THE UNIVERSITY OF MELBOURNE

MASTER THESIS

Hecke Operators on Quasimodular Forms

Author:
Dwi Setya WIBAWANTI

Supervisor:
Dr. Alex GHITZA

A thesis submitted in fulfillment of the requirements

for the degree of Master of Science

in the

Department of Mathematics and Statistics
The University of Melbourne

October 6, 2016

"Verily, with every hardship comes ease!"

Qur'an 94:5

THE UNIVERSITY OF MELBOURNE

Abstract

Faculty of Science

Department of Mathematics and Statistics

Master of Science

Hecke Operators on Quasimodular Forms

by Dwi Setya WIBAWANTI

Modular forms are special holomorphic functions that have many applications, particularly in Number Theory. There are linear transformations called Hecke operators preserving the space of modular forms of a given weight. Quasimodular forms are generalisations that contain both modular forms and their derivatives. The main objective of this thesis is to examine Hecke theory for quasimodular forms and its relation with the derivative D.

Contents

Dedicated to my beloved Mom and Dad

Chapter 1

Introduction

The group $SL(2,\mathbb{R})$ of 2×2 matrices with determinant one acts on the upper half plane $\mathcal{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ by Möbius transformations. A classical or elliptic modular form is a holomorphic function on the complex upper half-plane $\mathcal{H}$ which transforms in a certain way under the action of a discrete subgroup of $SL(2,\mathbb{R})$, for instance the full modular group $SL(2,\mathbb{Z})$. A holomorphic function is a complex differentiable function over some open, simply connected region in the complex plane. Hence, this topic at first seems to belong to Complex Analysis. However, modular forms, in fact, arise with a lot of applications in other fields such as Combinatorics, Differential Equations, Mathematical Physics, Geometry, and Number Theory especially. Important examples of modular forms include Eisenstein series, Ramanujan's discriminant function, theta series, and generating series of interesting sequences. We discuss some of these examples in Chapter 2.

The algebra of modular forms is not stable under differentiation. Therefore, we introduce quasimodular forms, which are an extension of modular forms. In Chapter 3, we give the definition of quasimodular functions and quasimodular forms and observe their behaviour under differentiation. There is also an Eisenstein series E_2 which is not a modular form but a quasimodular form. We finish the chapter with some structure theorems.

The set of all modular forms of a fixed weight is a complex vector space of finite dimension. There are linear transformations called Hecke operators preserving this space. We define these in Chapter 4. We also introduce eigenfunctions of the

Hecke operators and L-functions of Hecke eigenforms.

Chapter 5 addresses the main question of this thesis: is there a Hecke theory for quasimodular forms? We give Hecke operators acting on quasimodular forms which preserve the given weight and depth. The main ingredient is the relation between Hecke operators and the derivative operator D. We conclude by presenting some recent results on quasimodular eigenforms.

Our main source for the theory of modular forms is Serre's *Cours d'arithmétique*, with occasional references to the books of Bump and Koblitz. For the fundamentals of quasimodular forms, we relied heavily on Royer's *Un cours "Africain" sur les formes modulaires*. The classification of quasimodular eigenforms in Chapter 5 is an exposition of results of Meher (2012) and Das-Meher (2015).

Chapter 2

Classical Modular Forms

2.1 Basic Definitions

Let $\mathcal{H}$ be the upper half plane of $\mathbb{C}$, i.e. $\mathcal{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$.

Definition 1 (Modular Form)**.** A *modular form* of weight $k \in \mathbb{Z}$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that

1. $f\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^k f(z)$, for all $z \in \mathcal{H}$ and all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$
2. f is holomorphic as $z \to i\infty$.

Evaluating the first condition on the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ which are the generators of $SL(2,\mathbb{Z})$, we obtain $f(z+1) = f(z)$ and $f\left(-\frac{1}{z}\right) = z^k f(z)$ respectively. Since $f(z+1) = f(z)$, therefore the modular form f is periodic of period 1, and hence f can be represented by a Fourier series $f(z) = \sum_{n=0}^{\infty} a_n(f) e^{2\pi i n z} = \sum_{n=0}^{\infty} a_n(f) q^n$. There are no negative-index terms because of holomorphicity as $z \to i\infty$. The coefficients $a_n(f)$ bring arithmetic information which is important in number theory.

Definition 2 (Cusp Form)**.** A modular form which is zero at $i\infty$ is called a *cusp form*.

Consider a modular form f of weight k represented by a Fourier series $f(z) = \sum_{n=0}^{\infty} a_n(f) e^{2\pi i n z} = \sum_{n=0}^{\infty} a_n(f) q^n$. If $z \to i\infty$ then $q \to 0$. Hence f is a cusp form if

$a_0(f) = 0$. Let M_k denote the $\mathbb{C}$-vector space of modular forms of weight k and let S_k denote the $\mathbb{C}$-vector space of cusp forms of weight k. Clearly $S_k \subset M_k$.

Theorem 3. *(a) If $k < 12$ or k is odd, then $\dim S_k = 0$.*

(b) If $k \geq 12$ and k is even, then

$$\dim S_k = \begin{cases} \lfloor \frac{k}{12} \rfloor - 1, & \text{if } k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor, & \text{if } k \not\equiv 2 \pmod{12} \end{cases}$$

(c)

$$\dim M_k = \begin{cases} \dim S_k + 1, & \text{if } k = 0 \text{ or } k \geq 4 \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

See Corollary 1 of Proposition 2 in Zagier, 2008 and Section 1.3 in Bump, 1997. The above theorem shows that M_k and S_k are finite dimensional.

Remark 4. If $k = 12, 16, 18, 20, 22, 26$, then S_k has dimension 1.

There are some spaces of holomorphic functions. Let $Hol(\mathcal{H})$ denote the space of all holomorphic functions on $\mathcal{H}$, and let $Hol(\mathcal{H}/\mathbb{Z}) = \{f \in Hol(\mathcal{H}) | f(z+1) = f(z)\}$.

Definition 5. Let $Hol_\infty(\mathcal{H}/\mathbb{Z})$ be the $\mathbb{C}$-vector space of holomorphic functions $f : \mathcal{H} \to \mathbb{C}$ which are

1. periodic of period 1: $f(z+1) = f(z)$, and
2. holomorphic at ∞: $f(z) = \sum_{n=0}^{\infty} a_n(f) e^{2\pi i n z}$.

Note that $M_k \subset Hol_\infty(\mathcal{H}/\mathbb{Z})$ for all k.

2.2 Examples of Modular Forms

Example 6 (Eisenstein Series)**.** Let $k \geq 4$ be even integer and $z \in \mathcal{H}$. We define Eisenstein series G_k and E_k as follows.

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^k}.$$

This series converges absolutely to a holomorphic function of z in $\mathcal{H}$ and its Fourier expansion is given by

$$G_k(z) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right)$$

where $q = e^{2\pi iz}$, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$, $B_k \in \mathbb{Q}$ is the k-th Bernoulli number, and ζ denotes the Riemann zeta function. The Fourier expansion of Eisenstein series shows that it extends to a holomorphic function at $z = i\infty$. Moreover, it satisfies

$$G_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^k G_k(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Next, if we normalize it by setting

$$E_k(z) = \frac{G_k(z)}{2\zeta(k)},$$

then the Fourier expansion of $E_k(z)$ has rational coefficients and constant term 1.

Proposition 7. *$G_k \in M_k$ and $E_k \in M_k$ for $k \geq 4$.*

Note that it is important to assume that $k \geq 4$. However, there is a holomorphic Eisenstein series of weight $k = 2$ which is a quasimodular form. We will discuss quasimodular forms in Chapter 3. The following proposition is in Chapter VII Section 3.2 in Serre, 1970.

Proposition 8. *$M_k = S_k \oplus \mathbb{C}E_k$ for $k \geq 4$.*

Example 9 (The Discriminant Function Δ)**.** Since the Fourier expansion of $E_k(z)$ has non-zero constant term, $E_k \notin S_k$. Now, define

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728}.$$

It has integral Fourier coefficients

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} a_n(\Delta) q^n$$

where the sequence $a_n(\Delta)$ for $n \geq 1$: $1, -24, 252, -1472, \ldots$. The function $n \mapsto a_n(\Delta)$ is called the *Ramanujan function*. He calculated the first 30 values of $a_n(\Delta)$. It was also conjectured by Ramanujan (1915) and proved by Mordell (1916) that

(i) $a_{mn}(\Delta) = a_m(\Delta) a_n(\Delta), \quad$ if $(m, n) = 1$,

(ii) $a_{p^{n+1}}(\Delta) = a_p(\Delta) a_{p^n}(\Delta) - p^{11} a_{p^{n-1}}(\Delta), \quad$ if p is prime, $n \geq 1$.

The above identities are also mentioned in Corollary to Proposition 14 in Serre, 1970 Chapter VII, and were generalised by Hecke to the Theory of Hecke Operators which we will discuss later.

Proposition 10. $\Delta \in S_{12}$.

The proof of above proposition uses the identity of Dedekind eta function η in Lemma 28. We will prove it later in Section 3.4. More extensive discussion and examples of modular forms can be found in Zagier, 2008 Section 2 and 3 or in Weinstein, 2016 for more concise version.

Chapter 3

Quasimodular Forms

3.1 Quasimodular Functions

Define $D := \frac{1}{2\pi i}\frac{d}{dz}$. If $f(z) = \sum_{n=0}^{\infty} a_n(f)e^{2\pi i n z}$, we obtain $Df(z) = \sum_{n=0}^{\infty} n a_n(f)e^{2\pi i n z}$. If $f \in M_k$, then Df satisfies

$$(cz+d)^{-(k+2)} Df\left(\frac{az+b}{cz+d}\right) = Df(z) + \frac{k}{2\pi i} f(z)\frac{c}{cz+d}. \tag{3.1}$$

Proposition 11. *Let $f \in M_k$ and $m \in \mathbb{Z}_{\geq 0}$. For any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, the m^{th} derivative of f satisfies*

$$(cz+d)^{-(k+2m)} D^m f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{m} \binom{m}{j} \frac{(k+m-1)!}{(k+m-j-1)!}\left(\frac{1}{2\pi i}\right)^j D^{m-j} f(z)\left(\frac{c}{cz+d}\right)^j.$$

Note that for $m = 0$, we simply have $(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z)$ as in Definition 1 of modular form. Consider the Equation 3.1 above, we observe that the derivative of a modular form is not modular but almost, since should we regard only the first term, then we will obtain a modular form of weight $k+2$. Therefore, we now introduce the quasimodular forms which generalise the modular forms. Hence the derivatives of modular forms are an archetypal example of quasimodular forms. An introduction on quasimodular forms is studied in Royer, 2012. Now, let us first introduce quasimodular functions.

Definition 12 (Quasimodular Function)**.** A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is a *quasimodular function* of weight k and depth s with $k, s \in \mathbb{Z}$ and $s \geq 0$ if there exist holomorphic functions $f_0, ..., f_s$ over $\mathcal{H}$ with f_s non-identically zero, such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{s} f_j(z) \left(\frac{c}{cz+d}\right)^j \tag{3.2}$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and any $z \in \mathcal{H}$.

Let FM_k^s denote all quasimodular functions of weight k and depth s, and $FM_k^{\leq s}$ denote the $\mathbb{C}$-vector space of quasimodular functions of weight k and depth less than or equal to s. There is also

$$FM_k^{\infty} := \bigcup_{s \in \mathbb{N}} FM_k^{\leq s}.$$

Note:

1. If $s = 0$, we have $(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f_0(z)$. So, $M_k \subset FM_k^0$.
2. If $f \in M_k$, then $D^m f \in FM_{k+2m}^m$ where $f_j(z) = \binom{m}{j} \frac{(k+m-1)!}{(k+m-j-1)!} \left(\frac{1}{2\pi i}\right)^j D^{m-j} f(z)$.

By convention, zero function is quasimodular of depth 0 for any weight. We define $Q_j : FM_k^{\infty} \to Hol(\mathcal{H})$ given by $Q_j(f) := f_j$. It follows from Proposition 3.3 in Royer, 2012 that if f is a quasimodular function of weight k and depth s, then $Q_j(f)$ is a quasimodular function of weight $k - 2j$ and depth $s - j$.

3.2 Action of $SL(2,\mathbb{Z})$

$SL(2,\mathbb{Z}) \times Hol(\mathcal{H}) \to Hol(\mathcal{H})$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, f \mapsto (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ is a group action. We then write

$$\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) \tag{3.3}$$

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, we define

$$\begin{aligned} X(A) : \mathcal{H} &\to \mathbb{C} \\ z &\mapsto \frac{c}{cz+d}. \end{aligned}$$

Then Equation 3.2 becomes

$$f|_k A = \sum_{j=0}^{s} Q_j(f) X(A)^j \tag{3.4}$$

Lemma 13. *If $f_0, f_1, ..., f_s \in Hol(\mathcal{H})$ and $\sum_{j=0}^{s} f_j(z) X(A)^j = 0$ for all $A \in SL(2,\mathbb{Z})$ and $z \in \mathcal{H}$ then $f_0 = f_1 = ... = f_s = 0$.*

Proof. Let $f_0, f_1, ..., f_s \in Hol(\mathcal{H})$. Set $A = \begin{pmatrix} 1 & d-1 \\ 1 & d \end{pmatrix}$, we get

$$\sum_{j=0}^{s} f_j(z) \left(\frac{1}{z+d}\right)^j = f_0(z) + f_1(z)\frac{1}{z+d} + ... + f_s(z)\frac{1}{(z+d)^s} = 0$$

$$\implies P_z(d) = f_0(z)(z+d)^s + f_1(z)(z+d)^{s-1} + ... + f_s(z) = 0, \quad \forall z \in \mathcal{H}, d \in \mathbb{Z}.$$

Fix z. Then the polynomial

$$P_z(X) = \sum_{j=0}^{s} f_{s-j}(z)(X+z)^j \in \mathbb{C}[X]$$

has infinitely many roots since $P_z(d) = 0$ for all $d \in \mathbb{Z}$. Hence $P_z = 0$. Thus, the coefficients of the power series expansion of P_z at $X = -z$ are zero, which means

$$f_0(z), f_1(z), ..., f_s(z) = 0, \quad \forall z \in \mathcal{H}.$$

□

Remark 14. If $f \in M_k$, the Equation 3.1 is rewritten as

$$(Df|_{k+2}A) = Df + \frac{k}{2\pi i} fX(A).$$

In deriving Equation 3.3, for any function f holomorphic on $\mathcal{H}$ we have

$$D(f|_kA) = (Df|_{k+2}A) - \frac{k}{2\pi i}(f|_kA)X(A). \tag{3.5}$$

Lemma 15. *If* $A, B \in SL(2,\mathbb{Z})$ *then*

$$(X(A)|_2B) = X(AB) - X(B).$$

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Consider $X(AB) - X(B)$, where we can easily calculate that

$$X(AB) = \frac{c\alpha + d\gamma}{(c\alpha + d\gamma)z + c\beta + d\delta} \quad \text{and} \quad X(B) = \frac{\gamma}{\gamma z + \delta}.$$

Now, by Equation 3.3 we have

$$
\begin{aligned}
(X(A)|_2 B) &= X(A)|_2 \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (\gamma z+\delta)^{-2} X(A)\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) \\
&= \frac{c(\gamma z+\delta)^{-2}}{c\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)+d} = \frac{c}{(\gamma z+\delta)[(c\alpha+d\gamma)z+c\beta+d\delta]} \\
&= \frac{K}{(c\alpha+d\gamma)z+c\beta+d\delta} + \frac{L}{\gamma z+\delta},
\end{aligned}
$$

where

$$K = [(c\alpha+d\gamma)z+c\beta+d\delta]\,(X(A)|_2 B)_{|z=-\frac{c\beta+d\delta}{c\alpha+d\gamma}} = c\alpha+d\gamma$$

and

$$L = (\gamma z+\delta)\,(X(A)|_2 B)_{|z=-\frac{\delta}{\gamma}} = \frac{c}{-(c\alpha+d\gamma)\frac{\delta}{\gamma}+c\beta+d\delta} = \frac{-c\gamma}{(\alpha\delta-\beta\gamma)c} = -\gamma.$$

□

Remark 16.

1. Choosing $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in Equation 3.2 shows that

$$f \in FM_k^\infty \implies f_0(z) = f(z), \quad \text{i.e } Q_0(f) = f.$$

2. Similarly, the choice of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in Equation 3.2 implies

$$f \in FM_k^\infty \implies f \text{ is periodic of period } 1.$$

3. Let $depth(f)$ be the depth of f and $weight(f)$ be the weight of f.

4. If $f, g \in FM_k^\infty$ and $f, g \neq 0$ then

$$depth(fg) = depth(f) + depth(g) \text{ and } weight(fg) = 2k.$$

5. Let $f \in FM_k^{\infty}$. We have $Q_j(f) = f_j$ for $j = 0, 1, ..., depth(f)$. Set $Q_j(f) = 0$ for $j < 0$ and $j > depth(f)$. Then for all $n \in \mathbb{Z}, Q_n$ is linear and $Q_n(fg) = \sum_{j=0}^{n} Q_j(f)Q_{n-j}(g)$.

Lemma 17. *Consider the upper triangular nilpotent matrix*

$$M(x) = \left(\binom{\beta-1}{\alpha-1}x^{\beta-\alpha}\right)_{\substack{1\le\alpha\le s+1\\ \alpha\le\beta\le s+1}}$$

then

$$M(x+y) = M(x)M(y)$$

and

$$M(x)^{-1} = M(-x).$$

Proof. Let

$$M(x) = \left(\binom{\gamma-1}{\alpha-1}x^{\gamma-\alpha}\right)_{\substack{1\le\alpha\le s+1\\ \alpha\le\gamma\le s+1}} \quad \text{and} \quad M(y) = \left(\binom{\beta-1}{\gamma-1}y^{\beta-\gamma}\right)_{\substack{1\le\gamma\le s+1\\ \gamma\le\beta\le s+1}}.$$

Define

$$\delta(m \ge n) = \begin{cases} 1, & \text{if } m \ge n \\ 0, & \text{if } m < n. \end{cases}$$

Then the coefficient index (α, β) of the product $M(x)M(y)$ is

$$\begin{aligned}\sum_{\gamma=1}^{s+1} \delta(\gamma \ge \alpha)\binom{\gamma-1}{\alpha-1}\delta(\beta \ge \gamma)\binom{\beta-1}{\gamma-1}x^{\gamma-\alpha}y^{\beta-\gamma} &= \delta(\beta \ge \alpha)\binom{\beta-1}{\alpha-1}\sum_{\gamma=\alpha}^{\beta}\binom{\beta-\alpha}{\gamma-\alpha}x^{\gamma-\alpha}y^{\beta-\gamma} \\ &= \delta(\beta \ge \alpha)\binom{\beta-1}{\alpha-1}(x+y)^{\beta-\alpha}.\end{aligned}$$

□

Proposition 18. *Let $f \in FM_k^{\le s}$. For all $m \in \{0, 1, ..., s\}$ we have*

$$(Q_m(f)|_{k-2m}A) = \sum_{v=0}^{s-m}\binom{m+v}{v}Q_{m+v}(f)X(A)^v$$

for all $A \in SL(2,\mathbb{Z})$. In other words,

$$Q_m \circ Q_v = Q_m\left(Q_v(f)\right) = \binom{m+v}{v} Q_{m+v}.$$

Proof. Since $(f|_k AB) = ((f|_k A)|_k B)$. From Equation 3.4, we have

$$\begin{aligned}(f|_k AB) &= \left(\left(\sum_{n=0}^{s} Q_n(f) X(A)^n\right)\Bigg|_k B\right) \\ &= \sum_{n=0}^{s} \left(Q_n(f) X(A)^n|_k B\right) \\ &= \sum_{n=0}^{s} \left(Q_n(f)|_{k-2n} B\right)\left(X(A)|_2 B\right)^n.\end{aligned}$$

By Lemma 15,

$$\begin{aligned}(f|_k AB) &= \sum_{n=0}^{s} \left(Q_n(f)|_{k-2n} B\right)\left(X(AB) - X(B)\right)^n \\ &= \sum_{n=0}^{s} \left(Q_n(f)|_{k-2n} B\right) \sum_{j=0}^{n} \binom{n}{j} X(AB)^j (-X(B))^{n-j} \\ &= \sum_{n=0}^{s} \sum_{j=0}^{n} \binom{n}{j} (-X(B))^{n-j} \left(Q_n(f)|_{k-2n} B\right) X(AB)^j \\ &= \sum_{j=0}^{s} \left[\sum_{n=j}^{s} \binom{n}{j} (-X(B))^{n-j} \left(Q_n(f)|_{k-2n} B\right)\right] X(AB)^j \\ &= \sum_{j=0}^{s} Q_j(f) X(AB)^j.\end{aligned}$$

Then we get equations for $j = 0, 1, ..., s$:

$$Q_j(f) = \sum_{n=j}^{s} \binom{n}{j} (-X(B))^{n-j} \left(Q_n(f)|_{k-2n} B\right).$$

Rewriting these equations in form of matrix, we have

$$\begin{pmatrix} Q_0(f) \\ Q_1(f) \\ \vdots \\ Q_s(f) \end{pmatrix} = M(-X(B)) \begin{pmatrix} Q_0(f)|_k B \\ Q_1(f)|_{k-2} B \\ \vdots \\ Q_s(f)|_{k-2s} B \end{pmatrix}.$$

By Lemma 17,

$$\begin{pmatrix} Q_0(f)|_k B \\ \vdots \\ Q_s(f)|_{k-2s} B \end{pmatrix} = M(X(B)) \begin{pmatrix} Q_0(f) \\ \vdots \\ Q_s(f) \end{pmatrix}.$$

In other words,

$$(Q_n(f)|_{k-2n} B) = \sum_{n=j}^{s} \binom{n}{j} Q_n(f) X(B)^{n-j}.$$

The result follows. □

Corollary 19. *For any integer $r \geq 1$, $Q_r = \frac{1}{r!} \underbrace{Q_1 \circ \ldots \circ Q_1}_{r\text{-times}}$.*

Proof. Proof by induction on r. Base case: $r = 1$. By Proposition 18,

$$Q_1 \circ Q_1 = \binom{2}{1} Q_2 = 2Q_2 \implies Q_2 = \frac{1}{2} Q_1 \circ Q_1.$$

Induction step: Suppose true for $r-1$, i.e. $\underbrace{Q_1 \circ \ldots \circ Q_1}_{(r-1)\text{-times}} = (r-1)! Q_{r-1}$. By the induction hypothesis and Proposition 18,

$$\underbrace{Q_1 \circ \ldots \circ Q_1}_{r\text{-times}} = (\underbrace{Q_1 \circ \ldots \circ Q_1}_{(r-1)\text{-times}}) \circ Q_1 = (r-1)! \binom{r}{1} Q_r = r! Q_r.$$

□

Corollary 20. *If $m \leq s$ and $f \in FM_k^s$, then $Q_m(f) \in FM_{k-2m}^{s-m}$.*

Note that by Corollary 20, it follows that if $f \in FM_k^s$ then $Q_s(f) \in FM_{k-2s}^0$. So it satisfies the modularity equation of weight $k - 2s$, that is for all matrices

$A \in SL(2, \mathbb{Z})$, we have

$$(Q_s(f)|_{k-2s}A) = Q_s.$$

Since all $Q_j(f)$ are quasimodular functions whenever f is, hence by Remark 16, they are periodic of period 1, and hence admit a Fourier expansion. Thus, we add a condition to definition of quasimodular functions as follows.

3.3 Quasimodular Forms and Differentiation

Quasimodular forms were introduced by Kaneko and Zagier in 1995. They were motivated by the appearance of such forms as generating functions in Mathematical Physics.

Definition 21 (Quasimodular Form)**.** A *quasimodular form* f of weight k and depth s is a quasimodular function of weight k and depth s such that the Fourier expansions of each $Q_j(f)$ have no negative-index terms:

$$Q_j(f) = \sum_{n=0}^{\infty} a_n(Q_j(f))e^{2\pi inz}$$

for all $j \in \{0, 1, ..., s\}$.

Let M_k^s denote the set of quasimodular forms of weight k and depth s, and $M_k^{\leq s}$ denote the $\mathbb{C}$-vector space of quasimodular forms of weight k and depth less than or equal to s. There is also

$$M_k^{\infty} := \bigcup_{s \in \mathbb{N}} M_k^{\leq s}.$$

Proposition 22. *If $f \in M_k^{\leq s}$ and $g \in M_l^{\leq t}$, then $fg \in M_{k+l}^{\leq s+t}$.*

Proof. Let $f \in M_k^{\leq s}$ and $g \in M_l^{\leq t}$. Hence,

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = \sum_{j=0}^{s} Q_j(f)(z) \left(\frac{c}{cz + d}\right)^j, \text{ and}$$

$$(cz+d)^{-l} g\left(\frac{az+b}{cz+d}\right) = \sum_{i=0}^{t} Q_i(g)(z)\left(\frac{c}{cz+d}\right)^i.$$

Then

$$\begin{aligned}(cz+d)^{-(k+l)}(fg)\left(\frac{az+b}{cz+d}\right) &= \left[\sum_{j=0}^{s} f_j(z)\left(\frac{c}{cz+d}\right)^j\right]\left[\sum_{i=0}^{t} g_i(z)\left(\frac{c}{cz+d}\right)^i\right] \\ &= \sum_{m=0}^{s+t}\left(\sum_{j=0}^{m} f_j(z) g_{m-j}(z)\right)\left(\frac{c}{cz+d}\right)^m \\ &= \sum_{m=0}^{s+t} Q_m(fg)(z)\left(\frac{c}{cz+d}\right)^m,\end{aligned}$$

since $Q_m(fg) = \sum_{j=0}^{m} Q_j(f) Q_{m-j}(g)$. □

Remark 23.

1. A quasimodular form of weight k and depth 0 is a modular form of weight k, i.e. $M_k^0 = M_k$.
2. Since non-constant modular forms are of strictly positive weight, if $f \in M_k^s$ then $s \leq \frac{k}{2}$, for k even. Note that $Q_s(f) \in M_{k-2s}$ but if $l < 0$ then $M_l = 0$.
3. If k is odd then $M_k^s = 0$ (and hence $M_k^{\leq s} = 0$).

Theorem 24. *The sum of the spaces M_k^∞ as k varies is a direct sum. In other words, if $f_j \in M_{k_j}^\infty$ for $j \in \{1, 2, ..., r\}, k_1 < k_2 < ... < k_r$, and $f_1 + f_2 + ... + f_r = 0$ then $f_1 = f_2 = ... = f_r = 0$.*

Proof. Let $f_i \in M_{k_i}^{s_i} \subset M_{k_i}^s$, where $s = \max_i(s_i)$. Fix $z \in \mathcal{H}$ and $d \in \mathbb{Z}$. Let $\begin{pmatrix} 1 & d-1 \\ 1 & d \end{pmatrix} \in SL(2,\mathbb{Z})$. Then we have

$$\begin{aligned} f_i\left(\frac{z+d-1}{z+d}\right) &= (z+d)^{k_i}\sum_{j=0}^{s} Q_j(f_i)(z)\left(\frac{1}{z+d}\right)^j \\ &= \sum_{j=0}^{s} Q_j(f_i)(z)(z+d)^{k_i-j}.\end{aligned}$$

By the hypothesis,

$$0 = \sum_{i=1}^{r} f_i \left(\frac{z+d-1}{z+d} \right) = \sum_{i=1}^{r} \sum_{j=0}^{s} Q_j(f_i)(z)(z+d)^{k_i - j} = P(d).$$

Let $P(X) = \sum_{i=1}^{r} \sum_{j=0}^{s} Q_j(f_i)(z)(X+z)^{k_i-j} \in \mathbb{C}[X]$. Since $P(d) = 0$ for all $d \in \mathbb{Z}$, hence $P(X) = 0$. This implies all the $Q_j(f_i)(z) = 0$ for all $z \in \mathcal{H}$, for all $i = 1, 2, ..., r$ and for all $j = 0, 1, ..., s$. Consider the highest-degree term $Q_0(f_r)(z)(X+z)^{k_r}$. Then by Remark 16, we have $f_r(z) = Q_0(f_r)(z) = 0, \forall z \in \mathcal{H}$. So $f_r = 0$. By induction, $f_i = 0$ for all i. □

Theorem 25. *Let $f \in M_k^s$ be non-constant, then $Df \in M_{k+2}^{s+1}$. More precisely,*

$$Q_0(Df) = Df,$$

$$Q_n(Df) = D(Q_n f) + \frac{k-n+1}{2\pi i} Q_{n-1}(f), \text{ for } 1 \leq n \leq s,$$

$$Q_{s+1}(Df) = \frac{k-s}{2\pi i} Q_s(f).$$

Proof. Let $f \in M_k^s$ be non-constant and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. Recall that $Df := \frac{1}{2\pi i} \frac{df}{dz}$ and $X(A)(z) = \frac{c}{cz+d}$. Then

$$D(X(A))(z) = \frac{1}{2\pi i} \frac{d}{dz} \left(\frac{c}{cz+d} \right) = -\frac{1}{2\pi i} \left(\frac{c}{cz+d} \right)^2 = -\frac{1}{2\pi i} \left(X(A)(z) \right)^2.$$

On the other hand,

$$(f|_k A) = \sum_{j=0}^{s} Q_j(f) X(A)^j$$

implies

$$D(f|_k A) = \sum_{j=0}^{s} D(Q_j(f))X(A)^j + Q_j(f)jX(A)^{j-1}\left(-\frac{1}{2\pi i}\right)X(A)^2$$
$$= \sum_{j=0}^{s} D(Q_j(f))X(A)^j - \frac{j}{2\pi i}Q_j(f)X(A)^{j+1}.$$

By Equation 3.5,

$$\begin{aligned}(Df|_{k+2}A) &= D(f|_k A) + \frac{k}{2\pi i}(f|_k A)X(A)\\ &= \sum_{j=0}^{s} D(Q_j(f))X(A)^j - \frac{j}{2\pi i}Q_j(f)X(A)^{j+1} + \frac{k}{2\pi i}\sum_{j=0}^{s} Q_j(f)X(A)^{j+1}\\ &= \sum_{j=0}^{s} D(Q_j(f))X(A)^j + \frac{k-j}{2\pi i}Q_j(f)X(A)^{j+1}\\ &= \sum_{j=0}^{s+1} Q_j(Df)X(A)^j \in M_{k+2}^{s+1}.\end{aligned}$$

Hence, for $j = 0$: $\quad Q_0(Df) = D(Q_0(f)) = Df,$
for $1 \le j \le s$: $\quad Q_j(Df) = D(Q_j(f)) + \dfrac{k-j+1}{2\pi i}Q_{j-1}(f),$
for $j = s+1$: $\quad Q_{s+1}(Df) = \dfrac{k-s}{2\pi i}Q_s(f).$ □

The above theorem shows that the algebra of quasimodular forms is stable under differentiation.

Corollary 26. *If $f \in M_{k-2s}$ then*

$$Q_s(D^s f) = \frac{s!}{(2\pi i)^s}\binom{k-s-1}{s} f.$$

Proof. Let $f \in M_{k-2s} = M_{k-2s}^0$, then by Theorem 25

$$Q_1(Df) = \frac{k-2s}{2\pi i}Q_0(f) = \frac{k-2s}{2\pi i}f.$$

Since $f \in M_{k-2s}$, $Df \in M^1_{k-2s+2}$. Apply again Theorem 25, we get

$$Q_2(D^2 f) = \frac{k-2s+1}{2\pi i} Q_1(Df) = \frac{(k-2s+1)(k-2s)}{(2\pi i)^2} f.$$

By doing it recursively, we obtain

$$Q_s(D^s f) = \frac{(k-s-1)(k-s-2)...(k-2s)}{(2\pi i)^s} f = \frac{s!}{(2\pi i)^s} \binom{k-s-1}{s} f.$$

□

Define $[Q_n, D] := Q_n \circ D - D \circ Q_n$.

Remark 27. Let $n \in \{0, 1, ..., s+1\}$. The Theorem 25 is equivalent to

$$[Q_n, D] = \frac{k-n+1}{2\pi i} Q_{n-1}.$$

3.4 The Quasimodular Eisenstein Series E_2

Now, recall $\Delta \in S_{12}$ given by $\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24}$ as discussed in Example 9. To simplify the writing, we define a function

$$\begin{aligned} e : \mathcal{H} &\to \mathbb{C} \\ z &\mapsto e^{2\pi i z}. \end{aligned}$$

This function is periodic of period 1 and satisfies $De = e$. Next, let η be Dedekind eta function given by

$$\eta(z) = e\left(\frac{z}{24}\right) \prod_{n=1}^{\infty} (1 - e(nz)).$$

Then it satisfies the following lemma.

Lemma 28. *The function* η *satisfies the equation*

$$\eta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}}\eta(z).$$

For a proof, see Lemma 138 in Royer, 2013. Now, observe that

$$\Delta(z) = e(z)\prod_{n=1}^{\infty}(1-e(nz))^{24} = (\eta(z))^{24}.$$

Then by Lemma 28,

$$\Delta\left(-\frac{1}{z}\right) = \eta\left(-\frac{1}{z}\right)^{24} = \frac{z^{12}}{i^{12}}\eta(z)^{24} = z^{12}\Delta(z).$$

Note that $e(n(z+1)) = e^{2\pi inz}e^{2\pi in} = e^{2\pi inz} = e(nz)$. Hence,

$$\Delta(z+1) = e(z+1)\prod_{n=1}^{\infty}(1-e(n(z+1)))^{24} = e(z)\prod_{n=1}^{\infty}(1-e(nz))^{24} = \Delta(z).$$

Now we want to show:

$$\Delta(z)\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z), \quad \forall z \in \mathcal{H}, \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Evaluating the above equation on matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we obtain $\Delta\left(-\frac{1}{z}\right) = z^{12}\Delta(z)$ and $\Delta(z+1) = \Delta(z)$ respectively, which are true by our calculations above. Therefore, Δ satisfies modularity equation for $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which generate $SL(2,\mathbb{Z})$. Moreover,

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}a_n(\Delta)q^n,$$

i.e. $a_0(\Delta) = 0$. Hence, Δ is a cusp form of weight 12. This proves Proposition 10.

Now, we define a weight 2 Eisenstein series $E_2 := \frac{D\Delta}{\Delta}$. If $f \in M_k$, then dividing the Equation 3.1 by $(f|_k A) = f$, we have

$$(cz+d)^{-2}\frac{Df}{f}\left(\frac{az+b}{cz+d}\right) = \frac{Df}{f}(z) + \frac{k}{2\pi i}\frac{c}{cz+d}. \tag{3.6}$$

To ensure that $\frac{Df}{f}$ is holomorphic on $\mathcal{H}$, it suffices to show that f does not vanish on $\mathcal{H}$.

Proposition 29. *$E_2 \in M_2^1$ and $Q_1(E_2) = \frac{6}{\pi i}$.*

Proof. To prove that E_2 is holomorphic on $\mathcal{H}$, it suffices to show that Δ does not vanish on $\mathcal{H}$. Let $z = x + iy \in \mathcal{H}$.

Claim: $\Delta(z) \neq 0, \quad \forall z \in \mathcal{H}$.

Assume $\exists z \in \mathcal{H}$ such that $\Delta(z) = 0$. Then $e^{2\pi iz} = 0$ or $e^{2\pi inz} = 1$ for some $n \geq 1$. If $e^{2\pi iz} = 0$ then $e^{-2\pi y}(\cos 2\pi x + i \sin 2\pi x) = 0$, which implies $\cos 2\pi x = 0$ and $\sin 2\pi x = 0$ which are impossible. Hence, $e^{2\pi inz} = 1$ for some $n \geq 1$. So $e^{-2\pi ny}(\cos 2\pi nx + i\sin 2\pi nx) = 1 \implies \sin 2\pi nx = 0 \implies 2nx \in \mathbb{Z} \implies \cos 2\pi nx \in \{-1, 1\} \implies \cos 2\pi nx = 1$ and $e^{-2\pi ny} = 1 \implies y = 0$ contradicts $z \in \mathcal{H}$ $(y > 0)$. Thus, $\Delta(z) \neq 0$ for all $z \in \mathcal{H}$. Next, since $\Delta \in S_{12} \subset M_{12}$, hence it satisfies Equation 3.6. So,

$$(cz+d)^{-2}E_2\left(\frac{az+b}{cz+d}\right) = E_2(z) + \frac{12}{2\pi i}\frac{c}{cz+d}$$

$$\implies E_2 \in M_2^1 \text{ with } Q_0(E_2) = E_2 \text{ and } Q_1(E_2) = \frac{6}{\pi i}.$$

□

Corollary 30.

$$Q_s(D^{s-1}E_2) = \frac{(s-1)!}{(2\pi i)^{s-1}}\frac{6}{\pi i}.$$

Proposition 31. *The Fourier expansion of E_2 is given by*

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi inz}.$$

Proof. Recall $D := \frac{1}{2\pi i}\frac{d}{dz}$ and $\Delta(z) = e(z)\prod_{n=1}^{\infty}(1-e(nz))^{24}$. Note that $\frac{de(z)}{dz} = 2\pi i e(z)$ and $\frac{d}{dz}\left((1-e(nz))^{24}\right) = -48\pi i n e(nz)(1-e(nz))^{23}$. Hence,

$$
\begin{aligned}
D\Delta(z) &= \frac{1}{2\pi i}\left(\frac{de(z)}{dz}\prod_{n=1}^{\infty}(1-e(nz))^{24} + e(z)\sum_{n=1}^{\infty}\frac{d\left((1-e(nz))^{24}\right)}{dz}\prod_{\substack{m=1\\m\neq n}}^{\infty}(1-e(mz))^{24}\right)\\
&= e(z)\prod_{n=1}^{\infty}(1-e(nz))^{24} - 24e(z)\sum_{n=1}^{\infty}\frac{ne(nz)}{1-e(nz)}\prod_{m=1}^{\infty}(1-e(mz))^{24}\\
&= \Delta(z) - 24\Delta(z)\sum_{n=1}^{\infty}\frac{ne(nz)}{1-e(nz)}.
\end{aligned}
$$

Thus,

$$E_2(z) = \frac{D\Delta(z)}{\Delta(z)} = 1 - 24\sum_{n=1}^{\infty}\frac{ne(nz)}{1-e(nz)}.$$

Recall that $\frac{1}{1-x} = 1 + x + x^2 + ...$ for $|x| < 1$. Note that if $z \in \mathcal{H}$ then $|e(nz)| < 1$ for all $n \in \mathbb{N}$. Also note that $e(nz)^k = e(knz)$. Therefore,

$$
\begin{aligned}
E_2(z) &= 1 - 24\left(\frac{e(z)}{1-e(z)} + \frac{2e(2z)}{1-e(2z)} + \frac{3e(3z)}{1-e(3z)} + ...\right)\\
&= 1 - 24\left(e(z)[1 + e(z) + e(2z) + ...] + 2e(2z)[1 + e(2z) + e(4z) + ...] + ...\right)\\
&= 1 - 24\left(e(z) + e(2z) + e(3z) + ... + 2e(2z) + 2e(4z) + 2e(6z) + ... + 3e(3z) + ...\right)\\
&= 1 - 24(e(z) + (1+2)e(2z) + (1+3)e(3z) + (1+2+4)e(4z) + ...)\\
&= 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)e(nz).
\end{aligned}
$$

□

Another normalisation is

$$G_2 = -\frac{1}{24}E_2 = -\frac{1}{24} + \sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi i n z}.$$

3.5 Structure Theorems

Theorem 32. *For any $f \in M_k^s$, there exist unique modular forms $F_i \in M_{k-2i}$ such that*

$$f = F_0 + F_1 E_2 + F_2 E_2^2 + ... + F_s E_2^s.$$

Proof. Existence: Proof by induction on the depth.

Base case: $s = 0$. $f \in M_k^0 = M_k$, so $f = f$.

Induction step: Fix $s > 0$. Suppose true for depth $\leq s-1$. Let $f \in M_k^s$. Note that

$$f - \left(\frac{i\pi}{6}\right)^s Q_s(f) E_2^s \in M_k^{s-1}.$$

Then,

$$f = F_0 + F_1 E_2 + ... + F_{s-1} E_2^{s-1} + \left(\frac{i\pi}{6}\right)^s Q_s(f) E_2^s.$$

Uniqueness: Suppose $f = F_0 + F_1 E_2 + F_2 E_2^2 + ... + F_s E_2^s$ where $F_i \in M_{k-2i}$ and $f = G_0 + G_1 E_2 + G_2 E_2^2 + ... + G_s E_2^s$ where $G_i \in M_{k-2i}$. Then let

$$0 = (F_0 - G_0) + (F_1 - G_1) E_2 + ... + (F_s - G_s) E_2^s = H.$$

Note that $(F_s - G_s) E_2^s \in M_k^s$ implies $Q_s\left((F_s - G_s) E_2^s\right) = (F_s - G_s) Q_s(E_2^s)$. While for $t = 0, 1, 2, ..., s-1$, we have $(F_t - G_t) E_2^t$ implies $Q_s\left((F_t - G_t) E_2^t\right) = 0$. Hence,

$$0 = Q_s(H) = Q_s\left((F_s - G_s) E_2^s\right).$$

Since $F_s - G_s \in M_k^0$, $Q_s\left((F_s - G_s) E_2^s\right) = (F_s - G_s) Q_s(E_2^s) = (F_s - G_s)\left(\frac{i\pi}{6}\right)^s E_2^s$. Thus, $F_s = G_s$. □

The above theorem says that any quasimodular form is a polynomial in E_2 with coefficients being modular forms. The following theorem then says that any quasimodular form can be written uniquely as a linear combination of derivatives of modular forms and of E_2.

Theorem 33. *Let* $f \in M_k^s$.

(a) *If* $s < \frac{k}{2}$, *then* $f = F_0 + DF_1 + D^2F_2 + ... + D^sF_s$

for some modular forms $F_i \in M_{k-2i}$ *where* $i = 0, 1, ..., s$. *In fact,*

$$M_k^{\leq s} = \bigoplus_{i=0}^{s} D^i M_{k-2i}.$$

(b) *If* $s = \frac{k}{2}$, *then* $f = F_0 + DF_1 + ... + D^{\frac{k}{2}-2}F_{\frac{k}{2}-2} + \alpha D^{\frac{k}{2}-1}E_2$

for some modular forms $F_i \in M_{k-2i}$ *where* $i = 0, 1, ..., \frac{k}{2} - 2$, *and some non-zero* $\alpha \in \mathbb{C}$. *In fact,*

$$M_k^{\leq k/2} = \bigoplus_{i=0}^{k/2-2} D^i M_{k-2i} \oplus \mathbb{C}D^{\frac{k}{2}-1}E_2.$$

(c) *If* $s > \frac{k}{2}$, *then* $M_k^s = 0$.

Proof. (a) Proof by induction on the depth s. Let $f \in M_k^s$. Given $g \in M_{k-2s}$, then by Corollary 26

$$Q_s(D^s g) = \frac{s!}{(2\pi i)^s}\binom{k-s-1}{s} g.$$

So if $s \neq \frac{k}{2}$, let

$$g = \frac{(2\pi i)^s}{s!\binom{k-s-1}{s}} Q_s(f) \in M_{k-2s}.$$

Hence, $Q_s(D^s g) = Q_s(f)$. Then $Q_s(f - D^s g) = 0$ implies $f - D^s g \in M_k^{\leq s-1}$. By the induction hypothesis,

$$f - D^s g = \sum_{i=0}^{s-1} D^i F^i = F_0 + DF_1 + D^2F_2 + ... + D^{s-1}F_{s-1}.$$

Thus,

$$f = F_0 + DF_1 + D^2F_2 + ... + D^{s-1}F_{s-1} + D^s g.$$

For the direct sum part, let $f = 0$. Suppose $F_s \neq 0$, then by Corollary 26

$$Q_s(f) = Q_s(D^sF_s) \neq 0 \implies f \neq 0,$$

a contradiction. So, $F_s = 0$. Repeat to get $F_0 = F_1 = ... = F_s = 0$.

(b) If $s = \frac{k}{2}$, then

$$\binom{k-s-1}{s} = \binom{\frac{k}{2}-1}{\frac{k}{2}} = 0.$$

By Corollary 30,

$$Q_{\frac{k}{2}}\left(D^{\frac{k}{2}-1}E_2\right) = \frac{\left(\frac{k}{2}-1\right)!}{(2\pi i)^{\frac{k}{2}-1}} Q_1(E_2) = \frac{\left(\frac{k}{2}-1\right)!}{(2\pi i)^{\frac{k}{2}-1}} \frac{6}{\pi i}.$$

Since $f \in M_k^s$, $Q_{\frac{k}{2}}(f) \in M_0 = \mathbb{C}$. So let

$$\alpha = \frac{\pi i}{6} \frac{(2\pi i)^{\frac{k}{2}-1}}{\left(\frac{k}{2}-1\right)!} Q_{\frac{k}{2}}(f) \in \mathbb{C}.$$

Then,

$$Q_{\frac{k}{2}}\left(f - \alpha D^{\frac{k}{2}-1}E_2\right) = Q_{\frac{k}{2}}(f) - \alpha Q_{\frac{k}{2}}\left(D^{\frac{k}{2}-1}E_2\right) = 0.$$

So, $f - \alpha D^{\frac{k}{2}-1}E_2 \in M_k^{\leq \frac{k}{2}-1}$. Thus by part (a),

$$f = F_0 + DF_1 + ... + D^{\frac{k}{2}-2}F_{\frac{k}{2}-2} + \alpha D^{\frac{k}{2}-1}E_2.$$

Furthermore,

$$Q_{\frac{k}{2}}(f) = \alpha Q_{\frac{k}{2}}\left(D^{\frac{k}{2}-1}E_2\right) \neq 0 \implies f \neq 0.$$

(c) This was already discussed in Remark 23.

□

Corollary 34. *If $f \in M_k^s$ then $f \in Hol_\infty(\mathcal{H}/\mathbb{Z})$.*

Proof. Claim 1: If $F \in Hol_\infty(\mathcal{H}/\mathbb{Z})$ then $D^m F \in Hol_\infty(\mathcal{H}/\mathbb{Z})$ for all integers $m \geq 0$. Let $F(z) = \sum_{n=0}^{\infty} a_n(F) q^n \in Hol_\infty(\mathcal{H}/\mathbb{Z})$. Recall $D := \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$ with $q = e^{2\pi i z}$. Since F is holomorphic on $\mathcal{H}$, then DF is holomorphic on $\mathcal{H}$. Hence $D^m F \in Hol(\mathcal{H})$ for all $m \in \mathbb{Z}_{\geq 0}$. Since F is periodic of period one, DF is periodic of period one. Hence $D^m F \in Hol(\mathcal{H}/\mathbb{Z})$ for all $m \in \mathbb{Z}_{\geq 0}$. Since F is holomorphic at ∞ : $F(z) = \sum_{n=0}^{\infty} a_n(F)q^n$, $D^m F(z) = \sum_{n=0}^{\infty} a_n(D^m F)q^n$ with $a_n(D^m F) = n^m a_n(F)$. Hence $D^m F \in Hol_\infty(\mathcal{H}/\mathbb{Z})$ for all $m \in \mathbb{Z}_{\geq 0}$.

Claim 2: $E_2 \in Hol_\infty(\mathcal{H}/\mathbb{Z})$.

By Proposition 29, $E_2 \in Hol(\mathcal{H})$. Then by Proposition 31, $E_2 \in Hol_\infty(\mathcal{H}/\mathbb{Z})$.

Now, let $f \in M_k^s$. Then by Theorem 33:

(a) If $s < \frac{k}{2}$, then $f = F_0 + DF_1 + D^2F_2 + ... + D^sF_s$ for some $F_i \in M_{k-2i}$. For all $i \in \{0, 1, ..., s\}$, $F_i \in M_{k-2i} \implies F_i \in Hol_\infty(\mathcal{H}/\mathbb{Z})$ (since $M_{k-2i} \subset Hol_\infty(\mathcal{H}/\mathbb{Z})$) $\implies D^iF_i \in Hol_\infty(\mathcal{H}/\mathbb{Z})$ (by Claim 1) $\implies f \in Hol_\infty(\mathcal{H}/\mathbb{Z})$.

(b) If $s = \frac{k}{2}$, then $f = F_0 + DF_1 + ... + D^{\frac{k}{2}-2}F_{\frac{k}{2}-2} + \alpha D^{\frac{k}{2}-1}E_2$ for some $F_i \in M_{k-2i}$ and some $\alpha \in \mathbb{R}$. By part (a), $F_0 + DF_1 + ... + D^{\frac{k}{2}-2}F_{\frac{k}{2}-2} \in Hol_\infty(\mathcal{H}/\mathbb{Z})$. In the other hand, $E_2 \in Hol_\infty(\mathcal{H}/\mathbb{Z})$ (by Claim 2) $\implies D^{\frac{k}{2}-1}E_2 \in Hol_\infty(\mathcal{H}/\mathbb{Z})$ (by Claim 1). Hence, $f \in Hol_\infty(\mathcal{H}/\mathbb{Z})$.

□

Putting together Theorem 33 and Corollary 34, we have the following result.

Corollary 35. *If $f \in M_k^s$ then it has a Fourier expansion*

$$f = \sum_{n=0}^{\infty} a_n(f)e^{2\pi i n z}$$

where

$$a_n(f) = \begin{cases} a_n(F_0) + na_n(F_1) + ... + n^s a_n(F_s), & \text{if } s < \frac{k}{2} \\ a_n(F_0) + na_n(F_1) + ... + n^{\frac{k}{2}-2}a_n(F_{\frac{k}{2}-2}) + \alpha n^{\frac{k}{2}-1}a_n(E_2), & \text{if } s = \frac{k}{2} \\ 0, & \text{otherwise} \end{cases}$$

for some modular forms $F_i \in M_{k-2i}$ where $i = 0, 1, ..., s$, and some non-zero $\alpha \in \mathbb{C}$.

The following proposition shows that the space of quasimodular forms of weight k and depth less than or equal to s is finite dimensional and its dimension can be expressed as the sum of dimensions of some spaces of modular forms with decreasing weights.

Proposition 36.

$$\dim M_k^{\leq s} = \begin{cases} \dim M_k + \dim M_{k-2} + \ldots + \dim M_{k-2s}, & \textit{if } s < \frac{k}{2} \\ \dim M_k + \dim M_{k-2} + \ldots + \dim M_4 + 1, & \textit{if } s = \frac{k}{2} \\ 0, & \textit{otherwise} \end{cases}$$

Proof. By Theorem 33,

$$\dim M_k^{\leq s} = \begin{cases} \sum_{i=0}^{s} \dim D^i M_{k-2i}, & \text{if } s < \frac{k}{2} \\ \sum_{i=0}^{\frac{k}{2}-2} \dim D^i M_{k-2i} + \dim \mathbb{C} D^{\frac{k}{2}-1} E_2, & \text{if } s = \frac{k}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Claim: If $k > 0$ then $D : M_k^{\leq s} \to M_{k+2}^{\leq s+1}$ is an injective linear transformation.
Let $f = \sum_{n=0}^{\infty} a_n(f) q^n \in M_k^{\leq s}$ be such that $Df = 0$.
$Df = \sum_{n=0}^{\infty} n a_n(f) q^n = 0 \implies a_n(f) = 0$ for all $n \geq 1 \implies f = a_0(f)$.
If $a_0(f) \neq 0$ then $k = 0$ which is a contradiction. So $f = a_0(f) = 0$. This proves the claim. Hence, $\dim DM_k^{\leq s} = \dim M_k^{\leq s}$. We repeat to get $\dim D^i M_k^{\leq s} = \dim M_k^{\leq s}$ for all $i \in \mathbb{Z}_{\geq 0}$, and we are done. □

Chapter 4

Hecke Operators on Modular Forms

4.1 Definition and Basic Properties

There is a linear operator T_n for each integer $n \geq 1$, called nth *Hecke operator* acting on modular forms of a given weight. First, recall the space $Hol_\infty(\mathcal{H}/\mathbb{Z})$ from Definition 5. It is clear that $M_k \subset Hol_\infty(\mathcal{H}/\mathbb{Z})$ for all k.

Definition 37 (Hecke Operator)**.** Let $f \in M_k$ and p be prime; we define a linear map $T_{p,k} : Hol_\infty(\mathcal{H}/\mathbb{Z}) \to \mathcal{F}(\mathcal{H}, \mathbb{C})$ by the following formula:

$$T_{p,k}f(z) = p^{k-1}f(pz) + \frac{1}{p}\sum_{n=0}^{p-1} f\left(\frac{z+n}{p}\right).$$

For the purpose of convenience, we sometimes omit the k and write T_p instead. There are similar operators T_n for all $n \in \mathbb{N}$ which commute with one another and satisfy the identities:

$$T_m T_n = T_{mn}, \quad \text{if } (m,n) = 1, \tag{4.1}$$

$$T_p T_{p^n} = T_{p^{n+1}} + p^{k-1} T_{p^{n-1}}, \quad \text{if } p \text{ is prime}, n \geq 1. \tag{4.2}$$

See Section 5.3 Chapter VII in Serre, 1970 for more details.

Theorem 38. *If $f \in Hol_\infty(\mathcal{H}/\mathbb{Z})$, then $T_p f \in Hol_\infty(\mathcal{H}/\mathbb{Z})$.*

Proof. $T_p f$ is periodic of period 1:

$$T_p f(z+1) = p^{k-1} f(pz+p) + \frac{1}{p}\left[\sum_{n=0}^{p-1} f\left(\frac{z+n}{p}\right) - f\left(\frac{z}{p}\right) + f\left(\frac{z+p}{p}\right)\right] = T_p f(z).$$

Therefore $T_p f$ has a Fourier series expansion:

$$(T_p f)(z) = \sum_{n=-\infty}^{\infty} a_n(T_p f) e^{2\pi i n z}.$$

Its coefficients can be described explicitly in terms of the coefficients of f, see Proposition 39 below. Since f is holomorphic at ∞, $(T_p f)(z)$ has no negative terms, so $T_p f \in Hol_\infty(\mathcal{H}/\mathbb{Z})$. □

Proposition 39. *If $f \in Hol_\infty(\mathcal{H}/\mathbb{Z})$, then*

$$a_n(T_p f) = \begin{cases} p^{k-1} a_{n/p}(f) + a_{pn}(f), & \text{if } p \mid n \\ a_{pn}(f), & \text{if } p \nmid n. \end{cases}$$

for all $n \in \mathbb{N}$.

For a proof, see Royer, 2013 Proposition 68.

Lemma 40. $T_{p,k} G_k = \sigma_{k-1}(p) G_k = (1 + p^{k-1}) G_k$.

Proof.

$$G_k = c_0 + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

By Proposition 39, we have

$$a_n(T_{p,k} G_k) = \begin{cases} p^{k-1} a_{n/p}(G_k) + a_{pn}(G_k), & \text{if } p \mid n \\ a_{pn}(G_k), & \text{if } p \nmid n. \end{cases}$$

Hence,

$$a_0(T_{p,k} G_k) = p^{k-1} a_0(G_k) + a_0(G_k) = (1 + p^{k-1}) a_0(G_k),$$

and for $n > 0$,

$$a_n(T_{p,k}G_k) = \begin{cases} p^{k-1}\sigma_{k-1}(\frac{n}{p}) + \sigma_{k-1}(np), & \text{if } p \mid n \\ \sigma_{k-1}(np), & \text{if } p \nmid n. \end{cases}$$

Note that if $n = p^\beta m$ with $\gcd(p, m) = 1$, then

$$\sigma_{k-1}(np) = \sigma_{k-1}(p^{\beta+1}m) = \sigma_{k-1}(p^{\beta+1})\sigma_{k-1}(m)$$

and

$$\sigma_{k-1}\left(\frac{n}{p}\right) = \sigma_{k-1}(p^{\beta-1}m) = \sigma_{k-1}(p^{\beta-1})\sigma_{k-1}(m).$$

Hence, if $p \mid n$, we have

$$a_n(T_{p,k}G_k) = \sigma_{k-1}(m)\left[p^{k-1}\sigma_{k-1}(p^{\beta-1}) + \sigma_{k-1}(p^{\beta+1})\right].$$

On the other hand,

$$a_n(G_k) = \sigma_{k-1}(n) = \sigma_{k-1}(p^\beta)\sigma_{k-1}(m).$$

Observe that

$$p^{k-1}\sigma_{k-1}(p^{\beta-1})+\sigma_{k-1}(p^{\beta+1}) = 1^{k-1}+2p^{k-1}+...+2p^{\beta(k-1)}+p^{(\beta+1)(k-1)} = (1+p^{k-1})\sigma_{k-1}(p^\beta).$$

Thus,

$$a_n(T_{p,k}G_k) = (1 + p^{k-1})a_n(G_k) \text{ if } p \mid n.$$

If $p \nmid n$, we have

$$a_n(T_{p,k}G_k) = \sigma_{k-1}(np) = \sigma_{k-1}(n)\sigma_{k-1}(p)$$

and

$$a_n(G_k) = \sigma_{k-1}(n).$$

So,

$$(1+p^{k-1})a_n(G_k) = (1+p^{k-1})\sigma_{k-1}(n) = \sigma_{k-1}(n)\sigma_{k-1}(p) = a_n(T_{p,k}G_k).$$

Therefore for all $n \in \mathbb{N}$, $a_n(T_{p,k}G_k) = (1+p^{k-1})a_n(G_k)$ which implies

$$T_{p,k}G_k = (1+p^{k-1})G_k.$$

□

The last lemma shows that T_p sends Eisenstein series to Eisenstein series. More generally, we have the following theorem.

Theorem 41. *(a) If $f \in S_k$ then $T_p f \in S_k$.*

(b) If $f \in M_k$ then $T_p f \in M_k$.

The above theorem confirms that the function $T_p f$ is also a modular form of weight k. Thus, Hecke operators preserve the space of modular forms of a given weight. For a proof, see Theorem 90 in Royer, 2013.

4.2 Eigenfunctions of the Hecke Operator

Definition 42 (Eigenfunction)**.** Let $f \in M_k$ and $f \neq 0$. We call f an *eigenfunction* for the Hecke operator T_n if there exists $\lambda_n \in \mathbb{C}$ such that $T_n(f) = \lambda_n f$. We call λ_n the *eigenvalue* of T_n associated to f.

Definition 43 (Modular Eigenform)**.** A modular form is said to be an *eigenform* if it is an eigenfunction for all Hecke operators T_n for $n \in \mathbb{N}$.

Let $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n$. If f is an eigenform then $a_1(f) \neq 0$ (Proposition 40 in Koblitz, 1993), so we can multiply f by a suitable constant to get the coefficient $a_1(f)$ equal to 1.

Definition 44 (Normalized Eigenform)**.** If f is an eigenform, then f is called *normalized* if $a_1(f) = 1$.

Definition 45 (Primitive Form)**.** We call f a *primitive form* in S_k if it is a normalized eigenform in S_k.

Proposition 46. *All the primitive forms in S_k form an orthogonal basis of S_k. We denote this basis by H_k^*.*

For a proof, see Proposition 95 in Royer, 2013. The prime-indexed coefficients of primitive forms satisfy Deligne's bound (Theorem 8.2 in Deligne, 1974), a special case of Proposition 54.

Theorem 47. *If $f \in S_k$ is a primitive form and p is prime, then*

$$|a_p(f)| \leq 2p^{\frac{k-1}{2}}.$$

We have the following theorem (Proposition 40 in Koblitz, 1993).

Theorem 48. *Let $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n \in M_k$ be a normalized eigenform. If $\lambda_n(f)$ is the eigenvalue of T_n associated to f, then $\lambda_n(f) = a_n(f)$ for all $n > 1$.*

The following results are Corollaries 1 and 2 of Theorem 7 in Chapter VII of Serre, 1970.

Corollary 49 (Multiplicity One)**.** *Let $f, g \in M_k$ be two normalized eigenforms. If $\lambda_n(f) = \lambda_n(g)$ for all n, then $f = g$.*

Corollary 50. *If $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n \in M_k$ is a normalized eigenform, then*

$$a_m(f)a_n(f) = a_{mn}(f), \quad \text{if } (m, n) = 1 \tag{4.3}$$

$$a_p(f)a_{p^n}(f) = a_{p^{n+1}}(f) + p^{k-1}a_{p^{n-1}}(f), \quad \text{if } p \text{ is prime}, n \geq 1. \tag{4.4}$$

The two identities above were first discovered by Ramanujan (for $f = \Delta$) and proved by Mordell. We will also be using some results from linear algebra.

Lemma 51. *Let T be a linear operator defined on a finite-dimensional vector space over $\mathbb{C}$. Let $f = \sum_{i=1}^{r} c_i f_i$ (for some non-zero constants $c_i \in \mathbb{C}$) be such that f and all f_i are eigenvectors under T with eigenvalues a and a_i respectively. If all the f_i are linear independent, then $a = a_i$ for all i.*

Proof. Observe that $Tf_i = a_i f_i$ and $Tf = af = \sum_{i=1}^{r} ac_i f_i$. Since T is linear, $Tf = \sum_{i=1}^{r} c_i T f_i = \sum_{i=1}^{r} c_i a_i f_i$. Hence $\sum_{i=1}^{r} (a - a_i) c_i f_i = 0$. Since all the f_i are linear independent and $c_i \neq 0$ for all i, we have $a - a_i = 0$ for all i. □

Lemma 52. *Let $T : V \to V$ be a linear transformation. Suppose $V = U \oplus W$, where both U and W are T-invariant. Let $v \in V$ be an eigenvector of T with eigenvalue a. If $v = u + w$ for some $u \in U$ and $w \in W$, then either both u and v are eigenvectors of T with eigenvalue a or either u or w is zero.*

Proof. Let $v \in V = U \oplus W$. Then there exist $u \in U$ and $w \in W$ such that $v = u + w$. Suppose u and w are non zero. Observe that $T(u) + T(w) = T(u + w) = T(v) = av = au + aw \implies T(u) - au = -(T(w) - aw)$. Since both U and W are T-invariant, $T(u) - au \in U$ and $T(w) - aw \in W$. Since $U \cap W = \{0\}$, $T(u) = au$ and $T(w) = aw$. □

Some examples of eigenforms are the Eisenstein series and the Δ function. Eisenstein series are the only non-cuspidal eigenforms. By Lemma 40, the eigenvalue of T_p associated to E_k is $\lambda_p(E_k) = 1 + p^{k-1}$. More about eigenfunctions of the Hecke operator is discussed in Chapter VII Section 5.4 in Serre, 1970.

4.3 L-functions of Hecke Eigenforms

Definition 53 (L-function of Modular Form)**.** Let $f(z) = \sum_{n=0}^{\infty} a_n(f) q^n \in M_k$ and $s \in \mathbb{C}$. We define an associated *L-function*

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}.$$

We want to know the condition for which this series converges. So, we need the following estimates.

Proposition 54. *If $f \in S_k$ then there exists a constant $C > 0$ such that*

$$|a_n(f)| \leq Cn^{\frac{k}{2}}.$$

For a proof, see Proposition 1.3.5 in Bump, 1997. The more accurate estimate is $|a_n(f)| \leq Cn^{\frac{k-1}{2}+\epsilon}$ for any $\epsilon > 0$. This was conjectured by Ramanujan (1916) for $f = \Delta$ which is well known by *Ramanujan Conjecture*, and was proved by Deligne (1971). Further discussion for this conjecture can be found in Section 3.5 in Bump, 1997. A particular result for this is in Theorem 47. Next, if f is not a cusp form, we have the following proposition (Corollary of Theorem 5 in Chapter VII in Serre, 1970).

Proposition 55. *If $f \in M_k \backslash S_k$ then there exist two constants $C_1, C_2 > 0$ such that*

$$C_1 n^{k-1} \leq |a_n(f)| \leq C_2 n^{k-1}.$$

Corollary 56. *If $f \in M_k$ then there exists a constant $C > 0$ such that*

$$|a_n(f)| \leq Cn^{k-1}.$$

Now we are ready to prove the following result.

Corollary 57. *The L-function $L(f,s)$ converges absolutely if $\mathrm{Re}(s) > k$.*

Proof. By Corollary 56,

$$\sum_{n=1}^{\infty} \left| \frac{a_n(f)}{n^s} \right| \leq \sum_{n=1}^{\infty} \left| \frac{Cn^{k-1}}{n^s} \right| = C \sum_{n=1}^{\infty} \left| \frac{1}{n^{s-k+1}} \right|.$$

Note that the Dirichlet Series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges if $\mathrm{Re}(s) > 1$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^{s-k+1}}$ converges if $\mathrm{Re}(s-k+1) > 1$, i.e. $\mathrm{Re}(s) > k$. □

Hecke proved that by analytic continuation, L-function $L(f,s)$ can be extended to a meromorphic function on the whole $\mathbb{C}$ (entire if f is a cusp form) and satisfies a functional equation relating $L(f,k-s)$ to $L(\tilde{f},s)$ where $\tilde{f}(z) = f\left(-\frac{1}{z}\right)$. Hecke also proved the converse, that every $L(f,s)$ satisfying this functional equation and some regularity and growth hypothesis implies $f \in M_k$. See Section 33 in Hecke, 1959 for further discussion.

By Proposition 8 and Proposition 46, we know that M_k is spanned by normalized eigenforms $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n \in M_k$ where $a_n(f)$ satisfy the identities in Equation 4.3 and 4.4. Equation 4.3 says that the coefficients $a_n(f)$ are *multiplicative*, and hence the L-function of f has an Euler product

$$L(f,s) = \prod_{p \text{ prime}} \left(1 + \frac{a_p(f)}{p^s} + \frac{a_{p^2}(f)}{p^{2s}} + \ldots\right).$$

Putting Equation 4.3 and 4.4 together, we have

$$L(f,s) = \prod_{p \text{ prime}} \frac{1}{1 - a_p(f)p^{-s} + p^{k-1-2s}}$$

of $f \in M_k$ a normalized eigenform.

Example 58 ($L(E_k,s)$)**.** The L-function of the Eisenstein series E_k is given by

$$L(E_k,s) = \prod_{p \text{ prime}} \frac{1}{1 - (1+p^{k-1})p^{-s} + p^{k-1-2s}} = \zeta(s)\zeta(s-k+1).$$

For a proof, see Chapter VII Proposition 13 in Serre, 1970.

Further, if $f \in S_k$ is a primitive form, we let $A_n(f)$ be defined by

$$f(z) = \sum_{n=1}^{\infty} A_n(f) n^{\frac{k-1}{2}} q^n.$$

Then

$$\tilde{L}(f,s) = \sum_{n=1}^{\infty} \frac{A_n(f)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - A_p(f)p^{-s} + p^{-2s}}$$

is an example of an *automorphic L-function* on $GL(2)$, using the terminology in Iwaniec and Kowalski, 2004 Chapter 5. The *local components of $\tilde{L}(f,s)$ at p* are the complex numbers $\alpha_{1,p}(f)$ and $\alpha_{2,p}(f)$ in the factorisation

$$1 - A_p(f)z + z^2 = (1 - \alpha_{1,p}(f)z)\,(1 - \alpha_{2,p}(f)z)\,.$$

The following result is due to Jacquet and Shalika (see Proposition 5.43 in Iwaniec and Kowalski, 2004).

Theorem 59 (Strong Multiplicity One Principle). *Let $\tilde{L}(f,s)$ and $\tilde{L}(g,s)$ be two automorphic L-functions of cuspforms on $GL(2)$. If the local components of $\tilde{L}(f,s)$ and $\tilde{L}(g,s)$ coincide at all but finitely many primes, then $f = g$.*

Chapter 5

Hecke Operators on Quasimodular Forms

5.1 The Action of Hecke Operators on M_k^s

Recall the space $Hol_\infty(\mathcal{H}/\mathbb{Z})$ from Definition 5. Observe that $M_k^s \subset Hol_\infty(\mathcal{H}/\mathbb{Z})$ by Corollary 34. By Theorem 38, for each prime p we have a linear transformation $T_{p,k} : Hol_\infty(\mathcal{H}/\mathbb{Z}) \to Hol_\infty(\mathcal{H}/\mathbb{Z})$ given by the formula

$$T_{p,k} f(z) = p^{k-1} f(pz) + \frac{1}{p} \sum_{n=0}^{p-1} f\left(\frac{z+n}{p}\right). \tag{5.1}$$

There are also similar operators $T_{n,k}$ for all $n \in \mathbb{N}$. If we fix k, then by direct calculation with 5.1, $T_{n,k}$ and $T_{m,k}$ commute for all $n, m \in \mathbb{N}$, i.e.

$$T_{n,k} \circ T_{m,k} = T_{m,k} \circ T_{n,k}.$$

Further, these operators satisfy the identities 4.1 and 4.2 as before.

Define $D := \frac{1}{2\pi i}\frac{d}{dz}$. If $f(z) = \sum_{n=0}^{\infty} a_n(f) e^{2\pi i n z}$, we obtain $Df(z) = \sum_{n=0}^{\infty} n a_n e^{2\pi i n z}$.

Proposition 60. *For any $f \in Hol_\infty(\mathcal{H}/\mathbb{Z})$, we have $T_{p,k+2}(Df) = pD(T_{p,k} f)$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n(f) e^{2\pi n z}$. Then $T_{p,k} f(z) = \sum_{n=0}^{\infty} a_n(T_{p,k} f) e^{2\pi n z}$. First, note that if $f(z) = \sum_{n=0}^{\infty} a_n(f) e^{2\pi n z}$ then $Df(z) = \sum_{n=0}^{\infty} n a_n(f) e^{2\pi n z}$. Recall Proposition

39:

$$a_n(T_{p,k}f) = \begin{cases} p^{k-1}a_{n/p}(f) + a_{pn}(f), & \text{if } p \mid n \\ a_{pn}(f), & \text{if } p \nmid n. \end{cases}$$

Thus,

$$T_{p,k+2}(Df)(z) = \sum_{n=0}^{\infty} \begin{cases} \left(p^{k+1}\frac{n}{p}a_{n/p}(f) + pna_{pn}(f)\right) e^{2\pi nz}, & \text{if } p \mid n \\ (pna_{pn}(f))\, e^{2\pi nz}, & \text{if } p \nmid n. \end{cases}$$

On the other hand,

$$pD(T_{p,k}f)(z) = p\sum_{n=0}^{\infty} a_n(T_{p,k}f)e^{2\pi nz}.$$

Again, by Proposition 39,

$$pD(T_{p,k}f)(z) = \sum_{n=0}^{\infty} \begin{cases} \left(pnp^{k-1}a_{n/p}(f) + pna_{pn}(f)\right) e^{2\pi nz}, & \text{if } p \mid n \\ (pna_{pn}(f))\, e^{2\pi nz}, & \text{if } p \nmid n. \end{cases}$$

Simplifying both cases, we obtain the equality, i.e. $T_{p,k+2}(Df) = pD(T_{p,k}f)$. □

The above proposition shows that the Hecke operators and the derivatives commute up to multiplication by a scalar.

$$\begin{array}{ccc} Hol_\infty(\mathcal{H}/\mathbb{Z}) & \xrightarrow{\quad T_{p,k} \quad} & Hol_\infty(\mathcal{H}/\mathbb{Z}) \\ D \big\downarrow & \circlearrowleft & \big\downarrow pD \\ Hol_\infty(\mathcal{H}/\mathbb{Z}) & \xrightarrow{\quad T_{p,k+2} \quad} & Hol_\infty(\mathcal{H}/\mathbb{Z}) \end{array}$$

The following theorem confirms that the image of the Hecke operator of any quasimodular form of a given weight and depth is again a quasimodular form of the same weight and depth, i.e. $T_{p,k}(M_k^s) \subset M_k^s$.

Theorem 61. *If* $f \in M_k^s$ *then* $T_{p,k} f \in M_k^s$.

Proof. Set $G_2 = -\frac{1}{24}E_2$. Let $f \in M_k^s$ and let $s' = \min\left(s, \frac{k}{2} - 2\right)$. By Theorem 33, there exist modular forms $f_i \in M_{k-2i}$, complex numbers $c_i \in \mathbb{C}$ for $i = 0, 1, ..., s'$, and $c_{k/2} \in \mathbb{C}$ (note: $c_{k/2} \neq 0$ if and only if $s = \frac{k}{2}$) such that

$$f = \sum_{i=0}^{s'} c_i D^i f_i + c_{k/2} D^{k/2-1} G_2.$$

Then by Proposition 60,

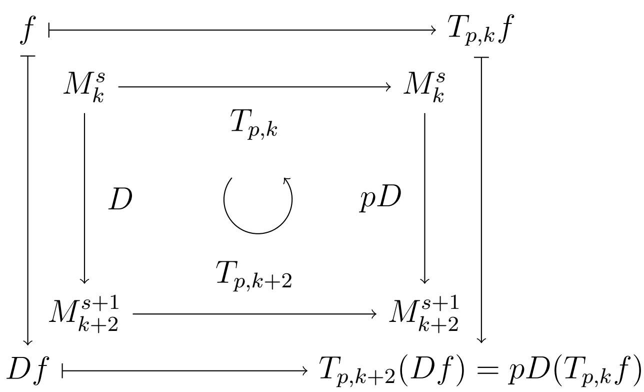
$$\begin{aligned} T_{p,k} f &= \sum_{i=0}^{s'} c_i T_{p,k}(D^i f_i) + c_{k/2} T_{p,k}(D^{k/2-1} G_2) \\ &= \sum_{i=0}^{s'} c_i p^i D^i (T_{p,k-2i} f_i) + c_{k/2} p^{k/2-1} D_{k/2-1}(T_{p,2} G_2). \end{aligned}$$

By Lemma 40, we have $T_{p,2} G_2 = \sigma_1(p) G_2 = (p+1) G_2$. And since $f_i \in M_{k-2i}$, hence $T_{p,k-2i} f_i \in M_{k-2i}$ by Theorem 41. Therefore,

$$T_{p,k} f = \sum_{i=0}^{s'} c_i p^i D^i g_i + c_{k/2} p^{k/2-1} (p+1) D_{k/2-1} G_2 \in M_k^s.$$

□

The Proposition 60 together with Theorems 25 and 61 show that the following diagram commutes.



5.2 Quasimodular Eigenforms

Definition 62 (Eigenfunction)**.** Let $f(z) \in M_k^{\leq\infty}$ and $f \neq 0$. We call f an *eigenfunction* for the Hecke operator T_n if there exists $\lambda_n \in \mathbb{C}$ such that $T_n(f) = \lambda_n f$. We call λ_n the *eigenvalue* of T_n associated to f.

Definition 63 (Quasimodular Eigenform)**.** A quasimodular form is said to be an *eigenform* if it is an eigenfunction for all of the Hecke operators T_n for $n \in \mathbb{N}$. Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n$, f is *normalized* if $a_1(f) = 1$.

Proposition 64. *Let $k \geq 2$. We define*

$$H_k^{\leq\infty} = \bigcup_{i=0}^{\frac{k}{2}-2} D^i H_{k-2i}^*$$

and

$$N_k^{\leq\infty} = \left\{ D^i G_{k-2i} \middle| 0 \leq i \leq \frac{k}{2} - 2 \right\} \bigcup \left\{ D^{\frac{k}{2}-1} G_2 \right\}.$$

Then $H_k^{\leq\infty} \bigcup N_k^{\leq\infty}$ forms a basis for $M_k^{\leq\infty}$ and it consists of quasimodular eigenforms.

For a proof, see Royer, 2013 Proposition 147. The following is a generalisation of Theorem 48.

Theorem 65. *Let $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n \in M_k^{\leq\infty}$ be a quasimodular eigenform. Then*

(a) $a_1(f) \neq 0$.

(b) If f is normalized and λ_n is the eigenvalue of T_n associated to f, then $\lambda_n = a_n(f)$ for all $n > 1$.

Since the Hecke operators of quasimodular forms satisfy the identities 4.1 and 4.2, by Theorem 65 above, we have a direct consequence which is a generalisation of Corollary 50 as follows:

Corollary 66. *If $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n \in M_k^s$ is a normalized eigenform, then*

$$a_m(f)a_n(f) = a_{mn}(f), \quad \textit{if } (m,n) = 1 \tag{5.2}$$

$$a_p(f)a_{p^n}(f) = a_{p^{n+1}}(f) + p^{k-1}a_{p^{n-1}}(f), \quad \text{if } p \text{ is prime}, n \geq 1. \tag{5.3}$$

Furthermore, one may check it immediately by direct computation using Proposition 70. The corollary below is a straightforward generalisation of Proposition 60 and Theorem 61.

Corollary 67. *If $f \in M_k^{\leq\infty}$, then*

$$T_{p,k+2m}(D^m f) = p^m D^m(T_{p,k}f),$$

for $m \geq 0$. Moreover, $D^m f$ is a quasimodular eigenform for T_p if and only if f is. Furthermore, if λ_p is the eigenvalue of T_p associated to f, then $p^m\lambda_p$ is the eigenvalue of T_p associated to $D^m f$.

The following two results are Proposition 2.4 and Proposition 2.5 respectively in Meher, 2012.

Proposition 68. *Let $\{f_i\}_i$ be a collection of non-zero modular forms of distinct weights k_i. Then for $a_i \in \mathbb{C}^*$, $\sum_{i=1}^t a_i D^{n-\frac{k_i}{2}} f_i$ is an eigenform if and only if each $D^{n-\frac{k_i}{2}} f_i$ is an eigenform and the eigenvalues are the same for all i.*

Proposition 69. *If $k > l$ and $f \in M_k$, $g \in M_l$ are eigenforms, then for all $r \geq 0$, $D^{\frac{k-l}{2}+r} g$ and $D^r f$ do not have the same eigenvalues.*

These are used in Das and Meher, 2015 to obtain the following classification of quasimodular eigenforms:

Proposition 70. *Let $f \in M_k^s$ be a quasimodular eigenform.*

(a) If $s < \frac{k}{2}$, then $f = D^s f_s$, where $f_s \in M_{k-2s}$ is an eigenform.

(b) If $s = \frac{k}{2}$, then $f \in \mathbb{C}D^{\frac{k}{2}-1}E_2$.

Proof. Let $f \in M_k^s$ be a quasimodular eigenform. By Theorem 33:

(a) If $s < \frac{k}{2}$, then

$$f = F_0 + DF_1 + D^2F_2 + ... + D^sF_s \tag{5.4}$$

for some modular forms $F_i \in M_{k-2i}$ where $i = 0, 1, ..., s$. We claim that there is only one non-zero term on the right hand side of Equation 5.4. Assume, on the contrary, that there are at least two non-zero terms D^iF_i and D^jF_j with $i < j$. By Proposition 68, D^iF_i and D^jF_j are eigenforms with $\lambda_n(D^iF_i) = \lambda_n(D^jF_j)$ for all $n \in \mathbb{N}$. By Proposition 67, F_i and F_j are (modular) eigenforms of distinct weights $k-2i > k-2j$. Applying Proposition 69 with $r = i$, we get $\lambda_n(D^jF_j) \neq \lambda_n(D^iF_i)$, contradiction. So, $f = D^aF_a$ for some $0 \leq a \leq s$ and some eigenform $F_a \in M_{k-2a}$. By Theorem 25, $depth(D^aF_a) = a$. In the other hand, $depth(f) = s$. Thus, $a = s$. In other words, $f = D^sF^s$ where $F_s \in M_{k-2s}$ is an eigenform.

(b) If $s = \frac{k}{2}$, then there exists a non-zero $\alpha \in \mathbb{C}$ such that

$$f = F_0 + DF_1 + ... + D^{\frac{k}{2}-2}F_{\frac{k}{2}-2} + \alpha D^{\frac{k}{2}-1}E_2 \tag{5.5}$$

for some modular forms $F_i \in M_{k-2i}$ where $i = 0, 1, ..., \frac{k}{2} - 2$. In fact,

$$M_k^{k/2} = \bigoplus_{i=0}^{k/2-2} D^iM_{k-2i} \oplus \mathbb{C}D^{\frac{k}{2}-1}E_2.$$

By Lemma 52, $g = F_0 + DF_1 + ... + D^{\frac{k}{2}-2}F_{\frac{k}{2}-2} \in M_k^{k/2-2}$ is either zero or an eigenform with $\lambda_n(g) = \lambda_n(f)$, and $\alpha D^{\frac{k}{2}-1}E_2 \in M_k^{k/2}$ is an eigenform with $\lambda_n\left(\alpha D^{\frac{k}{2}-1}E_2\right) = \lambda_n(f)$. Assume $g \neq 0$. By part (a), $g = D^{\frac{k}{2}-2}F_{\frac{k}{2}-2}$ with $F_{\frac{k}{2}-2} \in M_4$ an eigenform. Then we have

$$\lambda_n\left(D^{\frac{k}{2}-2}F_{\frac{k}{2}-2}\right) = \lambda_n\left(\alpha D^{\frac{k}{2}-1}E_2\right).$$

By Proposition 67,

$$n^{\frac{k}{2}-2}\lambda_n(F_{\frac{k}{2}-2}) = n^{\frac{k}{2}-1}\lambda_n(E_2), \quad \forall n \in \mathbb{N}.$$

By Lemma 40, for $n = p$ prime,

$$p^{\frac{k}{2}-2}\lambda_p(F_{\frac{k}{2}-2}) = p^{\frac{k}{2}-1}\lambda_p(E_2) = p^{\frac{k}{2}-1}(p+1).$$

Hence,

$$\lambda_p(F_{\frac{k}{2}-2}) = p(p+1) \tag{5.6}$$

where $F_{\frac{k}{2}-2} \in M_4$ an eigenform. By Proposition 8, $M_4 = S_4 \oplus \mathbb{C}E_4$. So, if $F_{\frac{k}{2}-2} \in S_4$ (without loss of generality, we assume that $F_{\frac{k}{2}-2}$ is a primitive form), then by Theorem 47 of Deligne,

$$\left|\lambda_p(F_{\frac{k}{2}-2})\right| \leq 2p^{\frac{3}{2}},$$

which contradicts Equation 5.6. In the other hand, if $F_{\frac{k}{2}-2} = \beta E_4$ for some non-zero $\beta \in \mathbb{C}$, then by Lemma 40,

$$\lambda_p(F_{\frac{k}{2}-2}) = 1 + p^3,$$

which also contradicts Equation 5.6. Therefore, $g = 0$. Hence, $f = \alpha D^{\frac{k}{2}-1}E_2$.

□

The description of quasimodular eigenforms given in the proposition above is crucial to the main result of Das and Meher, 2015:

Theorem 71 (Multiplicity One). *Let $f_1, f_2 \in M_k^{\leq\infty}$ be quasimodular eigenforms. If f_1, f_2 have same eigenvalues with respect to the Hecke operators T_p for all but finitely many primes p, then $f_1 = cf_2$ for some constant c.*

Proof. Let p be prime and let $f_1, f_2 \in M_k^{\leq\infty}$ be quasimodular eigenforms such that

$$T_p(f_1) = (\lambda_1)_p f_1, \quad \forall p,$$

$$T_p(f_2) = (\lambda_2)_p f_2, \quad \forall p.$$

Suppose $(\lambda_1)_p = (\lambda_2)_p$ for all but finitely many primes p. By Proposition 70, $\exists t_1, t_2 \in \mathbb{N}$ such that

$$f_1 = D^{t_1} g_1 \text{ and } f_2 = D^{t_2} g_2$$

where either $g_1 \in M_{k-2t_1}$ is an eigenform or $g_1 = \alpha E_2$ for some $\alpha \in \mathbb{C}$, and either $g_2 \in M_{k-2t_2}$ is an eigenform or $g_2 = \beta E_2$ for some $\beta \in \mathbb{C}$. Without loss of generality, we assume that g_1 and g_2 are normalized.

Since $M_k = S_k \oplus \mathbb{C}E_k$ by Proposition 8, so there are three cases:

(a) $g_1 = \alpha E_{k_1}$ and $g_2 = \beta E_{k_2}$, where $\alpha, \beta \in \mathbb{C}$ and $k_i \in \{2, k-2t_1, k-2t_2\}$.
By Lemma 40, $T_p(g_i) = (1 + p^{k_i - 1}) g_i$ for all $i \in \{1, 2\}$. Then by Proposition 67, we have

$$T_p(f_i) = p^{t_i}(1 + p^{k_i-1}) f_i = (\lambda_i)_p f_i.$$

Let p be such that $(\lambda_1)_p = (\lambda_2)_p$. This implies $t_1 = t_2$ and $k_1 = k_2$. Hence $g_1 = \frac{\alpha}{\beta} g_2$, so $f_1 = \frac{\alpha}{\beta} f_2$ follows directly.

(b) $g_1 = \alpha E_{k_1}$ where $k_1 \in \{2, k-2t_1\}$ and $g_2 \in S_{k-2t_2}$.
Again by Lemma 40 and Proposition 67, we have

$$(\lambda_1)_p = p^{t_1}(1 + p^{k_1-1}), \quad \forall p.$$

Let $g_2 = \sum_{n=1}^{\infty} a_n(g_2) q^n$. Since g_2 is normalized, by Theorem 65 and Proposition 67, we have

$$(\lambda_2)_p = p^{t_2} a_p(g_2), \quad \forall p.$$

In other hand, by Theorem 5 in Murty, 1983 we have the following fact: If $g \in S_k$ and the Fourier coefficients of g are real, then there exist infinitely many primes p such that $a_p(g) > 0$ and there exist infinitely many primes p' such that $a_{p'} < 0$. Thus, $(\lambda_1)_p = (\lambda_2)_p$ for all but finitely many primes p contradicts this fact. Hence, this case is ruled out.

(c) $g_1 \in S_{k-2t_1}$ and $g_2 \in S_{k-2t_2}$.
Let $g_1 = \sum_{n=1}^{\infty} a_n(g_1)q^n$ and $g_2 = \sum_{n=1}^{\infty} a_n(g_2)q^n$. By Theorem 47 of Deligne, $|a_p(g_i)| \leq 2p^{(k-2t_i-1)/2}$. Hence,

$$\left| \frac{a_p(g_i)}{p^{(k-2t_i-1)/2}} \right| \leq 2.$$

Let $A_p(g_i) = \left| \frac{a_p(g_i)}{p^{(k-2t_i-1)/2}} \right|$. Then $a_p(g_i) = A_p(g_i)p^{(k-2t_i-1)/2}$. Since both g_1 and g_2 are normalized, by Theorem 65 and Proposition 67, we have

$$(\lambda_i)_p = p^{t_i} a_p(g_i) = p^{\frac{k-1}{2}} A_p(g_i).$$

But since $(\lambda_1)_p = (\lambda_2)_p$ for all but finitely many primes p, so $A_p(g_1) = A_p(g_2)$ for all but finitely many primes p. Hence the local components of $\tilde{L}(g_1, s)$ and $\tilde{L}(g_2, s)$ coincide at all but finitely many primes p. By Theorem 59, $g_1 = g_2$, so $t_1 = t_2$ and hence $f_1 = f_2$.

□

5.3 L-functions of Quasimodular Eigenforms

We have seen L-functions of modular Hecke eigenforms in Section 4.3. The next question is: can we attach L-functions to quasimodular forms? By Corollary 35, any quasimodular form has a Fourier expansion. In this section, we want to make a generalisation of our previous results.

Definition 72 (L-function of Quasimodular Form)**.** Let $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n \in M_k^s$ and $t \in \mathbb{C}$. We define an associated *L-function*

$$L(f,t) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^t}.$$

We will see shortly under what condition this series converges. We know that the Fourier coefficients of modular forms are bounded above. The following propositions are a generalisation of Corollary 56.

Proposition 73. *If $f \in M_k^s$ with $k > 2$, then there exists a constant $C > 0$ such that*

$$|a_n(f)| \leq Cn^{k-1}.$$

Proof. By Corollary 35:

(a) If $s < \frac{k}{2}$, then $f = \sum_{n=0}^{\infty} a_n(f)q^n$ where $a_n(f) = a_n(F_0) + na_n(F_1) + ... + n^s a_n(F_s)$. Hence by Corollary 56, there exist positive constants C_i, C for all $i \in \{0, 1, ..., s\}$ such that

$$\begin{aligned} |a_n(f)| &\leq |a_n(F_0)| + |na_n(F_1)| + ... + |n^s a_n(F_s)| \\ &\leq C_0 n^{k-1} + nC_1 n^{k-3} + ... + n^s C_s n^{k-2s-1} \\ &= C_0 n^{k-1} + C_1 n^{k-2} + ... + C_s n^{k-s-1} \\ &\leq C(s+1)n^{k-1}. \end{aligned}$$

(b) If $s = \frac{k}{2}$, then $f = \sum_{n=0}^{\infty} a_n(f)q^n$ where $a_n(f) = a_n(F_0) + na_n(F_1) + ... + n^{\frac{k}{2}-2} a_n(F_{\frac{k}{2}-2}) + \alpha n^{\frac{k}{2}-1} a_n(E_2)$. Hence by Corollary 56 and Proposition 31, there exist a non-zero $\alpha \in \mathbb{C}$ and positive constants C_i, C for all $i \in \{0, 1, ..., \frac{k}{2}-$

2} such that for all $n \geq 1$,

$$\begin{aligned}|a_n(f)| &\leq |a_n(F_0)| + |na_n(F_1)| + ... + |n^{\frac{k}{2}-2}a_n(F_{\frac{k}{2}-2})| + |\alpha n^{\frac{k}{2}-1}a_n(E_2)| \\ &\leq C_0 n^{k-1} + nC_1 n^{k-3} + ... + n^{\frac{k}{2}-2}C_{\frac{k}{2}-2}n^3 + 24|\alpha|n^{\frac{k}{2}-1}\sigma_1(n) \\ &= C_0 n^{k-1} + C_1 n^{k-2} + ... + C_{\frac{k}{2}-2}n^{\frac{k}{2}+1} + 24|\alpha|n^{\frac{k}{2}-1}\sigma_1(n).\end{aligned}$$

Note that $\sigma_1(n) \leq n^2$ for all n. Since $k > 2$, $n^{\frac{k}{2}-1}\sigma_1(n) \leq n^{k-1}$. Therefore,

$$|a_n(f)| \leq \frac{Ck}{2}n^{k-1}.$$

□

Proposition 74. *If $f \in M_2^1$ then there exists a constant $C > 0$ such that*

$$|a_n(f)| \leq Cn^2.$$

Proof. By Corollary 35 and Proposition 31, there exists a non-zero $\alpha \in \mathbb{C}$ such that

$$|a_n(f)| = |\alpha a_n(E_2)| = 24|\alpha|\sigma_1(n) \leq 24|\alpha|n^2.$$

□

Corollary 75. *If $f \in M_k^s$ with $k > 2$, then the L-function $L(f,t)$ converges absolutely if $\mathrm{Re}(t) > k$.*

The proof is the same as of Corollary 57 replacing Corollary 56 with Proposition 73.

Corollary 76. *If $f \in M_2^1$ then the L-function $L(f,t)$ converges absolutely if $\mathrm{Re}(t) > 3$.*

Proof. By Proposition 74,

$$\sum_{n=1}^{\infty}\left|\frac{a_n(f)}{n^t}\right| \leq \sum_{n=1}^{\infty}\left|\frac{Cn^2}{n^t}\right| = C\sum_{n=1}^{\infty}\left|\frac{1}{n^{t-2}}\right|.$$

Note that $\sum_{n=1}^{\infty} \frac{1}{n^{t-2}}$ converges if $\mathrm{Re}(t-2) > 1$, i.e. $\mathrm{Re}(t) > 3$. □

Now by Proposition 64, we have M_k^s spanned by normalized quasimodular eigenforms $f(z) = \sum_{n=0}^{\infty} a_n(f) q^n \in M_k^s$ where by Corollary 66, $a_n(f)$ satisfy the identities in Equation 5.2 and 5.3. Thus, if $f \in M_k^s$ is a normalized eigenform,

$$L(f,t) = \prod_{p \text{ prime}} \left(1 + \frac{a_p(f)}{p^t} + \frac{a_{p^2}(f)}{p^{2t}} + \ldots\right) = \prod_{p \text{ prime}} \frac{1}{1 - a_p(f)p^{-t} + p^{k-1-2t}}.$$

Bibliography

Bump, Daniel (1997). *Automorphic forms and representations*. Vol. 55. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, pp. xiv+574. ISBN: 0-521-55098-X. DOI: 10.1017/CBO9780511609572. URL: http://dx.doi.org/10.1017/CBO9780511609572.

Das, Soumya and Jaban Meher (2015). "On quasimodular eigenforms". In: *Int. J. Number Theory* 11.3, pp. 835–842. ISSN: 1793-0421. DOI: 10.1142/S1793042115500451. URL: http://dx.doi.org/10.1142/S1793042115500451.

Deligne, Pierre (1974). "La conjecture de Weil. I". In: *Inst. Hautes Études Sci. Publ. Math.* 43, pp. 273–307. ISSN: 0073-8301.

Hecke, Erich (1959). *Mathematische Werke*. Herausgegeben im Auftrage der Akademie der Wissenschaften zu Göttingen. Vandenhoeck & Ruprecht, Göttingen, 955 pp. (1 plate).

Iwaniec, Henryk and Emmanuel Kowalski (2004). *Analytic number theory*. Vol. 53. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, pp. xii+615. ISBN: 0-8218-3633-1. DOI: 10.1090/coll/053. URL: http://dx.doi.org/10.1090/coll/053.

Koblitz, Neal (1993). *Introduction to elliptic curves and modular forms*. Second. Vol. 97. Graduate Texts in Mathematics. Springer-Verlag, New York, pp. x+248. ISBN: 0-387-97966-2. DOI: 10.1007/978-1-4612-0909-6. URL: http://dx.doi.org/10.1007/978-1-4612-0909-6.

Meher, Jaban (2012). "Some remarks on Rankin-Cohen brackets of eigenforms". In: *Int. J. Number Theory* 8.8, pp. 2059–2068. ISSN: 1793-0421. DOI: 10.1142/S1793042112501175. URL: http://dx.doi.org/10.1142/S1793042112501175.

Murty, M. Ram (1983). “Oscillations of Fourier coefficients of modular forms”. In: *Math. Ann.* 262.4, pp. 431–446. ISSN: 0025-5831. DOI: 10.1007/BF01456059. URL: http://dx.doi.org/10.1007/BF01456059.

Royer, Emmanuel (2012). “Quasimodular forms: an introduction”. In: *Ann. Math. Blaise Pascal* 19.2, pp. 297–306.

— (2013). “Un cours “Africain” sur les formes modulaires”.

Serre, Jean-Pierre (1970). *Cours d’arithmétique*. Vol. 2. Collection SUP: “Le Mathématicien”. Presses Universitaires de France, Paris, p. 188.

Weinstein, Jared (2016). “Reciprocity laws and Galois representations: recent breakthroughs”. In: *Bull. Amer. Math. Soc. (N.S.)* 53.1, pp. 1–39. ISSN: 0273-0979. DOI: 10.1090/bull/1515. URL: http://dx.doi.org/10.1090/bull/1515.

Zagier, Don (2008). “Elliptic modular forms and their applications”. In: *The 1-2-3 of modular forms*. Universitext. Springer, Berlin, pp. 1–103.