $\delta = \begin{pmatrix} w & x \\ yN & z \end{pmatrix}$ be an arbitrary matrix in $\Gamma_0(N)$. Define

$$\begin{aligned} \delta' &:= \gamma_\ell \delta \gamma_\ell^{-1} = \begin{pmatrix} \ell & a \\ N & \ell b \end{pmatrix} \begin{pmatrix} w & x \\ yN & z \end{pmatrix} \frac{1}{\ell} \begin{pmatrix} \ell b & -a \\ -N & \ell \end{pmatrix} \\ &= \begin{pmatrix} \ell b w + b a y N - N x - a z (N/\ell) & -a w + a^2 y (N/\ell) + \ell x + a z \\ N w b + \ell b^2 y N - (N/\ell) N x - N b z & -a (N/\ell) w - a b y N + \ell N x + \ell b z \end{pmatrix} \in \Gamma_0(N). \end{aligned}$$

Since $\ell b = 1 + a(N/\ell)$ we have that the lower right entry of δ' is congruent to $z \pmod{N/\ell}$. Therefore, for any $f \in \mathcal{M}_k^\chi(N)$ we have

$$(w_\ell f)|_k \delta = f|_k \gamma_\ell \delta = f|_k \delta' \gamma = \chi(z) f|_k \gamma,$$

so that $w_\ell(f) \in \mathcal{M}_k^\chi(N)$ as required. \square

Recall that when χ is trivial, the Atkin-Lehner operator is $\mathbb{Z}[1/\ell]$ -integral. For general χ with conductor d , we instead have that w_ℓ is $\mathbb{Z}[1/N, \zeta_m]$ -integral where $m = \gcd(d, \ell)$ (see [Loe])⁽ⁱ⁾. Now, suppose that χ is induced by a Dirichlet character mod N/ℓ as above. Then by Proposition A.10, we have that $m = 1$ and thus w_ℓ is $\mathbb{Z}[1/N]$ -integral. So, if B is a $\mathbb{Z}[1/N, \chi]$ -algebra then w_ℓ extends to a linear operator on $\mathcal{M}_k^\chi(N, B)$. If B is also a domain then the scaled Atkin-Lehner operator $W_\ell = \ell^{k/2} w_\ell$ is well-defined on $\mathcal{M}^\chi(N, B) = \sum_{k=0}^\infty \mathcal{M}_k^\chi(N, B)$ (cf. Proposition 5.4).

We now define the Hecke operators on $\mathcal{M}_k^\chi(N)$.

Definition 6.10. Let p be a prime number. The Hecke operators U_p and T_p are defined on $\mathcal{M}_k^\chi(N)$ by *their effect on the q -expansion coefficients*:

$$\begin{aligned} a_n(T_p f) &= a_n(U_p f) = a_{np}(f), & \text{if } p \mid N, \\ a_n(T_p f) &= a_{pn}(f) + \chi(r) r^{k-1} a_{n/p}(f), & \text{if } p \nmid N. \end{aligned}$$

The reason for defining the Hecke operators as such is given in [DS05, Chapter 5.2]. Importantly, for any prime p and $f \in \mathcal{M}_k^\chi(N)$ we have $T_p(f) \in \mathcal{M}_k^\chi(N)$. Also note that if $k > 0$ or $p \mid N$ then T_p is $\mathbb{Z}[\chi]$ -integral and this definition extends to $\mathcal{M}_k^\chi(N, B)$ for any

⁽ⁱ⁾Based on the case for trivial character, one would expect that w_ℓ is actually $\mathbb{Z}[1/\ell, \zeta_m]$ -integral. However, the author of this thesis is unaware of any proof of such a result.

$\mathbb{Z}[\chi]$ -algebra B . Otherwise, if $k = 0$ and $p \nmid N$ then we also require that p is invertible in B .

Other properties of the Atkin-Lehner and Hecke operators also naturally generalise from the case of trivial character. In particular, if $f \in \mathcal{M}_k^\chi(N)$ and χ is induced by a character $\chi' \pmod{N/\ell}$, then

$$(i) \quad T_p U_\ell(f) = U_\ell T_p(f),$$

$$(ii) \quad T_p w_\ell(f) = w_\ell T_p(f),$$

$$(iii) \quad w_\ell^2(f) = \overline{\chi'}(\ell)f.$$

Moreover, if $f \in \mathcal{M}_k^\chi(N/\ell) \subseteq \mathcal{M}_k^\chi(N)$, then

$$(i') \quad W_\ell(f) = \ell^k i_\ell(f),$$

$$(ii') \quad U_\ell W_\ell(f) = \ell^k f,$$

$$(iii') \quad T_\ell(f) = U_\ell(f) + \chi(\ell)\ell^{-1}W_\ell(f).$$

Here, we are assuming that p and ℓ are primes such that $p \nmid N$ and ℓ divides N exactly once. The proofs of these facts are essentially identical to the case for trivial character.

6.3 Generalising U_ℓ -new

Assume ℓ is a prime dividing N exactly once. Recall that for trivial character, U_ℓ -new forms were defined by considering the U_ℓ -eigenvalues of ℓ -new forms. In particular, if $f \in \mathcal{M}_k(N)^{\ell\text{-new}}$ then

$$U_\ell^2(f) = \ell^{k-2}f.$$

Now, suppose that χ is induced by a Dirichlet character $\chi' \pmod{N/\ell}$. If $f \in \mathcal{M}_k^\chi(N)^{\ell\text{-new}}$ then by [Li75, Theorem 3]

$$U_\ell^2(f) = \chi'(\ell)\ell^{k-2}f.$$

In light of this result, we now generalise our definition of the \mathcal{D}_ℓ operator.

Definition 6.11. Let B be a $\mathbb{Z}[\chi]$ -algebra. If χ is induced by a Dirichlet character $\chi' \pmod{N/\ell}$ then the operator $\mathcal{D}_\ell^\chi : \mathcal{M}_k^\chi(N, B) \rightarrow \mathcal{M}_k^\chi(N, B)$ is given by

$$\mathcal{D}_\ell^\chi = \ell^2 U_\ell^2 - \chi'(\ell)\ell^k.$$

If B is also a domain then \mathcal{D}_ℓ^χ extends to an operator on $\mathcal{M}^\chi(N, B)$.

Definition 6.12. Let B be a $\mathbb{Z}[\chi]$ -algebra. If χ is induced by a Dirichlet character mod N/ℓ then for any Hecke-invariant submodule \mathcal{C} of $\mathcal{M}_k^\chi(N, B)$ we define

$$\mathcal{C}^{U_\ell^{\chi\text{-new}}} := \ker(\mathcal{D}_\ell^\chi|_{\mathcal{C}}).$$

If B is a $\mathbb{Z}[\chi]$ -domain then we can take \mathcal{C} to be any Hecke-invariant submodule of $\mathcal{M}^\chi(N, B) = \sum_{k=0}^{\infty} \mathcal{M}_k^\chi(N, B)$.

As with the case for trivial character in Chapter 4, this definition of U_ℓ^χ -new agrees with the standard notion of ℓ -new in characteristic zero.

Theorem 6.13. Let B be a $\mathbb{Z}[\chi]$ -domain of characteristic zero. If χ is induced by a Dirichlet character χ' mod N/ℓ then

$$\mathcal{S}^\chi(N, B)^{\ell\text{-new}} = \mathcal{S}^\chi(N, B)^{U_\ell^{\chi\text{-new}}}.$$

Proof. Since $\mathcal{M}^\chi(N, B) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k^\chi(N, B)$ and \mathcal{D}_ℓ^χ is weight-preserving, it suffices to prove the theorem in fixed weight k . Moreover, since B is flat over $\mathbb{Z}[\chi]$ it suffices to prove the theorem for $B = \mathbb{Z}[\chi]$ or more simply, $B = \mathbb{C}$.

One can then repeat the argument from the proof of Theorem 3.9 and its preceding lemmas whilst keeping track of character. In particular, note that the Weil bound [Del74, Theorem 8.2] holds for modular forms with character. Moreover, for any eigenform $g \in \mathcal{S}_k^\chi(N/\ell)$, the characteristic polynomial of U_ℓ^2 on $\text{span}\{g, i_\ell g\}$ is given by

$$P_{\ell,g}(X) = X^2 - (a_\ell^2(g) - 2\ell^{k-1}\chi'(\ell))X + \ell^{2k-2}(\chi'(\ell))^2. \quad \square$$

Unfortunately, there does not appear to be a natural way to extend our definition of U_ℓ^χ -new to the case when ℓ divides N more than once. To see this, we will return to modular forms for $\Gamma_0(N)$ (or equivalently, modular forms with trivial character).

Let ℓ be a prime such that $\ell^2 \mid N$. By Corollary 3.4 we have $U_\ell(f) = 0$ for all eigenforms $f \in \mathcal{S}_k(N)^{\ell\text{-new}}$. In light of this fact, one may wish to define $\mathcal{S}_k(N)^{U_\ell\text{-new}}$ as the kernel of U_ℓ (restricted to $\mathcal{S}_k(N)$). Certainly $\mathcal{S}_k(N)^{\ell\text{-new}} \subseteq \ker(U_\ell)$ but the reverse inclusion may not hold.

Example 6.14. The space $\mathcal{S}_6(4)^{\text{new}}$ is 1-dimensional and spanned by

$$f = q - 12q^3 + 54q^5 - 88q^7 - 99q^9 + O(q^{11}).$$

Since $f \in \mathcal{S}_6(4)^{\text{new}}$ it follows that $f \in \ker(U_2)$ (Theorem 2.16). However, we also have $f \in \mathcal{S}_6(8)^{2\text{-old}}$ and thus $f \notin \mathcal{S}_6(8)^{2\text{-new}}$. In other words, $\mathcal{S}_6(8)^{2\text{-new}} \neq \ker(U_2)$.

6.4 Generalising Tr_ℓ -new

In order to generalise our notion of Tr_ℓ -new forms, we need to define a trace operator on $\mathcal{M}_k^\chi(N)$.

Definition 6.15. Let ℓ be a prime dividing N . The trace operator on $\mathcal{M}_k^\chi(N)$ is given by:

$$\begin{aligned} \text{Tr}_\ell: \mathcal{M}_k^\chi(N) &\rightarrow \mathcal{M}_k^\chi(N) \\ f &\mapsto \sum_{\gamma \in \Gamma_0(N) \backslash \Gamma_0(N/\ell)} \bar{\chi}(\gamma) f|_k \gamma, \end{aligned}$$

where $\bar{\chi}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \bar{\chi}(d)$.

The factor of $\bar{\chi}(\gamma)$ is included to ensure that the trace operator is independent of the choice of coset representatives in $\Gamma_0(N) \backslash \Gamma_0(N/\ell)$. Moreover, if χ is induced by a character χ' mod N/ℓ then Tr_ℓ maps $\mathcal{M}_k^\chi(N)$ to $\mathcal{M}_k^{\chi'}(N/\ell)$. To see this, one can generalise the argument in the proof of Proposition B.3 or refer to [CS17, Corollary 10.2.11].

For the rest of this section we assume that ℓ is a prime dividing N exactly once. In this case, $\text{Tr}_\ell(f)$ is again explicitly given by

$$\text{Tr}_\ell(f) = f + \ell^{1-k} U_\ell W_\ell f, \tag{6.3}$$

which is well-defined on $\mathcal{M}^\chi(N, B)$ for any $\mathbb{Z}[1/N, \chi]$ -domain B . Moreover, if χ is induced by a character χ' mod N/ℓ then for any $f \in \mathcal{M}_k^{\chi'}(N/\ell)$ we have

- (i) $\text{Tr}_\ell(f) = (\ell + 1)f$, and
- (ii) $\text{Tr}_\ell(W_\ell f) = \bar{\chi}'(\ell) \ell T_\ell(f)$

(cf. Lemma 3.13).

Definition 6.16. Let B be a $\mathbb{Z}[1/N, \chi]$ -algebra. If χ is induced by a Dirichlet character $\chi' \pmod{N/\ell}$ then for any Hecke-invariant submodule \mathcal{C} of $\mathcal{M}_k^\chi(N, B)$ we define

$$\mathcal{C}^{\mathrm{Tr}_\ell^\chi - \text{new}} = \ker(\mathrm{Tr}_\ell|_{\mathcal{C}}) \cap \ker(\mathrm{Tr}_\ell W_\ell|_{\mathcal{C}}).$$

If B is a $\mathbb{Z}[1/N, \chi]$ -domain then we can take \mathcal{C} to be any Hecke-invariant submodule of $\mathcal{M}^\chi(N, B) = \sum_{k=0}^\infty \mathcal{M}_k^\chi(N, B)$.

Again, we verify that this definition agrees with the standard notion of ℓ -new in characteristic zero.

Theorem 6.17. *Let B be a $\mathbb{Z}[1/N, \chi]$ -domain of characteristic zero. If χ is induced by a Dirichlet character $\pmod{N/\ell}$ then*

$$\mathcal{S}^\chi(N, B)^{\ell - \text{new}} = \mathcal{S}^\chi(N, B)^{\mathrm{Tr}_\ell^\chi - \text{new}}.$$

Proof. Since $\mathcal{M}^\chi(N, B) = \bigoplus_{k=0}^\infty \mathcal{M}_k^\chi(N, B)$ and Tr_ℓ is weight-preserving it suffices to prove the theorem in fixed weight k . As in the proof of Proposition 6.4 we note that $\mathbb{Z}[\chi] = \mathbb{Z}[\zeta_n]$ for some positive integer n . It then follows that $\mathbb{Z}[1/N, \chi] = \mathbb{Z}[1/N, \zeta_n]$. And so, B is flat over $\mathbb{Z}[1/N, \chi]$ by Corollary A.4. As a result, it suffices to prove the theorem for $B = \mathbb{Z}[1/N, \chi]$ and thus $B = \mathbb{C}$. As in Theorem 6.13, we also assume for non-triviality that χ is induced by a character $\chi' \pmod{N/\ell}$.

First we show that $\mathcal{S}_k^\chi(N, \mathbb{C})^{\ell - \text{new}} \subseteq \mathcal{S}_k^\chi(N, \mathbb{C})^{\mathrm{Tr}_\ell^\chi - \text{new}}$. So, let f be an eigenform in $\mathcal{S}_k^\chi(N, \mathbb{C})^{\ell - \text{new}}$. For any $p \nmid N$ we have that T_p commutes with U_ℓ and W_ℓ . Hence, T_p also commutes with Tr_ℓ and $\mathrm{Tr}_\ell W_\ell$. Suppose for a contradiction that $\mathrm{Tr}_\ell(f) \neq 0$ or $\mathrm{Tr}_\ell W_\ell(f) \neq 0$. Then $\mathrm{Tr}_\ell(f)$ (or $\mathrm{Tr}_\ell W_\ell(f)$) is an eigenform in $\mathcal{S}_k^\chi(N/\ell)$ with the same Hecke eigenvalues away from ℓ . This is impossible by [Li75, Theorem 5] so $f \in \ker(\mathrm{Tr}_\ell) \cap \ker(\mathrm{Tr}_\ell W_\ell)$ as required. To show that $\mathcal{S}_k^\chi(N, \mathbb{C})^{\mathrm{Tr}_\ell^\chi - \text{new}} \subseteq \mathcal{S}_k^\chi(N, \mathbb{C})^{\ell - \text{new}}$ one can repeat the argument in the proof of Theorem 3.14 whilst keeping track of character. \square

Combined with Theorem 6.13, the above result tells us that the notions of U_ℓ^χ -new and Tr_ℓ^χ -new agree in characteristic zero. The same is true in characteristic p .

Theorem 6.18. *Let B be a $\mathbb{Z}[1/N, \chi]$ -domain of characteristic p . If χ is induced by a Dirichlet character $\chi' \pmod{N/\ell}$ then*

$$\mathcal{S}^\chi(N, B)^{U_\ell^\chi - \text{new}} = \mathcal{S}^\chi(N, B)^{\mathrm{Tr}_\ell^\chi - \text{new}}.$$

Proof. First we note that a slight modification of the proof of Proposition 5.8 gives

$$\begin{aligned} (\mathcal{S}_k^\chi(N, B)^{\ell\text{-old}})^{U_\ell^\chi\text{-new}} &= (\mathcal{S}_k^\chi(N, B)^{\ell\text{-old}})^{\text{Tr}_\ell^\chi\text{-new}} \\ &= \{f + W_\ell(g) : \ell T_\ell f = -(\ell + 1)S_\ell g \text{ and } \ell T_\ell g = -(\ell + 1)\chi'(\ell)f\}. \end{aligned}$$

Here, k is some fixed weight, $S_\ell = \ell^k$, and we can take B to be any $\mathbb{Z}[1/N, \chi]$ -algebra⁽ⁱⁱ⁾.

To finish the proof, one can use the same argument as in Theorem 5.9. The main difference here is that we have to work with the ring extensions $\mathbb{F}_p[\chi]$, $\mathbb{Z}_p[\chi]$ and $\mathbb{Q}_p[\chi]$. \square

These results indicate that the notions of $U_\ell^\chi\text{-new}$ and $\text{Tr}_\ell^\chi\text{-new}$ are very natural generalizations of $U_\ell\text{-new}$ and $\text{Tr}_\ell\text{-new}$.

Now, at first glance, the notion of $U_\ell^\chi\text{-new}$ seems to be more useful than $\text{Tr}_\ell^\chi\text{-new}$. In particular, the definition of $U_\ell^\chi\text{-new}$ forms is arguably simpler and is defined over a larger class of commutative rings. However, as we saw in section 6.3, there does not seem to be a way to extend the idea of $U_\ell^\chi\text{-new}$ forms to the case where ℓ divides N more than once. On the other hand, it is actually possible to modify the definition of $\text{Tr}_\ell^\chi\text{-new}$ to such situations. This is the focus of the next section.

6.5 Beyond Squarefree Levels

In the previous section we defined a notion of $\text{Tr}_\ell^\chi\text{-new}$ for each Dirichlet character $\chi \pmod{N}$. However, since this definition relied on the Atkin-Lehner operator, we needed to assume that ℓ divided N exactly once. If one were to keep this assumption for all primes dividing N , then we are left with the requirement that N is squarefree.

To overcome this restriction, we suggest ^amodified definition of $\text{Tr}_\ell^\chi\text{-new}$ that no longer relies on the Atkin-Lehner operator. In particular, we take inspiration from [Li75, Theorem 4] by making use of the "Fricke" operator.

Definition 6.19. Let $\gamma_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in \text{GL}_2(\mathbb{Q})^+$. The *Fricke operator* h_N is the map that sends $f \in \mathcal{M}_k^\chi(N)$ to $f|_k \gamma_N$. We also write H_N for the scaling of the Fricke operator given by $H_N = N^{k/2} h_N$.

⁽ⁱⁱ⁾Note that if $f + W_\ell(g) \in (\mathcal{S}_k^\chi(N, B)^{\ell\text{-old}})^{U_\ell^\chi\text{-new}} = (\mathcal{S}_k^\chi(N, B)^{\ell\text{-old}})^{\text{Tr}_\ell^\chi\text{-new}}$, then $T_\ell^2(f) = T_\ell^2(g) = \chi'(\ell)\ell^{k-2}(\ell+1)^2$. These T_ℓ^2 eigenvalues are precisely those that appear in Diamond's paper on congruences between ℓ -old and ℓ -new forms [Dia91].

We now list some basic properties of the Fricke operator.

- (i) If $f \in \mathcal{M}_k^\chi(N)$ then $h_N(f)$ maps $\mathcal{M}_k^\chi(N)$ to $\mathcal{M}_k^{\bar{\chi}}(N)$.
- (ii) We have $h_N^2 = (-1)^k$.
- (iii) If f is a level 1 modular form, then $H_N(f(z)) = N^k f(Nz)$.
- (iv) If d is the conductor of χ then h_N and H_N extend to linear operators on $\mathcal{M}_k^\chi(N, B)$ for any $\mathbb{Z}[1/N, \zeta_d, \chi]$ -algebra B . If B is also a domain then H_N is well-defined on $\mathcal{M}^\chi(N, B)$.

The proof of the final property is discussed in [Loe] where the Fricke operator is denoted $w_{N,k}$. The other properties are analogous to those of the Atkin-Lehner operator and can be proven in a similar fashion.

Now, if ℓ divides N exactly once then the trace operator is given explicitly as in (6.3). However, if ℓ divides N more than once (i.e. $\ell^2 \mid N$), then for any $f \in \mathcal{M}_k^\chi(N)$,

$$\mathrm{Tr}_\ell(f) = \chi(-1)\ell N^{-k} H_{N/\ell} U_\ell H_N f.$$

This formula (although with slightly different notation and scaling) is derived in Weisinger's thesis [Wei77, Proposition 16]. So, if χ is a Dirichlet character with conductor d , then Tr_ℓ also extends to $\mathcal{M}^\chi(N, B)$ for any $\mathbb{Z}[1/N, \zeta_d, \chi]$ -domain B .

We now define another notion ^{of} newness. From here onwards we will no longer assume that ℓ divides N exactly once. That is, we allow ℓ ^{to} be **any** prime dividing N .

Definition 6.20. Let d denote the conductor of χ and B be a $\mathbb{Z}[1/N, \zeta_d, \chi]$ -algebra. If χ is induced by a Dirichlet character $\chi' \bmod N/\ell$ then for any Hecke-invariant submodule \mathcal{C} of $\mathcal{M}_k^\chi(N, B)$ we define

$$\mathcal{C}^{\mathrm{Tr}_\ell^\chi - \text{new}'} = \ker(\mathrm{Tr}_\ell|_{\mathcal{C}}) \cap \ker(\mathrm{Tr}_\ell H_N|_{\mathcal{C}}).$$

If B is a $\mathbb{Z}[1/N, \zeta_d, \chi]$ -domain then we can take \mathcal{C} to be any Hecke-invariant submodule of $\mathcal{M}^\chi(N, B) = \sum_{k=0}^\infty \mathcal{M}_k^\chi(N, B)$.

Theorem 6.21. Let d denote the conductor of χ and let B be a $\mathbb{Z}[1/N, \zeta_d, \chi]$ -domain of characteristic zero. If χ is induced by a Dirichlet character mod N/ℓ then

$$\mathcal{S}^\chi(N, B)^{\ell\text{-new}} = \mathcal{S}^\chi(N, B)^{\text{Tr}_\ell^\chi\text{-new}'},$$

and if ℓ divides N exactly once,

$$\mathcal{S}^\chi(N, B)^{\text{Tr}_\ell^\chi\text{-new}'} = \mathcal{S}^\chi(N, B)^{\text{Tr}_\ell^\chi\text{-new}} = \mathcal{S}^\chi(N, B)^{U_\ell\text{-new}}.$$

Proof. Arguing as in the proof of Theorem 6.17, it suffices to prove the theorem in fixed weight k and for $B = \mathbb{C}$. However, when $B = \mathbb{C}$, the theorem is well-known and a proof can be found in [Li75, Theorem 4] or [Lan76, Theorem 2.2 of Chapter 8].

If ℓ divides N exactly once, then the second statement in the theorem follows from Theorem 6.13 and Theorem 6.17. \square

Remark. Weisinger [Wei77, Proposition 19] gives a similar result for Eisenstein series in $\mathcal{M}_k^\chi(N, \mathbb{C})$.

We can also hope that our different notions of newness agree when B has characteristic p .

Question 6.22. If B is $\mathbb{Z}[1/N, \zeta_d, \chi]$ -domain of characteristic p do we also have

$$\mathcal{S}^\chi(N, B)^{\text{Tr}_\ell^\chi\text{-new}'} \stackrel{?}{=} \mathcal{S}^\chi(N, B)^{\text{Tr}_\ell^\chi\text{-new}} \stackrel{?}{=} \mathcal{S}^\chi(N, B)^{U_\ell\text{-new}}$$

I would make it visually clear that the equalities are not (claimed to be) known):

whenever ℓ divides N exactly once? Considering the proof of Theorem 6.18, it would suffice to show that

$$(\mathcal{S}_k^\chi(N, B)^{\ell\text{-old}})^{\text{Tr}_\ell^\chi\text{-new}'} \stackrel{?}{=} \{f + W_\ell(g) : \ell T_\ell f = -(\ell + 1)S_\ell g \text{ and } \ell T_\ell g = -(\ell + 1)\chi'(\ell)f\}.$$

A positive answer to this question would give further support for using the notion of $\text{Tr}_\ell^\chi\text{-new}'$.

As a final remark, we note that besides the Fricke operator, there are certainly other operators used in literature that can generalise or replace the Atkin-Lehner operator. For instance, there's Li's V_q^M operator [Li75, pp 287-288] or Weisinger's partial W -operators [Wei77, page 30]. Each of these operators can be used to define notions of newness. Thus, a direction

for further research would be to study these other notions of newness and see how they compare to our use of the Fricke operator here.

Appendices

APPENDIX A COMMUTATIVE ALGEBRA

A.1 Flat Modules

Flat modules are a widely studied class of modules that are well-behaved with respect to the tensor product. They are defined as follows.

Definition A.1. Let A be a commutative ring. An A -module M is *flat* if it satisfies any of the following equivalent conditions:

- (1) If $f: N \rightarrow N'$ is an injective morphism of A -modules then $f \otimes 1: N \otimes M \rightarrow N' \otimes M$ is also injective.
- (2) If $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ is a short exact sequence of A -modules then the sequence $0 \rightarrow N' \otimes M \rightarrow N \otimes M \rightarrow N'' \otimes M \rightarrow 0$ is also exact.
- (3) If \mathcal{S} is any exact sequence of A -modules then $\mathcal{S} \otimes M$ is also exact.

The equivalence of (1) and (2) follows from the right-exactness of the tensor product [AM69, Proposition 2.18]. And then, (2) and (3) are equivalent since any exact sequence can be constructed from short exact sequences.

Example A.2.

- A commutative ring A is flat as a module over itself since $M \otimes A \cong M$ for any A -module M . More generally, we have that any free A -module is flat.
- The \mathbb{Z} -module $\mathbb{Z}/2\mathbb{Z}$ is **not** flat. To see this, consider the injective map

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z},$$

that sends an integer x to $2x$. Tensoring with $\mathbb{Z}/2\mathbb{Z}$ we then obtain

$$\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2 \otimes 1} \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$$

which maps everything to zero and is therefore not injective.

If A is a Dedekind domain⁽ⁱ⁾, then a useful description of flat A -modules is well-known.

Proposition A.3. *Let A be a Dedekind domain. An A -module M is flat if and only if M is torsion-free.*

Proof. See [FS01, VI Theorem 9.10] for a more general proof for Prüfer domains (which include Dedekind domains). □

For our purposes, we are interested in the Dedekind domains \mathbb{Z} , $\mathbb{Z}[1/m]$, $\mathbb{Z}[\zeta_n]$ and $\mathbb{Z}[1/m, \zeta_n]$ where m and n are positive integers and ζ_n is a primitive n^{th} root of unity.

Corollary A.4. *Let $A = \mathbb{Z}$, $\mathbb{Z}[1/m]$, $\mathbb{Z}[\zeta_m]$ or $\mathbb{Z}[1/m, \zeta_n]$. If B is an A -domain of characteristic zero, then B is flat (as an A -module).*

Proof. Apply the previous proposition noting that any such B is torsion-free as an A -module. □

Finally, we show how flat modules "preserve" kernels.

Proposition A.5. *Let A be a commutative ring and $f: N \rightarrow N'$ be an A -module homomorphism. If M is a flat A -module then $\ker(f \otimes 1_M) \cong \ker(f) \otimes M$.*

Proof. Consider the exact sequence

$$0 \rightarrow \ker(f) \rightarrow N \xrightarrow{f} N' \rightarrow N'/\text{im}(f) \rightarrow 0.$$

Tensoring with M then gives the exact sequence

$$0 \rightarrow \ker(f) \otimes M \rightarrow N \otimes M \xrightarrow{f \otimes 1_M} N' \otimes M \rightarrow N'/\text{im}(f) \otimes M \rightarrow 0.$$

The exactness at $N \otimes M$ and injectivity of $\ker(f) \otimes M \rightarrow N \otimes M$ implies that $\ker(f \otimes 1_M) \cong \ker(f) \otimes M$ as required. □

A.2 The p -adic Numbers

The p -adic numbers are an increasingly important tool in modern mathematics that provide a link between the worlds of characteristic 0 and characteristic p . We begin by defining the p -adic integers; a subset of the p -adic numbers.

⁽ⁱ⁾A Dedekind domain is an integrally closed, Noetherian domain such that every nonzero prime ideal is maximal. Examples include principal ideal domains and rings of integers of algebraic number fields. For more information, see [AM69, Chapter 9].

Definition A.6. Let p be a prime number. A p -adic integer x is a formal power series of the form $x = \sum_{n=0}^{\infty} a_n p^n$ where $a_n \in \{0, 1, \dots, p-1\}$. The set of p -adic integers is denoted \mathbb{Z}_p .

Alternatively, we could more precisely define $\mathbb{Z}_p := \mathbb{Z}[[X]]/(X - p)$. Using the ring structure of $\mathbb{Z}[[X]]$, the p -adic integers form an integral domain of characteristic 0. Unlike the usual integers \mathbb{Z} , we are able to invert any element $x \in \mathbb{Z}_p$ provided p does not divide x (i.e. $a_0 \neq 0$). For instance, if $p = 3$ then

$$\frac{1}{2} = 2 + 2p + 2p^2 + 2p^3 + \dots$$

To see this we note that $2(2 + 2(3) + 2(3)^2 + 2(3)^3 + \dots) = 1 + 0p + 0p^2 + \dots = 1$.

If we now allow p to be invertible, we obtain the p -adic numbers.

Definition A.7. Let p be a prime number. A p -adic number x is a formal Laurent series of the form $x = \sum_{n=-m}^{\infty} a_n p^n$ for some $m \in \mathbb{Z}$. The set of p -adic numbers is denoted \mathbb{Q}_p .

Since we can now divide by p , the p -adic numbers \mathbb{Q}_p form a field. For our purposes, studying modular forms over \mathbb{Z}_p or \mathbb{Q}_p can provide insight into the theory of modular forms over \mathbb{F}_p .

A.3 Dirichlet Characters

Dirichlet characters are arithmetic functions frequently studied in analytic number theory. They are used when classifying modular forms for $\Gamma_1(N)$.

Definition A.8. Let N be a positive integer and $\chi: (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be a group homomorphism. We can extend χ to a function on all of \mathbb{Z} by setting

$$\chi(a) = \begin{cases} \chi(a \bmod N), & \text{if } \gcd(a, N) = 1 \\ 0, & \text{if } \gcd(a, N) > 1. \end{cases}$$

Such a function $\chi: \mathbb{Z} \rightarrow \mathbb{C}$ is called a *Dirichlet character mod N* .

For any Dirichlet character χ we have the following properties:

- $\chi(1) = 1$ since $\chi: (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ is a group homomorphism.

- $|\chi(a)| = 1$ if $\gcd(a, N) = 1$. To see this we let $\varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$ and then note that $a^{\varphi(N)} \equiv 1$ by Euler's theorem. Hence,

$$\chi(a)^{\varphi(N)} = \chi(a^{\varphi(N)}) = \chi(1) = 1,$$

so that $\chi(a)$ is a $\varphi(N)^{\text{th}}$ root of unity and $|\chi(a)| = 1$ as required.

- There are exactly $\varphi(N)$ Dirichlet characters mod N . A simple proof of this result can be found in [Sut15, Proposition 17.5].

Example A.9.

- (1) For any positive integer N we have the Dirichlet character

$$\chi_0(a) = \begin{cases} 1, & \text{if } \gcd(a, N) = 1 \\ 0, & \text{if } \gcd(a, N) > 1. \end{cases}$$

called the *trivial Dirichlet character* mod N .

- (2) There are two Dirichlet characters mod 4. The first is the trivial character from the previous example. The second is given by

$$\chi(a) = \begin{cases} 0, & \text{if } a \equiv 0, 2 \pmod{4} \\ 1, & \text{if } a \equiv 1 \pmod{4} \\ -1, & \text{if } a \equiv 3 \pmod{4} \end{cases}$$

- (3) Let M and N be positive integers such that M divides N . If χ is a Dirichlet character mod M then we can define a Dirichlet character χ' mod N by setting $\chi'(a) = \chi(a)$ whenever $\gcd(a, N) = 1$. In this situation, we say that χ' is *induced* by χ . The smallest value of M such that χ is induced by a character mod M is called the *conductor* of χ .

We now prove a couple of important results regarding Dirichlet characters that are used in Chapter 6 of this thesis.

Proposition A.10. *Let χ be a Dirichlet character mod N that is induced by a Dirichlet character mod M . Then, the conductor of χ divides M .*

Proof. We use an equivalent definition of induced characters shown in [Sut15, Lemma 17.20]. Namely, for any $N' \mid N$, we have that χ is induced by a character mod N' if and only if χ is

constant on all residue classes in $(\mathbb{Z}/N\mathbb{Z})^\times$ that are congruent mod N' .

Let d denote the conductor of χ . Since χ is induced by Dirichlet characters mod d and mod M , we have that χ is constant on all residue classes in $(\mathbb{Z}/N\mathbb{Z})^\times$ that are congruent mod $\gcd(d, M)$. That is, χ is induced by a character mod $\gcd(d, M)$. However, since d is the conductor of χ we must have $d \leq \gcd(M, d)$. This is only possible if $d = \gcd(M, d)$ or equivalently $d \mid M$. \square

Proposition A.11. *Let χ and ψ be two Dirichlet characters mod N . Then*

$$\sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(a)\psi(a) = \begin{cases} \varphi(n), & \text{if } \chi = \psi \\ 0, & \text{otherwise} \end{cases}$$

Proof. First we show that

$$\sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) = \begin{cases} \varphi(n), & \text{if } \chi = \chi_0 \\ 0, & \text{otherwise} \end{cases} \quad (\text{A.1})$$

If $\chi = \chi_0$ then (A.1) follows since $\chi_0(a) = 1$ for all $a \in (\mathbb{Z}/N\mathbb{Z})^\times$. Now suppose that $\chi \neq \chi_0$. Then there exists $b \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $\chi(b) \neq 0$. Since $(\mathbb{Z}/N\mathbb{Z})^\times$ is a group, ba runs over all the elements of $(\mathbb{Z}/N\mathbb{Z})^\times$ as a does. So, setting $S_\chi = \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)$ we have

$$\chi(b)S_\chi = \chi(b) \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) = \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(ba) = S_\chi.$$

Hence $S_\chi = 0$ as required.

Replacing χ with $\bar{\chi}\psi$ in (A.1) gives

$$\sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(a)\psi(a) = \begin{cases} \varphi(n), & \text{if } \bar{\chi}\psi = \chi_0, \\ 0, & \text{otherwise.} \end{cases}$$

Now, for any $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ we have $|\chi(a)| = 1$ and thus $\bar{\chi}(a) = \chi^{-1}(a)$. Hence $\bar{\chi}\psi = \chi_0$ is equivalent to $\chi = \psi$. \square

APPENDIX B
ASSORTED PROOFS

Proposition B.1. *Let $N > 1$ and $k \geq 0$. Then,*

$$\mathcal{M}_k(N)^{\text{old}} = \sum_{\substack{\ell \text{ prime} \\ \ell | N}} \mathcal{M}_k(N)^{\ell\text{-old}}.$$

Proof. We note that one inclusion is immediate, namely

$$\sum_{\substack{\ell \text{ prime} \\ \ell | N}} \mathcal{M}_k(N)^{\ell\text{-old}} \subseteq \mathcal{M}_k(N)^{\text{old}}.$$

For the other inclusion, we show that for each $M | N$ (with $M \neq N$) and $e | (N/M)$ there exists a prime ℓ such that $\ell | N$ and $i_e(\mathcal{M}_k(M)) \subseteq \mathcal{M}_k(N)^{\ell\text{-old}}$. So, let M be a proper divisor of N and $e | (N/M)$. We consider two cases.

Case 1: $e = 1$

In this case, take ℓ to be a prime such that $M | (N/\ell)$. We can do this since M is a proper divisor of N . Now,

$$i_1(\mathcal{M}_k(M)) = \mathcal{M}_k(M) \subseteq \mathcal{M}_k(N/\ell) \subseteq \mathcal{M}_k(N/\ell) + i_\ell(\mathcal{M}_k(N/\ell)) = \mathcal{M}_k(N)^{\ell\text{-old}},$$

as required.

Case 2: $e > 1$

In this case, we choose ℓ to be such that $\ell | e$. Since $e | (N/M)$ this means that in particular, $\ell | N$ and $M | (N/\ell)$. Now, let g be an arbitrary element of $i_e(\mathcal{M}_k(M))$. In particular, $g(z) := f(ez)$ for some form $f \in \mathcal{M}_k(M)$. We wish to show that $g \in \mathcal{M}_k(N)^{\ell\text{-old}}$.

So, let $h(z) := f(\frac{e}{\ell}z)$. Note that $h \in \mathcal{M}_k(N/\ell)$ as it is the image of f under the map $i_{e/\ell} : \mathcal{M}_k(M) \rightarrow \mathcal{M}_k(N/\ell)$. We then have $g = i_\ell(h) \in \mathcal{M}_k(N)^{\ell\text{-old}}$ as required. \square

Proposition B.2. *For any nonnegative integer N , the algebra $\mathcal{M}(N) = \sum_{k=0}^{\infty} \mathcal{M}_k(N)$ is a direct sum, i.e. $\mathcal{M}(N) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(N)$.*

Proof. The following proof is adapted from [Miy06, Lemma 2.1.1]. Let $n \in \mathbb{Z}_{\geq 0}$ and $\{f_0, \dots, f_n\}$ be a set of functions with $f_k \in \mathcal{M}_k(N)$. Now suppose that $\sum_{k=0}^n f_k = 0$. We want to show that this implies $f_k = 0$ for all $k \in \{0, \dots, n\}$. First we construct a sequence of matrices $\{\gamma_m\}_{m=0}^\infty$ in $\Gamma_0(N)$ given by

$$\gamma_m = \begin{pmatrix} mN + 1 & 1 \\ mN & 1 \end{pmatrix} \in \Gamma_0(N),$$

so that $j(\gamma_m, z) = mNz + 1$. This means that provided $z \neq 0$ we have $j(\gamma_m, z) \neq j(\gamma_{m'}, z)$ for $m' \neq m$. If $z = 0$ we could alternatively define $\gamma_m = \begin{pmatrix} 1 & \\ mN & mN+1 \end{pmatrix}$. As each f_k is in $\mathcal{M}_k(N)$ we then have for any $m \in \mathbb{Z}_{\geq 0}$ and $z \in \mathbb{H}$

$$\sum_{k=0}^n j(\gamma_m, z)^k f_k(z) = \sum_{k=0}^n f_k(\gamma_m z) = 0. \quad (\text{B.1})$$

Varying m from 0 to n in (B.1) leads to a system of linear equations that can be written in matrix form as

$$\begin{pmatrix} 1 & j(\gamma_0, z) & j(\gamma_0, z)^2 & \dots & j(\gamma_0, z)^n \\ 1 & j(\gamma_1, z) & j(\gamma_1, z)^2 & \dots & j(\gamma_1, z)^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & j(\gamma_n, z) & j(\gamma_n, z)^2 & \dots & j(\gamma_n, z)^n \end{pmatrix} \begin{pmatrix} f_0(z) \\ f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (\text{B.2})$$

If we let M denote the $(n+1) \times (n+1)$ matrix in (B.2) then

$$\det(M) = \prod_{1 \leq m' < m \leq n} (j(\gamma_{m'}, z) - j(\gamma_m, z)) \neq 0,$$

noting that M is a Vandermonde matrix. Since this holds for all $z \in \mathbb{H}$ we have $f_k = 0$ for all $k \in \{0, \dots, n\}$ as required. \square

Remark. More generally we have that $\mathcal{M}^\chi(N) = \bigoplus_{k=0}^\infty \mathcal{M}_k(N)$ for any Dirichlet character $\chi \pmod{N}$. The proof of this fact is almost identical to the one above. In particular, note that the lower right entry of each γ_m is congruent to 1 mod N so that if $f_k \in \mathcal{M}_k^\chi(N)$ then $f_k(\gamma_m z) = \chi(1)j(\gamma_m, z)^k f_k(z) = j(\gamma_m, z)^k f_k(z)$.

Proposition B.3. *Let M, N be integers such that $M|N$ and $R = \{R_i\}_{i=1}^n$ be a set of right coset representatives for $\Gamma_0(N) \backslash \Gamma_0(M)$. Let $f \in \mathcal{M}_k(N)$ and $F = \sum_{i=1}^n f|_k R_i$. Then $F \in \mathcal{M}_k(M)$.*

Proof. Let $\delta \in \Gamma_0(M)$. Note that right multiplication by δ permutes the coset representatives of $\Gamma_0(N) \backslash \Gamma_0(M)$. That is, $R_i \delta = \gamma_i R_{\sigma(i)}$ where $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is a bijection. We then have

$$F|\delta = \sum_{i=1}^n f|_k R_i|_k \delta = \sum_{i=1}^n f|_k R_i \delta = \sum_{i=1}^n f|_k \gamma_i R_{\sigma(i)} = \sum_{i=1}^n f|_k R_{\sigma(i)} = F,$$

so that $F \in \mathcal{M}_k(M)$ as required. \square

Remark. The above proposition also holds if replace $\Gamma_0(M)$ and $\Gamma_0(N)$ with arbitrary congruence subgroups Γ and Γ' satisfying $\Gamma \subseteq \Gamma'$.

Proposition B.4. *Let $k > 2$ and $N > 1$. We then have*

$$\mathcal{E}_k(N) = \mathcal{E}_k(N)^{\ell\text{-old}} = \mathcal{E}_k(N/\ell) \oplus W_\ell(\mathcal{E}_k(N/\ell))$$

whenever ℓ is a prime dividing N exactly once.

Proof. We proceed by a dimension counting argument. First note that W_ℓ is a \mathbb{C} -vector space isomorphism (with inverse $W_\ell^{-1} = \ell^{-k} W_\ell$) and thus $\dim_{\mathbb{C}}(\mathcal{E}_k(N/\ell)) = \dim_{\mathbb{C}}(W_\ell(\mathcal{E}_k(N/\ell)))$. It therefore suffices to prove that $\dim_{\mathbb{C}}(\mathcal{E}_k(N)) = 2 \dim_{\mathbb{C}}(\mathcal{E}_k(N/\ell))$.

Fortunately, there is an explicit formula for the dimension of $\mathcal{E}_k(N)$ [Ste07, Proposition 6.1]:

$$\dim_{\mathbb{C}}(\mathcal{E}_k(N)) = \sum_{d|N} \phi \left(\gcd \left(d, \frac{N}{d} \right) \right),$$

where ϕ is the Euler totient function. Hence,

$$\begin{aligned} \dim(\mathcal{E}_k(N)) &= \sum_{d|N} \phi \left(\gcd \left(d, \frac{N}{d} \right) \right) \\ &= \sum_{d|(N/\ell)} \phi \left(\gcd \left(d, \frac{N}{d} \right) \right) + \sum_{\substack{d=\ell e \\ e|(N/\ell)}} \phi \left(\gcd \left(d, \frac{N}{d} \right) \right) \quad (\text{Since } \ell \nmid N/\ell.) \\ &= \sum_{d|(N/\ell)} \phi \left(\gcd \left(d, \frac{(N/\ell)}{d} \right) \right) + \sum_{e|(N/\ell)} \phi \left(\gcd \left(e, \frac{(N/\ell)}{e} \right) \right) \\ &= 2 \dim(\mathcal{E}_k(N/\ell)), \end{aligned}$$

as required. □

Remark. The above proposition can also be proven using explicit bases for $\mathcal{E}_k(N)$ and $\mathcal{E}_k(N/\ell)$ as described in [DS05, Section 4.5].

Proposition B.5. *Let p be a prime number. Then $pB_{p-1} \equiv -1 \pmod{p}$, where B_{p-1} is the $(p-1)^{\text{th}}$ Bernoulli number. As a result, the denominator of B_{p-1} is divisible by p .*

Proof. Since $B_1 = -\frac{1}{2}$ the statement holds for $p = 2$ so we will assume $p \geq 3$. Now, recall that the Bernoulli numbers are defined implicitly by

$$\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1}.$$

Multiplying both sides by $(e^{pt} - 1)/t$ gives

$$\frac{e^{pt} - 1}{t} \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{e^{pt} - 1}{t} \frac{t}{e^t - 1} = \sum_{r=0}^{p-1} e^{rt}. \quad (\text{B.3})$$

Expanding e^{rt} as a Taylor series in t , we then have that the coefficient of $t^n/n!$ on the right-hand side of (B.3) is

$$\sum_{r=1}^{p-1} r^n. \quad (\text{B.4})$$

On the other hand, the left-hand side of (B.3) becomes

$$\frac{e^{pt} - 1}{t} \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \sum_{k=0}^{\infty} \frac{p^{k+1} t^k}{(k+1)!} \sum_{k=0}^{\infty} \frac{B_k t^k}{k!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{B_k p^{n-k+1}}{k!(n-k+1)!} t^n$$

so that the coefficient of $t^n/n!$ is

$$\sum_{k=0}^n \frac{n!}{k!(n-k+1)!} B_k p^{n-k+1} = pB_n + \sum_{k=0}^{n-1} \binom{n}{k} p B_k \frac{p^{n-k}}{n-k+1}.$$

Equating with (B.4) then gives

$$pB_n = - \sum_{k=0}^{n-1} \binom{n}{k} p B_k \frac{p^{n-k}}{n-k+1} + \sum_{r=1}^{p-1} r^n.$$

Since $p \geq 3$ we always have $p^{n-k} > n - k + 1$ so that $p^{n-k}/(n - k + 1)$ is p -integral ⁽ⁱ⁾ for each $k \in \{0, \dots, n - 1\}$. Then, noting that $B_0 = 1$ it follows by induction that pB_n is p -integral for all $n \geq 0$. Namely, this holds for our desired case $n = p - 1$ in which

$$pB_{p-1} = - \sum_{k=0}^{p-2} \binom{p-1}{k} pB_k \frac{p^{p-1-k}}{p-k} + \sum_{r=1}^{p-1} r^{p-1}. \quad (\text{B.5})$$

Again noting that $p^{p-1-k} > p - k$ we must have

$$\frac{p^{p-1-k}}{p-k} \equiv 0 \pmod{p}.$$

And so, reducing (B.5) mod p gives

$$pB_{p-1} \equiv \sum_{r=1}^{p-1} r^{p-1} \equiv \sum_{r=1}^{p-1} 1 \equiv -1 \pmod{p},$$

as required. □

Remark. The above proof was adapted from [Lan76, Chapter X Theorem 2.1]

⁽ⁱ⁾A rational number $x = \frac{a}{b}$ is said to be p -integral if $\nu_p(a) \geq \nu_p(b)$. Here, $\nu_p(a)$ and $\nu_p(b)$ denote the number of times that p divides a and b respectively.

NOTATION INDEX

\mathbb{H}	complex upper half plane,
\mathbb{Z}_p	the p -adic integers,
\mathbb{Q}_p	the p -adic numbers,
$j(\gamma, z)$	factor of automorphy,
$ _k \gamma$	slash operator of weight k ; $f _k \gamma = \det(\gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma \cdot z)$,
$E_k(z)$	weight k Eisenstein series normalised so that $a_0 = 1$,
Γ	congruence subgroup,
χ	a Dirichlet character mod N ,
T_m	the m^{th} Hecke operator,
U_m	the linear operator that maps $\sum_{n \geq 0} a_n q^n$ to $\sum_{n \geq 0} a_{mn} q^n$,
w_ℓ	the Atkin-Lehner operator at ℓ ,
W_ℓ	the scaled Atkin-Lehner operator with $W_\ell = \ell^{k/2} w_\ell$,
i_e	the embedding map that sends $f(z)$ to $f(ez)$,
\mathcal{D}_ℓ	the operator $\mathcal{D}_\ell = \ell^2 U_\ell^2 - \ell^k$,
\mathcal{D}_ℓ^χ	the operator $\mathcal{D}_\ell^\chi = \ell^2 U_\ell^2 - \chi'(\ell) \ell^k$ when χ is induced by $\chi' \bmod N/\ell$,
Tr_ℓ	the trace operator from level N to level N/ℓ ,
$\langle a \rangle$	the diamond operator evaluated at a ,
h_N	the level N Fricke involution,
H_N	the scaled Fricke involution $H_N = N^{k/2} h_N$,
$\mathcal{M}_k(\Gamma)$	space of weight k modular forms with respect to Γ ,
$\mathcal{M}(\Gamma)$	algebra of modular forms with respect to Γ ,
$\mathcal{S}_k(\Gamma)$	space of weight k cusp forms with respect to Γ ,
$\mathcal{S}(\Gamma)$	algebra of cusp forms with respect to Γ ,
$\mathcal{M}_k(N)$	space of weight k modular forms for $\Gamma_0(N)$,
$\mathcal{S}_k(N)$	space of weight k cusp forms for $\Gamma_0(N)$,
$\mathcal{M}_k(N, B)$	space of weight k modular forms for $\Gamma_0(N)$ with coefficients in B ,
$\mathcal{S}_k(N, B)$	space of weight k cusp forms for $\Gamma_0(N)$ with coefficients in B ,
$\mathcal{M}_k^\chi(N)$	space of weight k level N modular forms with character χ ,
$\mathcal{S}_k(N)^{\text{new}}$	space of new forms for $\mathcal{S}_k(N)$,

$\mathcal{S}_k(N)^{\text{old}}$	space of old forms for $\mathcal{S}_k(N)$,
$\mathcal{S}_k(N)^{\ell\text{-new}}$	space of ℓ -new forms for $\mathcal{S}_k(N)$,
$\mathcal{S}_k(N)^{\ell\text{-old}}$	space of ℓ -old forms for $\mathcal{S}_k(N)$,
$\mathcal{E}_k(N)$	the Eisenstein subspace of $\mathcal{M}_k(N)$,
$\mathcal{S}_k(N)^{U_\ell\text{-new}}$	elements of $\mathcal{S}_k(N)$ in the kernel of \mathcal{D}_ℓ ,
$\mathcal{S}_k(N)^{\text{Tr}_\ell\text{-new}}$	elements of $\mathcal{S}_k(N)$ in the kernel of Tr_ℓ and $\text{Tr}_\ell W_\ell$,
$\mathcal{S}_k^X(N)^{U_\ell^X\text{-new}}$	elements of $\mathcal{S}_k^X(N)$ in the kernel of \mathcal{D}_ℓ^X ,
$\mathcal{S}_k^X(N)^{\text{Tr}_\ell^X\text{-new}}$	elements of $\mathcal{S}_k^X(N)$ in the kernel of Tr_ℓ and $\text{Tr}_\ell W_\ell$,
$\mathcal{S}_k^X(N)^{\text{Tr}_\ell^X\text{-new}'}$	elements of $\mathcal{S}_k^X(N)$ in the kernel of Tr_ℓ and $\text{Tr}_\ell H_N$.

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