Modular forms and Rankin–Cohen brackets :
A representation-theoretic understanding

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ABSTRACT

This thesis gives a detailed survey on the existing literature regarding a representation-theoretic perspective of modular forms and Rankin–Cohen brackets. It delineates the association of a modular form to a representation of $\mathrm{SL}(2,\mathbb{R})$, synthesising a number of methods in the literature. It then outlines the way in which the Rankin–Cohen brackets arise naturally in such representations. Finally, generalisations of Rankin–Cohen brackets and a representation-theoretic understanding of Siegel modular forms are discussed, and we suggest areas for future research.

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INTRODUCTION

Initially arising from the study of elliptic functions, modular forms are a class of holomorphic functions on the upper half plane that are invariant under the action of a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Since their discovery, modular forms have been found to appear naturally in various areas of mathematics. One area with which they are deeply connected is number theory, where the study of these forms has been used to solve many arithmetic and counting problems. These include determining the representations of numbers as sums of squares and, perhaps most notably, Fermat's Last Theorem as proved by Andrew Wiles [Wil95]. It is not just number theory where these forms arise however - they have important uses in many other areas, from group theory and mathematical physics to representation theory, the latter of which we will use in this paper. In group theory, a modular form was used to prove the Monstrous Moonshine conjecture, while an application in mathematical physics relates modular forms to the counting of black hole states. The connections between modular forms and representation theory form a large part of the global Langlands Program, which in itself contains a number of conjectures connecting Galois representations and automorphic forms. Given the wide reach of modular forms and their applications, they are a valuable subject of research.

Since modular forms are holomorphic, we can consider differentiating them. While the derivative of a modular form is not a modular form, there are a number of ways to define differential operators which do produce modular forms. This research will focus on one such family of differential operators, known as the Rankin–Cohen brackets, which not only add to the theory of modular forms, but also arise naturally in a representation-theoretic setting.

In the 1950s, Rankin [Ran57] described types of differential operators which send modular forms to modular forms. These were later defined explicitly by Cohen [Coh75]. Using these results, Zagier [Zag94] defined a family of such differential operators, which were named Rankin-Cohen brackets after their earlier innovators. These brackets are bi-differential operators on modular forms which produce a third modular form of a higher weight. Since Zagier's paper formalising these brackets, much has been written on their structure and properties.

We focus here on the representation-theoretic perspective of modular forms and the Rankin–

Cohen brackets, beginning with the association of a representation of $\mathrm{GL}(2, \mathbb{A})$ to a modular form. This process is outlined in detail by Bump [Bum98], and Booher [Boo15], and from a slightly different approach by [Pev12]. This allows for the study of modular forms from a representation-theoretic point of view, which provides us with additional knowledge of their properties.

In the present paper, we look at this representation at infinity, to gain a representation of $\mathrm{GL}(2, \mathbb{R})$, and then $\mathrm{SL}(2, \mathbb{R})$, associated to a modular form. This space is chosen as the Rankin–Cohen brackets of two modular forms arise naturally in these representations – as projection maps of the tensor product of representations associated with the forms onto specific representations in their decomposition [Pev12] [Yao14]. Using a representation-theoretic approach in this way can provide alternative proofs for the modularity and uniqueness of the Rankin–Cohen brackets.

Given the insights gained from a representation-theoretic approach to modular forms, one may wish to generalise this approach. One generalisation of classical modular forms are Siegel modular forms, which act on a higher-dimensional version of the upper half plane. Both Rankin–Cohen brackets and the representations associated to modular forms have been generalised to classical Siegel modular forms by a number of authors. The study of differential operators on Siegel modular forms began with Ibukiyama's [Ibu+99] work on the relationship between pluri-harmonic polynomials and differential operators on automorphic forms. Using this, Eholzer and Ibukiyama [EI98] have shown that bi-differential operators on Siegel modular forms always exist. While a general closed form has not been found, a number of authors have given these operators for exact degrees and other specifications, such as can be seen in [CE98] and [IR06].

Turning to representation theory, a representation of $\mathrm{GSp}(2n, \mathbb{A})$ can be associated to a Siegel modular form of degree n. In this association, we see many similarities to the classical case. This was first shown by Asgari and Schmidt [AS01], and has been expanded upon recently by Pitale [Pit19]. The representation-theoretic understanding of the Rankin–Cohen brackets outlined in this paper has not yet been shown to generalise to Siegel modular forms.

The present research gives a detailed survey of the current literature regarding a representation-theoretic approach to modular forms and Rankin–Cohen brackets. It begins with a brief introduction of modular forms, Rankin–Cohen brackets and representation theory. Then, it works through the association of a modular form to a representation of $\mathrm{SL}(2, \mathbb{R})$, synthesising a number of different approaches in the existing literature. The natural occurrence of Rankin–

Cohen brackets in this setting is then delineated, along with two examples of the applications of this approach. The existing generalisations for Siegel modular forms are then given, first regarding the generalisations of Rankin–Cohen brackets, and second of a representation-theoretic approach to the study of Siegel modular forms. Finally, we suggest avenues for further research which follow the method taken here for classical modular forms.

CHAPTER 1

MODULAR FORMS

We begin by providing definitions of the objects of study in this thesis; modular forms and Rankin–Cohen brackets, as well as some properties which will be used in later chapters.

1.1 Preliminary definitions and notation

Let $\mathcal{H}$ represent the complex upper-half plane:

$$\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}.$$

The special linear group $\mathrm{SL}(2, \mathbb{R})$ is given by

$$\mathrm{SL}(2, \mathbb{R}) = \{\gamma \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \mid \det(\gamma) = 1\},$$

which is the group of 2×2 matrices with real entries and determinant 1. An important subgroup of $\mathrm{SL}(2, \mathbb{R})$ is those matrices with only integer entries, denoted $\mathrm{SL}(2, \mathbb{Z})$.

$\mathrm{SL}(2, \mathbb{R})$ acts on $\mathcal{H}$ by linear fractional transformations:

$$\gamma \cdot z := \frac{az + b}{cz + d},$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$ and $z \in \mathcal{H}$.

A quick calculation shows that for $z \in \mathcal{H}$ and $\gamma \in \mathrm{SL}(2, \mathbb{R})$, $\gamma \cdot z \in \mathcal{H}$, since:

$$\operatorname{Im}(\gamma \cdot z) = \frac{\operatorname{Im}(z)}{|cz + d|^2} > 0.$$

Additionally, this describes an action on $\mathcal{H}$ since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot z = z$ and $(\gamma\gamma') \cdot z = \gamma \cdot (\gamma' \cdot z)$ for all $\gamma, \gamma' \in \mathrm{SL}(2, \mathbb{R})$, and $z \in \mathcal{H}$.

1.2 Modular forms of level 1

Modular forms are particular holomorphic functions on the upper half plane, which are invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathcal{H}$, defined as follows.

Definition 1.1. Let k be a non-negative integer. A *modular form* of weight k and level 1 is a complex valued function $f : \mathcal{H} \to \mathbb{C}$ such that:

1. f is holomorphic.
2. f satisfies the following modularity condition:

$$f(\gamma \cdot z) = (cz+d)^k f(z)$$

   for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathcal{H}$.

3. f is holomorphic at infinity, i.e. $\lim_{z\to i\infty} f(z)$ exists.

The space of all modular forms of weight k defines a vector space, which we denote $\mathcal{M}_k$. We also define the *algebra of modular forms*, given by $\mathcal{M} := \bigoplus_k \mathcal{M}_k$.

Definition 1.2. Given $f \in \mathcal{M}_k$, $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$, we define the *slash operator* as

$$(f|_k\gamma)(z) := (cz+d)^{-k} f(\gamma \cdot z)$$

so that the modularity condition can be rewritten as

$$(f|_k\gamma)(z) = f(z) \tag{1.1}$$

for $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

The slash operator defines a group action, since $f|_k\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = f$, and $(f|_k\gamma)|_k(\gamma') = f|_k(\gamma\gamma')$.

Definition 1.3. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define the *automorphy factor* $j(\gamma, z) := cz + d$. It satisfies the following cocycle condition [Kud04]:

$$j(\gamma\gamma', z) = j(\gamma, \gamma'(z))j(\gamma', z). \tag{1.2}$$

Example 1.4. *Eisenstein Series*

A key family of modular forms are the Eisenstein series, denoted $E_k(z)$ for $k > 2$.

For each k, we define

$$E_k(z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k} \tag{1.3}$$

where (c, d) denotes the greatest common divisor of c and d.

For each $k > 2$, the Eisenstein series E_k is a modular form of weight k.

The Eisenstein series are the main examples of modular forms, and interestingly we have that E_4 and E_6 freely generate $\mathcal{M}$ [Zag08, Proposition 4].

By considering the modularity condition for specific matrices in $\mathrm{SL}(2, \mathbb{Z})$, we obtain a number of useful properties of modular forms.

First, $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is in $\mathrm{SL}(2, \mathbb{Z})$, and in this case the modularity condition gives

$$f(z) = (-1)^k f(z).$$

So if $k > 0$ is odd, we have $f(z) = -f(z)$ and thus f is the zero function. Hence, if $k > 0$, non-zero modular forms of weight k exist only when k is even.

When k is even, the dimension of $\mathcal{M}_k$ is also always finite, and we have the following result:

Proposition 1.5. *When $k < 0$ or k is odd,* $\dim(\mathcal{M}_k) = 0$. *Then, for even $k \geq 0$, we have*

$$\dim(\mathcal{M}_k) = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{otherwise} \end{cases}$$

Proof. See [Zag08, Corollary 1]. □

$\mathrm{SL}(2, \mathbb{Z})$ is generated by two elements, $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Thus f satisfies the modularity condition on all of $\mathrm{SL}(2, \mathbb{Z})$ when $f(z) = f|_S(z)$ and $f(z) = f|_T(z)$. These give:

$$f(z) = z^{-k} f\left(-\frac{1}{z}\right);$$
$$f(z) = f(z+1).$$

In particular, the condition for T means that any $f \in \mathcal{M}_k$ is periodic, and thus can be expressed by a Fourier series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n, \quad q = e^{2\pi i z}.$$

Furthermore, since f is holomorphic at infinity by definition, we have $a_n = 0$ for $n < 0$ [Bum98], so

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i z}. \tag{1.4}$$

A *cusp form* is a modular form where $a_0 = 0$ in its Fourier expansion. We denote the space of all cusp forms of weight k by $\mathcal{S}_k$.

Example 1.6. *The Discriminant function*

For $z \in \mathcal{H}$, we define the discriminant function by:

$$\Delta(q) = \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i z}.$$

This is a cusp form of weight 12.

The discriminant function is alternatively given in terms of Eisenstein series:

$$\Delta(z) = \frac{1}{1728}(E_4(z)^3 - E_6(z)^2).$$

Note. If k is even, we have $f|_\gamma = f|_{-\gamma}$ for all $f \in \mathcal{M}_k$ and $\gamma \in \mathrm{SL}(2, \mathbb{Z})$. The action given by the slash operator therefore descends to $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\{\pm I\}$ under the quotient map. So the space of modular forms over $\mathrm{SL}(2, \mathbb{Z})$ is isomorphic to that over $\mathrm{PSL}(2, \mathbb{Z})$. Some authors thus refer to only $\mathrm{PSL}(2, \mathbb{Z})$ when dealing with level 1 modular forms.

1.3 Modular forms of level N

Rather than requiring modularity on all of $\mathrm{SL}(2, \mathbb{Z})$ as in the previous section, we consider expanding this definition to only require the modularity condition hold for a specific subgroup $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$. This allows for the study of a larger and more generalised group of functions. In particular, we are interested in the congruence subgroups of $\mathrm{SL}(2, \mathbb{Z})$, defined as follows.

Definition 1.7. Let $N \geq 1$ be an integer. The principal congruence subgroup of level N is:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Definition 1.8. Let H be a subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Then H is a congruence subgroup if for some $N \geq 1$ we have $\Gamma(N) \subseteq H$. The level of H is the smallest such N such that this is true.

When studying modular forms of higher level, we are particularly interested in the following two congruence subgroups.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

where the $*$ can be any integer. Note that $\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$.

We can now define higher-level modular forms.

Definition 1.9. Let k be a non-negative integer, and $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ a congruence subgroup of level N. A *modular form* of weight k and level N is a complex valued function $f : \mathcal{H} \to \mathbb{C}$ such that:

1. f is holomorphic.
2. f satisfies the following:
$$(f|_k\gamma)(z) = f(z)$$
for all $\gamma \in \Gamma$.
3. $f|_k\alpha$ is holomorphic at infinity for all $\alpha \in \mathrm{SL}(2,\mathbb{Z})$.

We denote the space of all modular forms of weight k for a particular congruence subgroup Γ as $\mathcal{M}_k(\Gamma)$. The algebra of modular forms for Γ is given by $\mathcal{M}(\Gamma) := \sum_k \mathcal{M}_k(\Gamma)$.

We have $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0$ and Γ_1, and so any $f \in \mathcal{M}_k(\Gamma_0)$ or $f \in \mathcal{M}_k(\Gamma_1)$ is periodic and has a

Fourier expansion given by

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \qquad q = e^{2\pi i z}.$$

For general Γ, we do not necessarily have $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$. However, there is always an element $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ with t minimal. Substituting this into the modularity condition tells us that f is periodic with period t:

$$f(z) = f(z+t).$$

So we have the following Fourier expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z/t}. \tag{1.5}$$

Again, by the holomorphicity of f we have $a_n = 0$ for all $n < 0$ in (1.5). If $a_0 = 0$ in the Fourier expansion of $f|_k\alpha$ for every $\alpha \in \mathrm{SL}(2,\mathbb{Z})$, then f is a cusp form. The space of all cusp forms of weight k on Γ is denoted $\mathcal{S}_k(\Gamma)$.

We have the following inner product on the spaces of modular forms.

Definition 1.10. The map given by

$$\langle\ ,\ \rangle : \mathcal{M}_k(\Gamma) \times \mathcal{S}_k(\Gamma) \to \mathbb{C};$$

$$\langle\, f, g\,\rangle = \int_{\mathcal{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}, \tag{1.6}$$

where $z = x + iy$, is an inner product on the space of modular forms, known as the *Petersson inner product*. We will also use this inner product defined for general functions on $\mathcal{H}$.

1.4 Rankin–Cohen brackets

The product of two modular forms $f \in \mathcal{M}_k(\Gamma)$ and $g \in \mathcal{M}_\ell(\Gamma)$ gives a modular form of weight $k+\ell$. However, when we take the derivative of a modular form, this does not work so easily, and we get a function that does not satisfy the modularity condition. There are a number of ways to account for the lack of modularity and to define differential operators which return modular forms.

We first introduce the differential operator for $f \in \mathcal{M}_k(\Gamma)$:

$$Df := f' = \frac{1}{2\pi i}\frac{d}{dz}f(z). \tag{1.7}$$

In terms of the Fourier expansion of f, this is given by $q\frac{d}{dq}$. The factor $\frac{1}{2\pi i}$ is used so that rational coefficients in the Fourier expansion remain rational.

Using this operator, taking the derivative of f gives:

$$f'(\gamma \cdot z) = j(\gamma, z)^{k+2}f'(z) + \frac{k}{2\pi i}c(cz+d)^{k+1}f(z), \tag{1.8}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. The second term prevents the modularity condition from being satisfied, and so f' is not a modular form,. However, correcting for this term allows us to define differential operators which do produce modular forms.

When we consider two modular forms $f \in \mathcal{M}_k(\Gamma)$ and $g \in \mathcal{M}_\ell(\Gamma)$, the non-modularity of $f'g$ is given by the term

$$\frac{k}{2\pi i}c(cz+d)^{k+1}f(z)g(z). \tag{1.9}$$

Multiplying by ℓ makes (1.9) symmetric in both f and g. Hence, the difference

$$[f, g] := kfg' - \ell f'g \tag{1.10}$$

is a modular form.

The difference in (1.10) is the first Rankin–Cohen bracket. The family of operators which generalise this is defined as follows.

Definition 1.11. The n^{th} Rankin–Cohen bracket is a bi-differential operator which takes two modular forms, $f \in \mathcal{M}_k(\Gamma), g \in \mathcal{M}_\ell(\Gamma)$ and produces a third modular form of weight $k + \ell + 2n$:

$$[f, g]_n : \mathcal{M}_k(\Gamma) \times \mathcal{M}_\ell(\Gamma) \to \mathcal{M}_{k+\ell+2n}(\Gamma).$$

The bracket is given by:

$$[f, g]_n(z) = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s}\binom{n+l-1}{r} D^r f(z) D^s g(z),$$

where $z \in \mathcal{H}$ and D^r is the differential operator $\left(\frac{1}{2\pi i}\frac{d}{dz}\right)^r f(z)$.

We include a proof of the modularity of the Rankin–Cohen bracket using a representation-theoretic approach in Section 5.1. This can also be proven without representation theory, for example using Cohen–Kuznetsov lifting, which associates modular forms to a formal power series, as in [Zag94].

1.4.1 The Shimura Operator

An important operator on modular forms is the Shimura operator – a differential operator which takes a modular form to a function that satisfies the modularity condition but is not holomorphic. Denote $\widetilde{M}_k(\Gamma)$ the space of such functions over Γ for weight k.

Definition 1.12. Let $k \in \mathbb{Z}$. The *Shimura operator* is defined by:

$$\partial_k : M_k(\Gamma) \to \widetilde{M}_{k+2}(\Gamma),$$

$$\partial_k f(z) := \frac{1}{2\pi i}\frac{df}{dz} - \frac{k}{4\pi \operatorname{Im}(z)} f(z).$$

Then the n^{th} power of Shimura operator is the differential operator [Lan08]:

$$\partial_k^n : M_k(\Gamma) \to \widetilde{M}_{k+2n}(\Gamma),$$

$$\partial_k^n := \partial_{k+2n-2} \circ \partial_{k+2n-4} \circ ... \circ \partial_k. \tag{1.11}$$

We have the following reformulation of the Rankin–Cohen brackets in terms of the Shimura operator [Lan08, Corollary 1]:

$$[f,g]_n(z) = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s}\binom{n+l-1}{r} \partial_k^r f(z) \partial_\ell^s g(z), \tag{1.12}$$

for $f \in \mathcal{M}_k(\Gamma), g \in \mathcal{M}_\ell(\Gamma)$ and $z \in \mathcal{H}$.

1.5 Siegel modular forms

What we have worked with so far are called classical modular forms. Siegel modular forms generalise classical modular forms to functions on the Siegel upper half space. This introductory section primarily follows work by Pitale [Pit19] and van Der Geer [Gee08].

Definition 1.13. Let $n \in \mathbb{N}$. The *Siegel upper half space of genus* n is given by:

$$\mathcal{H}_n := \{Z \in M_n(\mathbb{C}) \mid Z = Z^T, \mathrm{Im}(Z) \text{ is positive definite}\}.$$

Here, $Z \in M_n(\mathbb{C})$ is an $n \times n$ matrix, which we can decompose as $Z = X + iY$, $X, Y \in M_n(\mathbb{R})$, so $Y = \mathrm{Im}(Z)$.

Definition 1.14. The *symplectic similitude group* is given by:

$$\mathrm{GSp}(2n, \mathbb{R}) := \left\{ g \in \mathrm{GL}(2n, \mathbb{R}) \mid g^T J g = \mu(g) J,\ \mu(g) \in \mathbb{R}^\times, J = \left[\begin{smallmatrix} 0_n & I_n \\ -I_n & 0_n \end{smallmatrix}\right] \right\},$$

where I_n denotes the $n \times n$ identity matrix.

The function $\mu : \mathrm{GSp}(2n, \mathbb{R}) \to \mathbb{R}^\times$ is called the *multiplier*.

When $\mu(g) = 1$, we have the *symplectic group*

$$\mathrm{Sp}(2n, \mathbb{R}) := \{g \in \mathrm{GSp}(2n, \mathbb{R}) \mid \mu(g) = 1\}.$$

We define an action of $\mathrm{GSp}(2n, \mathbb{R})$ on $\mathcal{H}_n$:

$$\gamma \cdot Z := (AZ + B)(CZ + D)^{-1},$$

where $Z \in \mathcal{H}_n$, and $\gamma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{GSp}(2n, \mathbb{R})$.

For all γ, $(CZ + D)$ is invertible so the action is well defined [Gee08]. Also, it is a group action since $I_{2n} \cdot Z = Z$, and $(\gamma\gamma') \cdot Z = \gamma \cdot (\gamma' \cdot Z)$.

As in the classical case, we use the notation $J(\gamma, Z) := CZ + D$. We then have the cocycle condition [Pit19]:

$$J(\gamma\gamma', Z) = J(\gamma, \gamma' \cdot Z) J(\gamma', Z), \tag{1.13}$$

for all $\gamma, \gamma' \in \mathrm{GSp}(2n, \mathbb{R})$ and $Z \in \mathcal{H}_n$.

Definition 1.15. A *scalar-valued Siegel modular form* (or classical Siegel modular form) of degree $g \in \mathbb{N}$ and weight $k \in \mathbb{Z}$ is a holomorphic function

$$F : \mathcal{H}_g \to \mathbb{C}$$

such that

$$F(\gamma \cdot Z) = \det(J(\gamma, Z))^k F(Z)$$

for all $\gamma \in \mathrm{Sp}(2g, \mathbb{Z})$ and $Z \in \mathcal{H}_g$.

We denote the space of all scalar-valued Siegel modular forms of degree $g \in \mathbb{N}$ and weight k by $M_k(\Gamma_g)$.

Example 1.16. We can define Siegel modular forms analogous to the Eisenstein series given in Example 1.4. Let $k \in \mathbb{Z}_{>0}$. Define:

$$E_k^{(n)}(Z) := \sum_{G \in \Gamma_{0,n} \backslash \Gamma_n} \det(CZ + D)^{-k},$$

where $G = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, and $\Gamma_{0,n} := \left\{\left[\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right] \in \Gamma_n\right\}$.

Then if k is even and $k > n+1$, $E_k^{(n)} \in M_k(\Gamma_n)$.

To define the second kind of Siegel modular forms, we require the definition of a representation, which we omit here but cover in detail in Chapter 2 – see Definition 2.1.

Definition 1.17. Let (ρ, V) be a rational representation of $\mathrm{GL}(n, \mathbb{C})$ where V is a finite dimensional $\mathbb{C}$-vector space. A *vector-valued Siegel modular form* of degree $g \in \mathbb{N}$ and weight ρ is a holomorphic function

$$F : \mathcal{H}_g \to V$$

such that

$$F(\gamma \cdot Z) = \rho(J(\gamma, Z)) f(Z)$$

for all $\gamma \in \mathrm{Sp}(2g, \mathbb{Z})$ and $Z \in \mathcal{H}_g$.

Note that the irreducible rational representations of $\mathrm{GL}(n, \mathbb{C})$ are parameterised by integers $r_1 \geq r_2 \geq \ldots \geq r_n$ [AS01]. We denote the corresponding representation $\rho_{r_1, r_2, \ldots, r_n}$.

Denote the space of all vector-valued Siegel modular forms of degree $g \in \mathbb{N}$ and weight ρ by $M_\rho(\Gamma_g)$.

In both types of Siegel modular forms, when $g = 1$ (i.e. a classical modular form), it is also required that F is holomorphic at infinity.

In order to define cusp forms in $M_\rho(\Gamma_g)$, we require the following definition.

Definition 1.18. The *Siegel operator* is a linear map given on scalar-valued Siegel modular forms by:

$$\Phi : \mathcal{M}_k(\Gamma_g) \to \mathcal{M}_k(\Gamma_{g-1}) \tag{1.14}$$

$$\Phi(F)(Z) = \lim_{t\to\infty} F \begin{bmatrix} Z & 0 \\ 0 & it \end{bmatrix} \tag{1.15}$$

with $Z \in \mathcal{H}_{g-1}, t \in \mathbb{R}$.

When defined on vector-valued Siegel modular forms, the same function gives a linear map $\mathcal{M}_\rho(\Gamma_g) \to \mathcal{M}_{\rho'}(\Gamma_{g-1})$. That is,

$$\Phi : \mathcal{M}_\rho(\Gamma_g) \to \mathcal{M}_{\rho'}(\Gamma_{g-1}) \tag{1.16}$$

$$\Phi(F)(Z) = \lim_{t\to\infty} F \begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix} \tag{1.17}$$

with $Z \in \mathcal{H}_{g-1}, t \in \mathbb{R}$. We use Φ for both maps, as the domain is given by context.

Definition 1.19. A scalar-valued Siegel modular form $F \in \mathcal{M}_k(\Gamma_g)$ is called a *cusp form* if $\Phi(F) = 0$. The space of cusp forms is denoted $\mathcal{S}_k(\Gamma_g)$.

Definition 1.20. Similarly, a vector-valued Siegel modular form $F \in \mathcal{M}_\rho(\Gamma_g)$ is called a *cusp form* if $\Phi(F) = 0$. The space of cusp forms is denoted $\mathcal{S}_\rho(\Gamma_g)$.

CHAPTER 2

THE REPRESENTATION THEORY OF $\mathrm{SL}(2, \mathbb{R})$

This chapter will cover some preliminary definitions and concepts from representation theory. In particular, we focus on the representation theory of $\mathrm{SL}(2, \mathbb{R})$, which will be required for the following chapters. An in-depth study of the representations of $\mathrm{SL}(2, \mathbb{R})$ can be found in [Lan85] and [Kna01].

2.1 Preliminary definitions

We begin with some preliminaries of representation theory, including basic definitions and concepts which will be drawn on throughout this paper. In the following, G denotes a Lie group.

Definition 2.1. Let G be a Lie group and V a complex vector space. A *representation* of G on V is a group homomorphism

$$\pi : G \to \mathrm{Aut}(V)$$

$$g \mapsto \pi_g,$$

where $\mathrm{Aut}(V)$ is the group of automorphisms of V.

We use the notation (π, V) for a representation as above.

Representations can be equivalently defined by modules, which will sometimes be easier to work with.

Definition 2.2. Given a representation (π, V) on a Lie group G, V is also a G-module defined by the operation:

$$g \cdot v := \pi(g)v$$

for $g \in G$ and $v \in V$.

We define a *sub-representation* of (π, V) to be a representation $(\pi|_W, W)$ where W is a G-invariant subspace of V, and $\pi|_W(g) = \pi(g)|_W$ for each $g \in G$.

When studying the representations of a group, of particular interest are the irreducible, unitary and admissible representations, which we will define below.

Definition 2.3. A representation (π, V) of G is *irreducible* if it has no non-trivial proper sub-representations.

Definition 2.4. Suppose V is endowed with an inner product. A representation (π, V) of G is *unitary* if π_g is a unitary operator on V for all $g \in G$.

Every finite dimensional unitary representation on a Hilbert space V can be decomposed as the direct sum of irreducible representations. Thus, studying the irreducible representations of a group allows us to study all of its finite dimensional unitary representations.

Definition 2.5. Let K be a maximal compact subgroup of G, and (π, V) a representation of G. Then (π, V) is *admissible* if π is unitary when restricted to K, and if each irreducible unitary representation of K occurs in it with finite multiplicity.

From now on, we set K to be a maximal compact subgroup of G, and V a Hilbert Space.

Definition 2.6. Let (π, V) be a representation of G. We call an element $v \in V$ *K-finite* if the space spanned by $\{\pi(k)v \mid k \in K\}$ is finite-dimensional. The space of all K-finite vectors is denoted V_K.

Associated to a Lie group $G \subset \mathrm{GL}(n, \mathbb{C}))$ is the Lie algebra $\mathfrak{g}$, defined by the following correspondence:

$$\mathfrak{g} = \{X \in M_{n\times n}(\mathbb{C}) \mid e^{tX} \in G \text{ for all } t \in \mathbb{R}\}.$$

A representation of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

We want to define a representation on $\mathfrak{g}$ given a representation on G. In order to do so, we need to work in the space of smooth vectors relative to a representation (π, V):

$$V^\infty := \{v \in V \mid g \mapsto \pi(g)(v) \in C^\infty\}.$$

Definition 2.7. Let (π, V) be a representation of G. Then since we have $e^{tX} \in G$ for all $X \in \mathfrak{g}$, we can define a representation of $\mathfrak{g}$ on V^∞ by:

$$d\pi(X)v := \left(\frac{d}{dt}\pi(\exp(tX))v\right)\bigg|_{t=0}. \tag{2.1}$$

This is called the *derived representation* of $\mathfrak{g}$.

Working in V^∞ ensures that $\pi(X)v$ is differentiable for all X, so that the formula (2.1) is valid.

Definition 2.8. Assume π is the right regular representation of G, given by $(\pi(g)f)(x) = f(xg)$ for $g, x \in G$ and $f \in C^\infty(G)$. Then for $X \in \mathfrak{g}$, $d\pi$ is given by:

$$(d\pi(X)f)(g) = \left(\frac{d}{dt} f(g \exp(tX))\right)\bigg|_{t=0}. \tag{2.2}$$

In this case, we denote $d\pi(X)f$ by dXf, and refer to it as the *derived action* of X.

2.2 $(\mathfrak{g}, K)$-modules

Given a real reductive Lie group (such as $\mathrm{SL}(2,\mathbb{R})$), Harish-Chandra [HC54] showed that the representation theory of this group is completely determined by its $(\mathfrak{g}, K)$-modules, defined as follows.

Definition 2.9. Let G be a Lie group, $\mathfrak{g}$ its associated Lie algebra, and K a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Then a $(\mathfrak{g}, K)$-module is a vector space V together with representations π of K and $\mathfrak{g}$ such that:

i. V decomposes into an algebraic direct sum of finite dimensional invariant subspaces under the action of K, i.e. as a K-module we have

$$V = \bigoplus V(\sigma). \tag{2.3}$$

Here, σ ranges over equivalence classes of irreducible unitary representations of K, and $V(\sigma)$ is the sum of all K-submodules of V that are in the class σ.

ii. The representations of $\mathfrak{g}$ and K be compatible, i.e. the following holds for all $X \in \mathfrak{k}$ and $v \in V$:

$$\pi(X)v = \left(\frac{d}{dt}\pi(\exp(tX))v\right)\bigg|_{t=0}.$$

iii. The following holds for $g \in K$, $X \in \mathfrak{g}$, and $v \in V$:

$$\pi(g)\pi(X)\pi(g^{-1})v = \pi(\mathrm{Ad}(g)X)v,$$

where $\mathrm{Ad}(g)X = gXg^{-1}$.

Definition 2.10. A $(\mathfrak{g}, K)$-module is called *admissible* if in the decomposition

$$V = \bigoplus V(\sigma)$$

we have $\dim(V(\sigma)) < \infty$ for all σ.

Proposition 2.11. *If (π, V) is an admissible representation of G, this gives representations of both K and $\mathfrak{g}$ on V_K, the latter of which is the derived representation. Using these representations, V_K is a $(\mathfrak{g}, K)$-module.*

We first note that since V is admissible, $V_K \subset V^\infty$, and so we can define the derived representation on $\mathfrak{g}$ [Bum98, Proposition 2.4.5].

Proof. For clarity, we will denote the representation on K by π and on $\mathfrak{g}$ by $d\pi$. In order to show that V_K is a $(\mathfrak{g}, K)$-module, we first need to show that (π, V_K) and $(d\pi, V_K)$ do indeed give representations of K and $\mathfrak{g}$ respectively.

For the representation of K, we restrict π to K, and set $k \in K, v \in V_K$. Then for every $k' \in K$:

$$\pi(k')(\pi(k)(v)) = \pi(k'k)(v).$$

Since $v \in V_K$, the space spanned by $\{\pi(k'k)(v) \mid k' \in K\}$ is certainly finite. So $(\pi(k)(v)) \in V_K$, and thus π is a representation of K on V_K.

We know that $(d\pi, V)$ is a representation on $\mathfrak{g}$, so need only to check that $(d\pi, V_K)$ is a representation on $\mathfrak{g}$. Note that a vector v is K-finite if and only if the space spanned by $\{d\pi(Y)v \mid Y \in \mathfrak{k}\}$ is finite [Bum98, Proposition 2.4.5]. Let $v \in V_K$. Let R be a finite dimensional subspace of V that is stable under $\mathfrak{k}$ and such that $v \in R$. Define

$$R_1 := \{d\pi(X)v \mid X \in \mathfrak{g}, v \in R\}. \tag{2.4}$$

Then R_1 is finite dimensional.

If we have $Y \in \mathfrak{k}$ and $d\pi(X)v \in R_1$,

$$d\pi(Y)(d\pi(X)v) = d\pi(X)(d\pi Y)(v) + d\pi([Y, X])(v).$$

Then since $d\pi(X)(d\pi Y)(v)$ and $d\pi([Y,X])(v)$ are in R_1, so too is $d\pi(Y)(d\pi(X)v)$. So R_1 is stable under $\mathfrak{k}$.

Thus, for arbitrary $X \in \mathfrak{g}$, we have

$$\operatorname{span}\{d\pi(Y)d\pi(X)v \mid Y \in \mathfrak{k}\} \subset R_1.$$

Hence $d\pi(X)v$ is K finite when v is, and we have the desired representation:

$$d\pi : \mathfrak{g} \to \operatorname{Aut}(V_K).$$

Considering the three conditions in Definition 2.9:

i. The condition
$$V_K = \bigoplus V(\sigma),$$
is equivalent to V_K being defined by the K-finite vectors, so this is immediate [Bum98, Theorem 2.4.4].

ii. Since we are using the derived representation $d\pi$, the representations of $\mathfrak{g}$ and K are compatible by definition.

iii. Let $g \in K$, $X \in \mathfrak{g}$, and $v \in V_K$. Then

$$\begin{aligned}
\pi(g)d\pi(X)\pi(g^{-1})v &= \pi(g)\frac{d}{dt}\pi(\exp(tX))\pi(g^{-1})v|_{t=0} \\
&= \frac{d}{dt}\pi(g)\pi(\exp(tX))\pi(g^{-1})v|_{t=0} \\
&= \frac{d}{dt}\pi(g\exp(tX)g^{-1})v|_{t=0} \\
&= \frac{d}{dt}\pi(\exp\bigl(tgXg^{-1}\bigr)v|_{t=0} \\
&= \pi(\operatorname{Ad}(g)X)v.
\end{aligned}$$

So V_K is a $(\mathfrak{g}, K)$-module. □

Definition 2.12. The $(\mathfrak{g}, K)$-module V_K defined in Proposition 2.11 is called the *underlying* $(\mathfrak{g}, K)$*-module* of (π, V) [Kob15].

If (π, V) and (π', V') are two admissible representations of G, we say that they are *infinitesimally equivalent* if their underlying $(\mathfrak{g}, K)$-modules are isomorphic. Note that two admissible representations are infinitesimally equivalent if and only if they are equivalent (i.e. isomorphic as representations) [Bum98, Theorem 2.6.6].

2.3 The representation theory of $\mathrm{SL}(2,\mathbb{R})$

We now turn to the representation theory of $\mathrm{SL}(2,\mathbb{R})$, where we find the representations associated to modular forms. As above, we are primarily interested in the irreducible, unitary representations. These fall into five distinct families of representations, up to unitary equivalence [Kna01, Theorem 16.3].

As above, we set G to be a Lie group, K to be its maximal compact subgroup, and V a Hilbert Space. We are now specifically interested in $G = \mathrm{SL}(2,\mathbb{R})$.

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ has basis given by the matrices

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which have the commutator relations:

$$[H, X_+] = 2X_+, \qquad [H, X_-] = -2X_-, \qquad [X_+, X_-] = H. \tag{2.5}$$

In order to classify representations of $\mathfrak{g}$, it is often useful to work in its complexification, $\mathfrak{g}_\mathbb{C}$. The complexification will be required for classifications and computations in Sections 3,4 and 5. Any representation (π, V) on $\mathfrak{g}$ can be extended to one on $\mathfrak{g}_\mathbb{C}$ in the following way. If we write $Z \in \mathfrak{g}_\mathbb{C}$ as $Z = X + iY$ with $X, Y \in \mathfrak{g}$, we can define the extension of a representation π on $\mathfrak{g}$ by:

$$\pi(Z) := \pi(X) + i\pi(Y). \tag{2.6}$$

Since V is a complex vector space, it is a $\mathfrak{g}_\mathbb{C}$ module under this action.

The following matrices give an alternative basis for the complexification:

$$E_+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad E_- = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \qquad \hat{H} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

These are closely related to X_+, X_- and H. Let $\mathcal{C} = -\frac{1+i}{2}\left(\begin{smallmatrix} i & 1 \\ i & -1 \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$. Then:

$$E_+ = \mathcal{C}X_+\mathcal{C}^{-1}, \qquad E_- = \mathcal{C}X_-\mathcal{C}^{-1}, \qquad \hat{H} = \mathcal{C}H\mathcal{C}^{-1}. \tag{2.7}$$

Since these are all given by conjugation of the same element, the basis $\{E_+, E_-, \hat{H}\}$ satisfies the same commutator relations as $\{X_+, X_-, H\}$:

$$[\hat{H}, E_+] = 2E_+, \qquad [\hat{H}, E_-] = -2E_-, \qquad [E_+, E_-] = \hat{H}.$$

Definition 2.13. Any basis of $\mathfrak{sl}(2)$ which satisfies the commutator relations in (2.5) is called a $\mathfrak{sl}(2)$-*triple*.

Definition 2.14. The *universal enveloping algebra* of $\mathfrak{g}$ is the universal associative algebra with an embedding of $\mathfrak{g}$. It is denoted $\mathcal{U}(\mathfrak{g})$.

Definition 2.15. The Casimir element[(i)] is given by

$$C = -\frac{1}{4}(H^2 + 2X_+X_- + 2X_-X_+) \in Z(\mathcal{U}(\mathfrak{g})),$$

where $Z(\mathcal{U}(\mathfrak{g}))$ denotes the center of $\mathcal{U}(\mathfrak{g})$.

Definition 2.16. A representation (π, V) on G is a called a *quasisimple* representation if the Casimir element C acts as a multiple of the identity on V.

Proposition 2.17. *If (π, V) is an irreducible representation on G, then π is quasisimple.*

We note that a maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$ is given by $K = \mathrm{SO}(2,\mathbb{R})$, which is the group of matrices

$$\left\{ \kappa_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \middle| \, \theta \in [0, 2\pi] \right\}.$$

Let V be an irreducible admissible $(\mathfrak{g}, K)$-module. The irreducible representations of K are one dimensional, and are given by their characters $\sigma_k(\kappa_\theta) = e^{ik\theta}$ with $k \in \mathbb{Z}$ [Lan85, §1]. Hence the equivalence classes of irreducible representations are indexed by the integer k, so

[(i)]The Casimir element is often normalized to $H^2 + 2X_+X_- + 2X_-X_+$.

the decomposition in (2.3) becomes:

$$V = \bigoplus_{k\in\mathbb{Z}} V(k). \tag{2.8}$$

Proposition 2.18. *Consider $V(k)$ as above, where $V(k)$ is defined to be the components isotypic to each σ_k, and σ_k is the irreducible representation of $\mathrm{SO}(2)$ with character $k \in \mathbb{Z}$. Then $V(k)$ is also given by the space of H-eigenvectors with eigenvalue k.*

Proof. See [Bum98, Proposition 2.5.2]. □

Let $\sum$ be the set of all integers such that $V(k) \neq 0$, called the *set of K-types* of V. The set of K-types of V consists of either all even or all odd integers, and so we can define the parity of V in this way [Bum98].

For $k \in \mathbb{Z}$, we define the following specific sets of K-types of V:

$$\begin{aligned}
\Sigma^+(k) &= \{\ell \in \mathbb{Z} \mid \ell \equiv k \pmod 2,\ \ell \geq k\};\\
\Sigma^-(k) &= \{\ell \in \mathbb{Z} \mid \ell \equiv k \pmod 2,\ \ell \leq -k\};\\
\Sigma^0(k) &= \{\ell \in \mathbb{Z} \mid \ell \equiv k \pmod 2,\ -k < \ell < k\}.
\end{aligned}$$

We now have the required definitions and notation for the following result.

Theorem 2.19. *The following is a complete list of the irreducible admissible $(\mathfrak{g}, K)$-modules for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and $K = SO(2)$. Let λ be a complex number, and let $\epsilon = 0$ or 1.*

i) if λ is not of the form $\frac{k}{2}\left(1 - \frac{k}{2}\right)$ where $k \in \mathbb{Z}$ and $k \equiv \epsilon \pmod 2$, then there exists a unique irreducible $(\mathfrak{g}, K)$-module of parity ϵ on which C acts by scalar λ. The set of K-types is the set of all integers congruent to ϵ modulo 2.

ii) if λ is of the form $\frac{k}{2}\left(1 - \frac{k}{2}\right)$ where $k \in \mathbb{Z}_{>1}$ and $k \equiv \epsilon \pmod 2$, then there exist three irreducible admissible $(\mathfrak{g}, K)$-modules of parity ϵ on which C acts by λ. The set of K-types of these representations are $\sum^{\pm}(k)$ and $\sum^{0}(k)$.

iii) if $\lambda = \frac{1}{4}$ and $1 \equiv \epsilon \pmod 2$, then there exist two irreducible admissible $(\mathfrak{g}, K)$-modules of parity ϵ on which C acts by λ. The set of K-types of these representations are $\sum^{\pm}(k)$.

This is proven in [Bum98, Theorem 2.5.4] for $\mathrm{GL}(2,\mathbb{R})$, and the proof carries over in a straightforward way to $\mathrm{SL}(2,\mathbb{R})$.

The infinitesimal equivalence classes of representations in item (i) are denoted $\mathcal{P}(\lambda,\epsilon)$ and are known as the principal series representations. Those in item (ii) with K-types $\sum^{\pm}(k)$ are denoted $\mathcal{D}^{\pm}(k)$, and are called discrete series representations. When $k=1$ (item (iii)), these are $\mathcal{D}^{\pm}(1)$ and are called the limit of discrete series.

By classifying the isomorphism classes of $(\mathfrak{g},K)$-modules, or infinitesimal equivalence classes, we can classify the irreducible admissible unitary representations, since each class contains at most one unitary representative [Bum98]. Thus, by determining which of the $(\mathfrak{g},K)$-modules gives a unitary representation of $\mathrm{SL}(2,\mathbb{R})$, we have the following result.

Theorem 2.20. *The following is a complete list of the isomorphism classes of irreducible admissible unitary representations for* $\mathrm{SL}(2,\mathbb{R})$*. Each class has a unique representative that is a unitary representation. Throughout, we take* $\epsilon\in\{0,1\}$*, and* $\lambda\in\mathbb{R}$*.*

i) The trivial representation given by $g\mapsto 1$ *for all* g*;*

ii) The unitary principal series $\mathcal{P}(\lambda,\epsilon)$ *with* $\lambda\geq\frac{1}{4}$*;*

iii) The complementary series representations, which are given by $\mathcal{P}(\lambda,0)$ *with* $0<\lambda<\frac{1}{4}$*;*

iv) The discrete series representations $\mathcal{D}^{\pm}(k)$*, with* $k\geq 2$*;*

v) The limits of discrete series $\mathcal{D}^{\pm}(1)$*.*

For a full proof, refer to [Kna01, Theorem 16.3] and [Bum98, Theorem 2.6.7].

CHAPTER 3

ASSOCIATING A REPRESENTATION TO A MODULAR FORM

There are a number of strategies employed in order to relate a modular form to a representation. In this section I will follow methods outlined in Pevzner [Pev12] and Bump [Bum98], however other methods are used in different texts. We use Pevzner's method to compare the infinitesimal equivalence classes of π_k and $\mathcal{D}^+(k)$ (Sections 3.1-3.3) and Bump's method to justify the association of a representation to modular form (Sections 3.4 and 3.5). Note that for the parts of this paper which follow Bump's method, we use $\mathrm{SL}(2,\mathbb{R})$ rather than $\mathrm{GL}(2,\mathbb{R})$ as in Bump, which results in some small differences in calculations.

It is important to note that while we are working here with representations of $\mathrm{SL}(2,\mathbb{R})$, when we are talking about the representation associated to a modular form this is actually a representation of $\mathrm{GL}(2,\mathbb{A})$. We look to the point at infinity to obtain a representation of $\mathrm{GL}(2,\mathbb{R})$, and then reduce this to a representation of $\mathrm{SL}(2,\mathbb{R})$. This is done due to the natural relationship between this representation and the Rankin–Cohen brackets, outlined in Chapter 4. Please see remark at the end of Section 3.5 for more details on this construction.

The main result of this section is as follows.

Theorem 3.1. *To each modular form $f \in \mathcal{M}_k(\Gamma)$, we can associate a representation of $\mathrm{SL}(2,\mathbb{R})$, given by*

$$\pi_k : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Aut}(\mathfrak{H})$$

$$\pi_k(g)(F) = F|_k(g^{-1}),$$

where $\mathfrak{H}$ is the space of holomorphic, square integrable functions on $\mathcal{H}$ with respect to

$$\int_{\mathcal{H}} f(z) y^k \frac{dxdy}{y^2}$$

for $z = x + iy$.

Moreover, π_k is in the equivalence class $\mathcal{D}^+(k)$ of holomorphic discrete series representations of $\mathrm{SL}(2,\mathbb{R})$.

We will prove Theorem 3.1 in a number of steps. First, we show that π_k is an irreducible admissible representation in the infinitesimal equivalence class of $\mathcal{D}^+(k)$. We will do this by directly comparing the underlying $(\mathfrak{g}, K)$-modules of these representations, and showing that they are isomorphic.

Finally, we will justify the fact that we can associate a representation to a modular form through introducing a Hilbert space isomorphism between spaces of modular forms and functions on G. Using this, we show that the multiplicities of $\mathcal{D}^+(k)$ are equal to the dimensions of the spaces of modular forms $\mathcal{M}_k(\Gamma)$.

3.1 Classifying $\mathfrak{sl}(2, \mathbb{R})$-modules

Throughout, let $G = \mathrm{SL}(2, \mathbb{R})$, $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ and V be a $\mathfrak{g}$-module. We want to work towards classifying all $\mathfrak{g}$-modules of a certain type, so that we can determine the infinitesimal equivalence class of particular representations. Note that we can extend the action of $\mathfrak{g}$ to an action of $\mathcal{U}(\mathfrak{g})$, and since V is also a $\mathfrak{g}_\mathbb{C}$ module (by (2.6)), this can be extended to $\mathcal{U}(\mathfrak{g}_\mathbb{C})$. Calculations are done in $\mathcal{U}(\mathfrak{g}_\mathbb{C})$, since the elements E_-, E_+ and $\hat{H}$ are needed in the classification of $\mathfrak{g}$-modules, and the relation of these to modular forms.

Definition 3.2. The *generalized H-eigenspace of V* for $\lambda \in \mathbb{C}$ is given by

$$V_\lambda = \{v \in V \mid (H - \lambda \mathrm{Id})^n v = 0 \text{ for some } n \in \mathbb{N}\}.$$

The generalised eigenvalues of H are called *weights*.

If V can be decomposed as a direct sum of its H-eigenspaces, then V is called a *weight module*. In particular, we have that when V is an irreducible admissible $(\mathfrak{g}, K)$-module, V must be a weight module.

Definition 3.3. If $V = \sum_\lambda V_\lambda$ and $\dim(V_\lambda) < \infty$ for all λ, then we say that V is *H-admissible*.

Definition 3.4. If V is H-admissible, and V_λ gives us exactly the H-eigenspace of V, that is if

$$V_\lambda = \{v \in V \mid Hv = \lambda v\},$$

then we say that V is *H-semisimple*.

Proposition 3.5. *If V is a indecomposable, quasisimple, H-admissible and H-semisimple $\mathfrak{g}$-module, then V is isomorphic to a standard module, defined below.*

Proof. See [HT92, Theorem 1.3.1] □

3.1.1 Standard Modules

There are five standard modules, each representing an isomorphism class of irreducible, H-admissible and H-semisimple $\mathfrak{g}$-modules, which we define below.

1. A *lowest weight module* V_k with $k \in \mathbb{C}$ has a basis of H-eigenvectors $\{v_j \,|\, j \in \mathbb{N}\}$ such that

$$\begin{aligned} Hv_j &= (k+2j)v_j, && j \in \mathbb{N} \\ X_+v_j &= v_{j+1}, && j \in \mathbb{N} \\ X_-v_j &= -j(k+j-1)v_{j-1}, && j \in \mathbb{N}_{>0} \\ X_-v_0 &= 0 && \\ Cv &= \frac{k}{2}\left(1-\frac{k}{2}\right)v, && v \in V_k. \end{aligned}$$

   The element v_0 is the *lowest weight vector* and k is the *lowest weight* of the module.

2. A *highest weight module* $\overline{V}_k$ with $k \in \mathbb{C}$ has a basis of H-eigenvectors $\{\overline{v}_j \mid j \in \mathbb{N}\}$ such that

$$\begin{aligned} H\overline{v}_j &= (k-2j)\overline{v}_j, && j \in \mathbb{N} \\ X_+\overline{v}_j &= \overline{v}_{j+1}, && j \in \mathbb{N} \\ X_-\overline{v}_j &= -j(k-j-1)\overline{v}_{j-1}, && j \in \mathbb{N}_{>0} \\ X_-\overline{v}_0 &= 0 && \\ Cv &= -\frac{k}{2}\left(1+\frac{k}{2}\right)v, && v \in \overline{V}_k. \end{aligned}$$

   The element $\overline{v}_0$ is the *highest weight vector* and k is the *highest weight* of the module.

3. $W(\mu,\lambda)$ with $\mu,\lambda \in \mathbb{C}$, with a basis of H-eigenvectors $\{v_j \mid j \in \mathbb{Z}\}$ such that

$$\begin{aligned} Hv_j &= (\lambda+2j)v_j, && j \in \mathbb{Z} \\ X_+v_j &= v_{j+1}, && j \in \mathbb{Z} \\ X_-v_j &= \frac{1}{4}\left(\mu-(\lambda+2j-1)^2+1\right)v_{j-1}, && j \in \mathbb{Z} \end{aligned}$$

$$Cv = -\frac{1}{4}\mu v, \qquad v \in W(\mu, \lambda)$$

4. $\overline{W}(\mu, \lambda)$ with $\mu, \lambda \in \mathbb{C}$, with a basis of H-eigenvectors $\{\overline{v}_j \mid j \in \mathbb{Z}\}$ such that

$$\begin{aligned}
H\overline{v}_j &= (\lambda + 2j)\overline{v}, && j \in \mathbb{Z} \\
X_+\overline{v}_j &= \frac{1}{4}\left(\mu - (\lambda + 2j - 1)^2 + 1)\right)\overline{v}_{j+1}, && j \in \mathbb{Z} \\
X_-\overline{v}_j &= \overline{v}_{j-1}, && j \in \mathbb{Z} \\
Cv &= -\frac{1}{4}\mu v, && v \in \overline{W}(\mu, \lambda).
\end{aligned}$$

5. $U(\nu^+, \nu^-)$ with $\nu^+, \nu^- \in \mathbb{C}$, with a basis of H-eigenvectors $\{v_j \mid j \in \mathbb{Z}\}$ such that

$$\begin{aligned}
Hv_j &= (\nu^+ - \nu^- + 2j)v_j, && j \in \mathbb{Z} \\
X_+v_j &= (\nu^+ + j)v_{j+1}, && j \in \mathbb{Z} \\
X_-v_j &= (\nu^- - j)v_{j-1}, && j \in \mathbb{Z} \\
Cv &= (\nu^+\nu^-)(\nu^+ + \nu^- - 2)v, && v \in W(\mu, \lambda).
\end{aligned}$$

Note that while we have defined the standard modules in terms of H, X_+ and X_-, these definitions hold for weight modules with respect to any $\mathfrak{sl}(2)$-triple.

3.2 Infinitesimal equivalence class of $\mathcal{D}^+(k)$

In order to classify the holomorphic discrete series representations $\mathcal{D}^+(k)$, we want to consider the $(\mathfrak{g}, K)$-module defined in Theorem 2.19 (ii), which is the representative of its infinitesimal equivalence class.

First, we need to show that we can indeed use the classification used in the previous section.

Proposition 3.6. *If V is an irreducible admissible $(\mathfrak{g}, K)$-module, then V is H-admissible and H-semisimple.*

Proof. Let V be an irreducible admissible $(\mathfrak{g}, K)$-module. Then from Proposition 2.18 we have

$$V = \bigoplus_{\lambda \in \mathbb{Z}} V(\lambda). \tag{3.1}$$

Where $V(\lambda)$ is given by the space of H-eigenvectors with eigenvalue λ. Note first that this necessarily covers all H-eigenvalues λ.

Also, we have for $\lambda \neq \mu$, $V(\lambda) \cap V(\mu) = \{0\}$.

Then, for H-semisimplicity we have that $V(\lambda)$ is exactly the H-eigenspace of V, and so we want to show that $V(\lambda) = V_\lambda$. Clearly $V(\lambda) \subset V_\lambda$. Let $v \in V_\lambda \subset V$. Then by 3.1 we have $v \in V(\lambda')$ for some λ'. We want to show that $\lambda = \lambda'$. If $\lambda \neq \lambda'$ then $v \in V_\lambda \cap V_{\lambda'} = 0$, a contradiction. So $\lambda = \lambda'$, which means that $v \in V(\lambda)$, so $V_\lambda = V(\lambda)$. Therefore V is H-semisimple.

Then, since $V_\lambda = V(\lambda)$ and all the V_λ are disjoint, 3.1 gives us $V = \sum_\lambda V_\lambda$.

Moreover, $\dim(V_\lambda) = \dim(V(\lambda)) < \infty$ for all λ since V is admissible.

Therefore, V is H-admissible. □

Let V be the $(\mathfrak{g}, K)$ module corresponding to $\mathcal{D}^+(k)$ from Theorem 2.19 (ii). Then V is irreducible and admissible by definition. Thus V is irreducible, H-admissible and H-semisimple, and we can use the classification of standard modules in order to determine its infinitesimal equivalence class, which gives us the following result.

Proposition 3.7. *The underlying $(\mathfrak{g}, K)$-modules of the holomorphic discrete series representations are all lowest weight modules.*

Proof. Let $k \in \mathbb{Z}$. Using the classification of Theorem 2.19, the infinitesimal equivalence class of the holomorphic discrete series representations $D^+(k)$ is classified by a complex number λ, with $\lambda = \frac{k}{2}(1 - \frac{k}{2})$ and the set of K-types is $\Sigma^+(k)$.

So, for the underlying $(\mathfrak{g}, K)$-module of the holomorphic discrete series representation V_K, we have

$$V_K = \bigoplus_{k \in \mathbb{Z}} V_K(k).$$

Recalling that $V_K(k)$ gives the space of H-eigenvectors with eigenvalue k, we have that Σ gives the set of integers $\ell \in \mathbb{Z}$ such that $V(\ell) \neq 0$, that is, ℓ is an eigenvector of H.

Since the set of K-types of V_K is given by

$$\Sigma^+(k) = \{\ell \in \mathbb{Z} \mid \ell \equiv k \mod 2, \ell \geq k\},$$

the set of H-eigenvectors of V_K has eigenvalues given by $\{k, k+2, k+4, ...\}$. Also, it follows from the direct sum decomposition that these H-eigenvectors form a basis of V_K.

Finally, C acts on V_K by the scalar

$$\lambda = \frac{k}{2}\left(1 - \frac{k}{2}\right).$$

Together, these facts show us that the underlying $(\mathfrak{g}, K)$-module of $\mathcal{D}^+(k)$ is a lowest weight module, with lowest weight k. □

3.3 Underlying $(\mathfrak{g}, K)$-modules of π_k

Proposition 3.8. *For each $k \in \mathbb{Z}$, we define*

$$\pi_k : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{Aut}(\mathfrak{H})$$

$$\pi_k(g)(F) = F|_k(g^{-1})$$

where $\mathfrak{H}$ is the space of holomorphic, square integrable functions on $\mathcal{H}$ with respect to

$$\int_{\mathcal{H}} f(z) y^k \frac{dxdy}{y^2}$$

for $z = x + iy$.

Then π_k is a irreducible unitary representation of $\mathrm{SL}(2, \mathbb{R})$.

Proof. First we verify that π_k is a representation of $\mathrm{SL}(2, \mathbb{R})$.

If $F \in \mathfrak{H}$ and $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$,

$$F|_k(g^{-1})(z) = (cz + d)^{-k} F(g^{-1}z).$$

We have $(cz + d)^{-k} F(g^{-1}z) \in \mathfrak{H}$ since $F \in \mathfrak{H}$. Also, the slash operator is an action (as shown in Section 1), so π_k gives a representation on $\mathfrak{H}$.

To show that π_k is unitary, we will show that π_k is preserves the inner product on $\mathfrak{H}$ defined by

$$\langle f, g\rangle = \int_{\mathcal{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}, \tag{3.2}$$

for $z = x + iy \in \mathcal{H}$.

Let $f, g \in \mathcal{H}$, and $\gamma \in \mathrm{SL}(2, \mathbb{R})$, $\gamma^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then:

$$\begin{aligned}
\langle \pi_k(\gamma) f, \pi_k(\gamma) g \rangle &= \int_{\mathcal{H}} \pi_k(\gamma) f(z) \overline{\pi_k(\gamma) g(z)} y^k \frac{dxdy}{y^2} \\
&= \int_{\mathcal{H}} (cz+d)^{-k} f(\gamma^{-1} \cdot z) \overline{(cz+d)^{-k} g(\gamma^{-1} \cdot z)} y^k \frac{dxdy}{y^2} \\
&= \int_{\mathcal{H}} f(\gamma^{-1} \cdot z) \overline{g(\gamma^{-1} \cdot z)} |cz+d|^{-2k} y^k \frac{dxdy}{y^2} \\
&= \int_{\mathcal{H}} f(\gamma^{-1} \cdot z) \overline{g(\gamma^{-1} \cdot z)} \left(\frac{y}{|cz+d|^2} \right)^k \frac{dxdy}{y^2} \\
&= \int_{\mathcal{H}} f(z') \overline{g(z')} (y')^k \frac{dx'dy'}{y'^2} \\
&= \langle f, g \rangle .
\end{aligned}$$

Here, the second last line is given by a change of variable from z to $z' = \gamma^{-1} \cdot z = x' + iy'$. We note that $y' = \frac{y}{|cz+d|^2}$, and the measure $\frac{dxdy}{y^2}$ is invariant under the action of $\mathrm{SL}(2, \mathbb{R})$ [DS06, §5.4]. So π_k is unitary.

A full proof of the irreducibility of π_k can be found in [Bum98, Theorem 2.6.5]. □

Since π_k is a unitary irreducible representation it is necessarily admissible [Kna01, Theorem 8.1]. So, we can use the construction in Proposition 2.11 to find the underlying $(\mathfrak{g}, K)$-module of π_k, which is given by $V_K = \mathfrak{H}_{\mathrm{SO}(2)}$. We have the following result.

Proposition 3.9. *The underlying* $(\mathfrak{g}, K)$ *module of* π_k *is a irreducible,* H*-admissible and* H*-semisimple* $\mathfrak{sl}(2, \mathbb{R})$*-module.*

Proof. That $\mathfrak{H}_{\mathrm{SO}(2)}$ is irreducible follows immediately from the fact that π_k is irreducible since elements in $\mathfrak{H}_{\mathrm{SO}(2)}$ are the $\mathrm{SO}(2)$-finite vectors in $\mathfrak{H}$. So $\mathfrak{H}_{\mathrm{SO}(2)}$ is indecomposable, and quasisimple (Proposition 2.17).

Moreover, each irreducible representation of $\mathrm{SO}(2)$ is one dimensional and occurs only once, we have that the dimension of each summand is finite in (2.3). So $\mathfrak{H}_{\mathrm{SO}(2)}$ is admissible.

Therefore, $\mathfrak{H}_{\mathrm{SO}(2)}$ is an indecomposable, quasisimple admissible $(\mathfrak{g}, K)$-module and is thus H-admissible and H-semisimple by Proposition 3.6.

□

Now we can use Proposition 3.5 to classify the isomorphism class of $\mathfrak{H}_{\mathrm{SO}(2)}$.

Proposition 3.10. *The underlying* $(\mathfrak{g}, K)$ *module of* π_k *is a lowest weight module* V_k.

Proof. The following proof is an expansion of that given in [Pev12] for the same result.

Fix $k \in \mathbb{Z}$. Define $\mathfrak{H}_{\mathrm{SO}(2)}$ to be the underlying $(\mathfrak{g}, K)$-module of π_k. This is a H-admissible H-semisimple and irreducible module by Proposition 3.9 and so one of the five standard modules.

In order to classify this module, we compute the action of H, X_- and X_+ on a vector in $\mathfrak{H}_{\mathrm{SO}(2)}$ under the representation $d\pi$. Since vectors in $\mathfrak{H}_{\mathrm{SO}(2)}$ are functions in $\mathfrak{H}$, we will call an arbitrary element f.

So for $f \in \mathfrak{H}_{\mathrm{SO}(2)}$, the action of H is given by:

$$\begin{aligned}
H \cdot f(z) &= \left(\frac{d}{dt}\pi_k(\exp(tH))f(z)\right)\bigg|_{t=0} \\
&= \left(\frac{d}{dt}\pi_k \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} f(z)\right)\bigg|_{t=0} \\
&= \left(\frac{d}{dt}(e^t)^{-k} f(e^{-2t}z)\right)\bigg|_{t=0} \\
&= -ke^{-tk} f(e^{-2t}z) + e^{-tk} f'(e^{-2t}z)(-2e^{-2t}z)\big|_{t=0} \\
&= -kf(z) - 2zf'(z).
\end{aligned}$$

The action of X_+ is given by:

$$\begin{aligned}
X_+ \cdot f(z) &= \left(\frac{d}{dt}\pi_k(\exp(tX_+))f(z)\right)\bigg|_{t=0} \\
&= \left(\frac{d}{dt}\pi_k \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} f(z)\right)\bigg|_{t=0} \\
&= \left(\frac{d}{dt} f(z-t)\right)\bigg|_{t=0} \\
&= (-1)f'(z-t)\big|_{t=0} \\
&= -f'(z).
\end{aligned}$$

And the action of X_- is given by:

$$\begin{aligned}
X_- \cdot f(z) &= \left(\frac{d}{dt}\pi_k(\exp(tX_-))f(z)\right)\bigg|_{t=0} \\
&= \left(\frac{d}{dt}\pi_k\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}f(z)\right)\bigg|_{t=0} \\
&= \left(\frac{d}{dt}(-tz+1)^{-k}f\left(\frac{z}{-tz+1}\right)\right)\bigg|_{t=0} \\
&= zk(-tz+1)^{-k-1}f\left(\frac{z}{-tz+1}\right) + (-tz+1)^{-k}f'\left(\frac{z}{-tz+1}\right)\frac{z^2}{(-tz+1)^2}\bigg|_{t=0} \\
&= zkf(z) + z^2f'(z) \qquad .
\end{aligned}$$

We now define a basis $\{v_j \mid j \in \mathbb{N}\}$ by

$$v_j := \frac{(k+j-1)!}{(k-1)!}z^{-k-j}. \tag{3.3}$$

Using the basis $\{v_j \mid j \in \mathbb{N}\}$ and our above calculations, we can prove that $\mathfrak{H}_{\mathrm{SO}(2)}$ is a lowest weight module by computing the actions of H, X_- and X_+ on our basis.

We have:

$$\begin{aligned}
H \cdot v_j &= -kv_j - 2z\frac{d}{dz}v_j \\
&= -k\frac{(k+j-1)!}{(k-1)!}z^{-k-j} - 2z\frac{d}{dz}\frac{(k+j-1)!}{(k-1)!}z^{-k-j} \\
&= -k\frac{(k+j-1)!}{(k-1)!}z^{-k-j} - 2z\frac{(k+j-1)!}{(k-1)!}(-k-j)z^{-k-j-1} \\
&= -k\frac{(k+j-1)!}{(k-1)!}z^{-k-j} - 2\frac{(k+j-1)!}{(k-1)!}(-k-j)z^{-k-j} \\
&= (-k+2(k+j))\left(\frac{(k+j-1)!}{(k-1)!}z^{-k-j}\right) \\
&= (k+2j)v_j.
\end{aligned}$$

$$\begin{aligned}
X_+ \cdot v_j &= -\frac{d}{dz}v_j \\
&= -\frac{d}{dz}\frac{(k+j-1)!}{(k-1)!}z^{-k-j}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(k+j-1)!}{(k-1)!}(k+j)z^{-k-j-1} \\
&= \frac{(k+j)!}{(k-1)!}z^{-k-j-1} \\
&= v_{j+1}.
\end{aligned}
$$

For $j = 0$, we have $v_0 = z^{-k}$, so

$$
\begin{aligned}
X_- \cdot v_0 &= zkv_0 + z^2 \frac{d}{dz} v_0 \\
&= zkz^{-k} + z^2 \frac{d}{dz} z^{-k} \\
&= kz^{-k+1} + (-k)z^{-k+1} \\
&= 0.
\end{aligned}
$$

Then for $j > 1$:

$$
\begin{aligned}
X_- \cdot v_j &= zkv_j + z^2 \frac{d}{dz} v_j \\
&= k\frac{(k+j-1)!}{(k-1)!}z^{-k-j+1} + \frac{(k+j-1)!}{(k-1)!}(-k-j)z^{-k-j+1} \\
&= (k + (-k-j))\frac{(k+j-1)!}{(k-1)!}z^{-k-j+1} \\
&= (-j)(k+j-1)\frac{(k+j-2)!}{(k-1)!}z^{-k-j+1} \\
&= (-j)(k+j-1)v_{j-1}.
\end{aligned}
$$

So, using the basis $\{v_j : j \in \mathbb{N}\}$, $\mathfrak{H}_{\mathrm{SO}(2)}$ satisfies the properties for a lowest weight module V_k. □

Hence, the underlying $(\mathfrak{g}, K)$ modules of both $\mathcal{D}^+(k)$ and π_k are lowest weight modules V_k. Since we can construct an isomorphism between two lowest weight modules by mapping the basis elements to each other, they are isomorphic. Therefore, $\mathcal{D}^+(k)$ and π_k are infinitesimally equivalent.

3.4 Introducing an isomorphism between modular forms and functions on $\mathrm{SL}(2,\mathbb{R})$

In order to associate the representation π_k to a modular form $f \in \mathcal{M}_k(\Gamma)$ we need to introduce a Hilbert space isomorphism between modular forms and a space of smooth functions over $\mathcal{H}$. Using this isomorphism will allow us to show that the dimensions of the spaces of modular forms and the multiplicity of the discrete series representation are equal, completing the proof of Theorem 3.1.

We define the space

$$C^\infty(\Gamma \setminus \mathcal{H}, k) := \{f \in C^\infty(\Gamma \setminus \mathcal{H}) \mid f(z) = f|_k\gamma(z) \text{ for all } \gamma \in \Gamma\}.$$

Let $L^2(\Gamma \setminus \mathcal{H}, k)$ to be the Hilbert space completion of $C^\infty(\Gamma \setminus \mathcal{H}, k)$ with respect to the inner product given in (3.2).

Proposition 3.11. *There is a Hilbert space isomorphism between:*

$$L^2(\Gamma \setminus \mathcal{H}, k)$$

and

$$L^2(\Gamma \setminus G, k) := \{F \in L^2(\Gamma \setminus G) \mid F(g\kappa_\theta) = \exp(ik\theta)F(g) \textit{ for all } g \in G\}.$$

This isomorphism is given by:

$$\begin{aligned} \sigma_k : L^2(\Gamma \setminus \mathcal{H}, k) &\to L^2(\Gamma \setminus G, k) \\ \sigma_k(f)(g) &= (f|_k g)(i), \quad g \in G. \end{aligned}$$

Proof. Starting with a function $f \in L^2(\Gamma \setminus \mathcal{H}, k)$, we show that $\sigma_k(f)(g) \in L^2(\Gamma \setminus G, k)$. Note that $\sigma_k(f)$ is a smooth function and square-integrable on G since $f \in L^2(\Gamma \setminus \mathcal{H}, k)$.

Then

$$\sigma_k(f)(\gamma g) = f|_k(\gamma g \cdot i) = (f|_k\gamma)|_k(g \cdot i) = (f|_k g)(i) = \sigma_k(f)(g), \tag{3.4}$$

so $\sigma_k(f)$ is in $L^2(\Gamma \setminus G)$.

Also,

$$\sigma_k(f)(g\kappa_\theta) = f|_k(g\kappa_\theta{\cdot}i) = f(g\kappa_\theta{\cdot}i)j(g\kappa_\theta, i)^{-k} = f(g{\cdot}i)j(g, \kappa_\theta{\cdot}i)^{-k}j(\kappa_\theta, i)^{-k} = \sigma_k(f)(g)\text{exp}(ik\theta),$$

using the cocycle property of $j(g, z)$. So $\sigma_k(f) \in L^2(\Gamma \setminus G, k)$.

Now, for a function $F \in L^2(\Gamma \setminus G, k)$, we define the map:

$$\phi_k : L^2(\Gamma \setminus G, k) \to L^2(\Gamma \setminus \mathcal{H}, k);$$
$$\phi_k(F)(z) = F(g)j(g, i)^k,$$

where $z = g \cdot i$ for any $g \in \text{SL}(2, \mathbb{R})$.

We first verify that ϕ_k is well defined. Let $F \in L^2(\Gamma \setminus G, k)$, and define

$$f(z) = \phi_k(F)(z) = F(g)j(g, i)^k.$$

Let $\gamma \in \Gamma$. Then

$$\begin{aligned} f|_k\gamma(z) &= j(\gamma, z)^{-k} f(\gamma \cdot z) = j(\gamma, z)^{-k} F(\gamma g)j(\gamma g, i)^k = j(\gamma, z)^{-k} F(g)j(g, i)^k j(\gamma, g \cdot i)^k \\ &= F(g)j(g, i)^k = f(z). \end{aligned}$$

So $f \in L^2(\Gamma \setminus \mathcal{H}, k)$.

Now, we show that ϕ_k and σ_k are inverse to each other.

Let $f \in L^2(\Gamma \setminus \mathcal{H}, k)$ and $F(g) := \sigma_k(f)(g) = (f|_k g)(i)$. Then

$$\begin{aligned} (\phi_k \circ \sigma_k)(f)(z) &= \phi_k(F)(z) = F(g)j(g, i)^k = (f|_k g)(i)j(g, i)^k \\ &= f(g \cdot i)j(g, i)^{-k} j(g, i)^k = f(z). \end{aligned}$$

For $F \in L^2(\Gamma \setminus G, k)$, let $f(z) = \phi_k(F)(z) = F(g)j(g, i)^k$. We have

$$\begin{aligned} (\sigma_k \circ \phi_k)(F)(g) &= \sigma_k(f)(g) = (f|_k g)(i) = j(g, i)^{-k} f(g \cdot i) = j(g, i)^{-k} f(z) \\ &= j(g, i)^{-k} j(g, i)^k F(g) = F(g). \end{aligned}$$

So ϕ_k and σ_k are inverse to each other, and thereby define an isomorphism. Moreover, they give a Hilbert space isomorphism, which Bump proves using the Haar measure on these spaces [Bum98, Proposition 2.18]. □

Note that the space $C^\infty(\Gamma \setminus \mathcal{H}, k)$ is the space of functions on the upper half plane which satisfy the modularity condition. When we also require $f \in C^\infty(\Gamma \setminus \mathcal{H}, k)$ to be holomorphic, we have a finer isomorphism. In order to establish this isomorphism, we need to use the basis $\{E_+, E_-, \hat{H}\}$ of $\mathfrak{g}_\mathbb{C}$ for the calculations. We establish the following results to work with these elements.

Lemma 3.12. *We have the following decomposition for any element in* $\mathrm{SL}(2, \mathbb{R})$*:*

$$g = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \kappa_\theta. \tag{3.5}$$

For g as above, denote $g_0 = \left(\begin{smallmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{smallmatrix}\right)$.

Proof. For proof, see [Bum98, §2.1]. □

Lemma 3.13. *The derived actions of the basis elements E_+ E_- and $\hat{H} \in \mathfrak{g}_\mathbb{C}$ on $f \in C^\infty(G)$ are:*

$$(dE_+ f)(g) = e^{2i\theta} \left(iy\frac{df}{dx} + y\frac{df}{dy} + \frac{1}{2i}\frac{df}{d\theta} \right), \tag{3.6}$$

$$(dE_- f)(g) = e^{-2i\theta} \left(-iy\frac{df}{dx} + y\frac{df}{dy} - \frac{1}{2i}\frac{df}{d\theta} \right), \tag{3.7}$$

$$(d\hat{H} f)(g) = -i\frac{df}{d\theta}, \tag{3.8}$$

where g is given by the decomposition (3.5).

Proof. For proof, see [Bum98, Proposition 2.2.5]. □

Working with these results, we can introduce the second isomorphism.

Proposition 3.14. *There is a bijection between the space of holomorphic modular forms $\mathcal{M}_k(\Gamma)$ and the space*

$$\mathcal{A}_k := \{F \in L^2(\Gamma \setminus G, k) \mid dE_- F = 0\}, \tag{3.9}$$

where dE_- is the the derived action of $E_- \in \mathfrak{g}_\mathbb{C}$ on functions on G.

This bijection is given by

$$f \mapsto F;$$
$$F(g) = f|_k(g)(i).$$

Proof. Let $f \in \mathcal{M}_k(\Gamma)$, and F its image under the given map. The map here is given by the same as in Proposition 3.11, so we have already shown that $F \in L^2(\Gamma \setminus G, k)$.

Using the notation from Lemma 3.12, for any $g \in \mathrm{SL}(2, \mathbb{R})$:

$$F(g) = F(g_0\kappa_\theta) = e^{ik\theta}F(g_0) = e^{ik\theta}f|_k(g_0)(i) = e^{ik\theta}y^{k/2}f(x+iy). \tag{3.10}$$

Then, using Lemma 3.13, for $F \in \mathcal{A}_k$:

$$\begin{aligned} dE_-F(g) &= e^{-2i\theta}\left(-iy\frac{d}{dx} + y\frac{d}{dy} - \frac{1}{2i}\frac{d}{d\theta}\right)F \\ &= e^{-2i\theta}\left(-iy\frac{dF}{dx} + y\frac{dF}{dy} - \frac{1}{2i}\frac{dF}{d\theta}\right) = e^{-2i\theta}\left(-iy\frac{dF}{dx} + y\frac{dF}{dy} - \frac{k}{2}F\right). \end{aligned}$$

For the last line, we use that for $F \in \mathcal{A}_k$:

$$F(g\kappa_\theta) = e^{ik\theta}F(g),$$

which gives

$$\frac{d}{d\theta}F(g\kappa_\theta) = ikF(g\kappa_\theta).$$

So $dE_-F = 0$ when

$$-iy\frac{dF}{dx} + y\frac{dF}{dy} = \frac{k}{2}F. \tag{3.11}$$

Using (3.10) for F, (3.11) becomes:

$$\begin{aligned} 0 &= -iy\frac{d}{dx}(e^{ik\theta}y^{k/2}f(x+iy)) + y\frac{d}{dy}(e^{ik\theta}y^{k/2}f(x+iy)) - \frac{k}{2}(e^{ik\theta}y^{k/2}f(x+iy)) \\ &= -iy(e^{ik\theta}y^{k/2}\frac{d}{dx}f(x+iy)) + \frac{k}{2}e^{ik\theta}y^{k/2}f(x+iy) + ye^{ik\theta}y^{k/2}\frac{d}{dy}f(x+iy) - \frac{k}{2}(e^{ik\theta}y^{k/2}f(x+iy)) \\ &= -ie^{ik\theta}y^{1+k/2}\frac{d}{dx}f(x+iy) + e^{ik\theta}y^{1+k/2}\frac{d}{dy}f(x+iy) \end{aligned}$$

$$= -e^{ik\theta}y^{1+k/2}\left(i\frac{d}{dx}f(x+iy) - \frac{d}{dy}f(x+iy)\right).$$

Therefore, $dE\_F = 0$ if and only if

$$i\frac{d}{dx}f(x+iy) = \frac{d}{dy}f(x+iy), \tag{3.12}$$

which is precisely the Cauchy-Riemann equations for f. □

3.5 Assigning a modular form to the representation π_k

Using the Hilbert space isomorphism of the previous section, we can now justify why the representation π_k can be associated to a modular form.

Lemma 3.15. *Let $k \in \mathbb{Z}_{\geq 1}$. The multiplicity of the representation $\mathcal{D}^+(k)$ in $L^2(\Gamma \setminus G)$ is equal to the dimension of the space $\mathcal{M}_k(\Gamma)$.*

Proof. Assume that $\lambda = \frac{k}{2}\left(1 - \frac{k}{2}\right)$ where $k \in \mathbb{Z}_{\geq 1}$. Then if ρ is an infinite dimensional irreducible sub-representation of $\mathfrak{H}$ it is unitary since it inherits an inner product from $L^2(\Gamma \setminus G)$, and hence admissible. By Theorem 2.19, since $C(\rho) = \lambda\rho$ and $k \geq 1$, either $\rho \cong \mathcal{D}^+(k)$ or $\rho \cong \mathcal{D}^-(k)$.

Consider the representations $\mathcal{D}^+(k)$ in $L^2(\Gamma \setminus G)$. Each has underlying $(\mathfrak{g}, K)$-module given by V_k, which is a lowest weight module with lowest weight k. Since each representation is generated uniquely by the lowest weight vectors, we need only to consider the lowest weight vector. Let $v_0 \in V_k$ be the lowest weight vector. Then

$$d\hat{H}v_0 = kv_0. \tag{3.13}$$

By Lemma 3.13, the action of H is given by

$$d\hat{H} = -i\frac{d}{d\theta}. \tag{3.14}$$

So for any $g\kappa_\theta \in \Gamma \setminus G$:

$$\frac{d}{d\theta}v_0(g\kappa_\theta) = ikv_0(g\kappa_\theta). \tag{3.15}$$

Hence, $v_0(g\kappa_\theta) = \exp(ik\theta)v_0(g)$, so $v_0 \in L^2(\Gamma \setminus G, k)$. Moreover, since V_k is the lowest weight module, we also have

$$Cv_0 = \frac{k}{2}\left(1 - \frac{k}{2}\right) v_0. \tag{3.16}$$

So v_0 is in the λ-eigenspace of C in $L^2(\Gamma \setminus G, k)$. Denote this eigenspace $C_G(\lambda)$. Note that each representation of $L^2(\Gamma \setminus G)$ gives a linearly independent v_0, and so the multiplicity of $\mathcal{D}^+(k)$ is equal to the dimension of $C_G(\lambda)$.

Since $L^2(\Gamma \setminus G, k) \cong L^2(\Gamma \setminus \mathcal{H}, k)$, the dimension of the $C_G(\lambda)$ is equal to the dimension of the λ-eigenspace of C in $L^2(\Gamma \setminus \mathcal{H}, k)$. We will denote the latter $C_{\mathcal{H}}(\lambda)$.

For $v \in C_G(\lambda)$,

$$d\hat{H}dE_-v = d[\hat{H}, E_-]v + dE_-d\hat{H}v = -2dE_-v + dE_-kv = (k-2)dE_-v.$$

Hence dE_-v is in the $\hat{H}$-eigenspace with eigenvalue $k-2$. However, since the K-type for $\mathcal{D}^+(k)$ is $\Sigma^+(k)$, the dimension of any eigenspace with eigenvalue less than k is 0. So the λ-eigenspace is annihilated by dE_-. Therefore, any $v \in C_G(\lambda)$ maps to a holomorphic modular form $f \in \mathcal{M}_k$ by Proposition 3.14.

This map is an isomorphism, whose inverse is obtained as follows. Given a modular form $f \in \mathcal{M}_k(\Gamma)$, we have a map $f \mapsto F$, where $F \in \mathcal{A}_k$ is given by

$$F(g) = f|_k(g)(i). \tag{3.17}$$

Then $dE_-F = 0$, since $F \in \mathcal{A}_k$, and we note that $F \in L^2(\Gamma \setminus \mathcal{H}, k)$ with

$$CF = \frac{k}{2}\left(1 - \frac{k}{2}\right) F. \tag{3.18}$$

For calculations, see [Boo15, Corollary 3.3].

So, given a space of modular forms $\mathcal{M}_k(\Gamma)$, we map isomorphically to $C_G(\lambda)$, which then maps to a copy of $\mathcal{D}^+(k)$. Therefore the dimension of the space of modular forms is equal to the multiplicity of $\mathcal{D}^+(k)$. □

Thus, if we choose a basis of modular forms for $\mathcal{M}_k(\Gamma)$, each of these basis forms will correspond to one copy of the representation $D^+(k)$ in the space of representations. This representation is then infinitesimally equivalent to π_k. Therefore, we can associate the space of modular forms of weight k to the holomorphic discrete series representation $D^+(k)$.

Remark. We note that when we talk about a representation associated to a modular form, it is in the context of a one-to-one correspondence between Hecke eigenforms and automorphic representations over $\mathrm{GL}(2, \mathbb{A})$. This is derived via an equivalent isomorphism to that in Proposition 3.14, considered over all of the $\mathrm{GL}(2, \mathbb{A})$. The function on $\mathrm{GL}(2, \mathbb{A})$ obtained then gives a representation on a quotient of $\mathrm{GL}(2, \mathbb{A})$, which we can denote as $\pi = \otimes\pi_p$ where each π_p is a representation of $\mathrm{GL}(2, \mathbb{Q}_p)$. We can then examine the representation by examining each representation π_p.

In particular, the representation at infinity is a representation of $\mathrm{GL}(2, \mathbb{R})$. This can be reduced to a representation of $\mathrm{SL}(2, \mathbb{R})$, which is π_k. For a full discussion of the relation between Hecke eigenforms and automorphic representations of $\mathrm{GL}(2, \mathbb{A})$, see [Bum98], [Kud04] and [Boo15].

While these other resources begin with the representation of $\mathrm{GL}(2, \mathbb{A})$ in order to construct a view of the bijective association between eigenforms and representations, we have chosen to work purely in the setting of π_∞. Focusing here allows us to go into the details of the representation where the Rankin–Cohen brackets arise. In particular, the classification using weight modules has been chosen as this is integral to the working throughout Chapters 4 and 5.

CHAPTER 4

RANKIN–COHEN BRACKETS IN REPRESENTATIONS

In this chapter, we outline how Rankin–Cohen brackets arise naturally in the representations associated to modular forms. Using notation from the previous chapter, we denote the holomorphic discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ associated to a modular form $f \in \mathcal{M}_k(\Gamma)$ as π_k, with underlying $(\mathfrak{g}, K)$-module V_k. We proceed by first decomposing the tensor product of two such modules, showing that this decomposes into a sum of lowest weight modules. It will then be shown that the projection map from this tensor product to the module $V_{k+\ell+2n}$ in its decomposition is given by the n^{th} Rankin–Cohen bracket. The main papers which this section follows are by Pevzner [Pev12] and El Gradechi [EG06].

4.1 Decomposing the tensor product of two modules

Let $f \in \mathcal{M}_k(\Gamma)$ and $g \in \mathcal{M}_\ell(\Gamma)$. Then their associated representations π_k and π_ℓ are both discrete series representations of $\mathrm{SL}(2,\mathbb{R})$. In order to derive the relationship between the Rankin–Cohen bracket of these modular forms and their associated representations, we first introduce an important fact about the tensor products of these representations and their underlying modules.

We want to consider the tensor product $V_k \otimes V_\ell$ of lowest weight modules as a diagonal $\mathfrak{sl}(2,\mathbb{R})$ module. This diagonalisation is given via the embedding

$$\begin{aligned} \Delta :& \mathfrak{sl}(2,\mathbb{R}) \to \mathcal{U}(\mathfrak{sl}(2,\mathbb{R})) \otimes \mathcal{U}(\mathfrak{sl}(2,\mathbb{R})) \\ &\Delta(x) = x \otimes 1 + 1 \otimes x, \end{aligned}$$

for $x \in \mathfrak{sl}(2,\mathbb{R})$. We will use $\mathfrak{sl}(2,\mathbb{R})^\Delta$ to denote the diagonal modules.

Proposition 4.1. *Let the basis of V_k be given by $\{v_i \mid i \in \mathbb{N}\}$ and the basis of V_ℓ be given by $\{\widetilde{v}_j \mid j \in \mathbb{N}\}$. A $\Delta(H)$-eigenvector in $V_k \otimes V_\ell$, considered as a $\mathfrak{sl}(2,\mathbb{R})^\Delta$-module, is given by*

$$v_i \otimes \widetilde{v}_j,$$

with corresponding eigenvalue $k + \ell + 2(i + j)$. Here, $\Delta(H)$ is the induced diagonal action of H on $V_k \otimes V_\ell$, given by:

$$\Delta(H) := H \otimes \mathrm{Id} + \mathrm{Id} \otimes H.$$

Proof. Let $v_i \in V_k$ and $\widetilde{v_j} \in V_\ell$. We will show that

$$\Delta(H)(v_i \otimes \widetilde{v_j}) = (k + \ell + 2(i + j))(v_i \otimes \widetilde{v_j}).$$

Since $v_i \in V_k$ and $\widetilde{v}_j \in V_\ell$,

$$H \cdot v_i = (k + 2i)v_i; \qquad \text{and} \qquad H \cdot \widetilde{v}_j = (\ell + 2j)\widetilde{v}_j.$$

Considering the diagonal action of H on $(v_j \otimes \widetilde{v_i})$ gives:

$$\begin{aligned}\Delta(H)\,(v_i \otimes \widetilde{v_j}) &= H \cdot v_i \otimes \widetilde{v_j} + v_i + H \cdot \widetilde{v_j} \\ &= (k + 2i)v_i \otimes \widetilde{v_j} + v_i \otimes (\ell + 2j)\widetilde{v_j} \\ &= k + \ell + 2(i + j)[v_i \otimes \widetilde{v_j}].\end{aligned}$$

So $v_i \otimes \widetilde{v_j}$ is a $\Delta(H)$-eigenvector with eigenvalue $k + \ell + 2(i + j)$. □

Now we can prove the following result.

Proposition 4.2. *The tensor product $V_k \otimes V_\ell$ of lowest weight modules, considered as a $\mathfrak{sl}(2, \mathbb{R})^\Delta$-module, decomposes into a direct sum of lowest weight modules:*

$$V_k \otimes V_\ell = \bigoplus_{n \geq 0} V_{k+\ell+2n}. \tag{4.1}$$

Proof. We will call j the index of the vector v_j, so as not to confuse this with the weights of the modules. Then if a vector in V_k has index j, it has H-eigenvalue $k + 2j$.

If $u \in V_k$ has index i, and $v \in V_\ell$ has index j, then $u \otimes v$ has index $(i + j)$ from Proposition 4.1. Thus, there are no vectors with index less than $k+\ell$ in $V_k \otimes V_\ell$. Also, the space of vectors with index n has basis given by $\{u_j \otimes v_{n-j} \mid j \in [0, n]\}$, so has dimension $n + 1$.

In particular, the space with index 0 has dimension 1, and so the vector with weight $k + \ell$ must be a lowest weight vector. Therefore $V_k \otimes V_\ell$ contains an $\mathfrak{sl}(2, \mathbb{R})^\Delta$-module isomorphic to a submodule of $V_{k+\ell}$. However, since k and ℓ are strictly positive, $V_{k+\ell}$ is irreducible and so its only non-zero submodule is itself. So the tensor product contains one copy of $V_{k+\ell}$.

Considering the orthogonal complement of this subspace, we have that the space of vectors with index 1 now has dimension 1, so the vector with weight $k+\ell+2$ must be a lowest weight vector. The tensor product thus contains one copy of $V_{k+\ell+2}$. Continuing this way we reach the statement of (4.1). □

Note. In the following constructions, many authors refer to the following decomposition of $\pi_k \otimes \pi_\ell$ when discussing the Rankin–Cohen brackets in representations. This was obtained by Repka [Rep78] using Proposition 4.2.

Proposition 4.3. *Given two discrete series representations of* $\mathrm{SL}(2,\mathbb{R})$*,* π_k *and* π_ℓ*, their tensor product decomposes as:*

$$\pi_k \otimes \pi_\ell = \bigoplus_{n\geq 0} \pi_{k+\ell+2n}.$$

Proof. See [Rep79, §(7)]. □

However, we are working with the underlying $(\mathfrak{g}, K)$-modules, so the result as is formulated in Proposition 4.1 is more relevant for the current study.

4.2 The lowest weight vector of $V_{k+\ell+2n}$

For each $n \in \mathbb{N}$, there is a lowest weight module $V_{k+\ell+2n}$ in the decomposition of the tensor product $V_k \otimes V_\ell$. We want to determine the lowest weight vector for this module.

By Proposition 4.1, the $\Delta(H)$-eigenspace corresponding to the eigenvalue $(k+\ell+2n)$ has basis:

$$v_j \otimes \widetilde{v}_{n-j}, \quad j \in [0, n].$$

Denote this eigenspace $W_n^{k,\ell}$.

Then $W_n^{k,\ell}$ contains the lowest weight vector of $V_{k+\ell+2n}$.

The lowest weight vector is thus:

$$\sum_j a_j (v_j \otimes \widetilde{v}_{n-j}) \in V_k \otimes V_\ell,$$

such that

$$\Delta(X_-)\left(\sum_j a_j(v_j \otimes \widetilde{v}_{n-j})\right) \in V_k \otimes V_\ell = 0.$$

We have the following result.

Theorem 4.4. *The lowest weight vector of the module $V_{k+\ell+2n}$ for $n \in \mathbb{N}$ in the tensor decomposition of $V_k \otimes V_\ell$ is given by:*

$$\sum_{j=0}^{n}\left((-1)^j \binom{k+n-1}{n-j}\binom{\ell+n-1}{j}\right)(v_j \otimes \widetilde{v}_{n-j}). \tag{4.2}$$

Proof. To show that (4.2) is the lowest weight vector of the module $V_{k+\ell+2n}$, we need to show that it is in the kernel of $\Delta(X_-)$.

To do this, we find $\{a_j\}$ such that $\sum_j a_j(v_j \otimes \widetilde{v}_{n-j})$ is annihilated by $\Delta(X_-)$.

We have:

$$\begin{aligned}
&\Delta(X_-)\left(\sum_j a_j(v_j \otimes \widetilde{v}_{n-j})\right) = \sum_j a_j\left(X_- \cdot v_j \otimes \widetilde{v}_{n-j} + v_j \otimes X_- \cdot \widetilde{v}_{n-j}\right)\\
&= \sum_j a_j\left(-j(k+j-1)v_{j-1} \otimes \widetilde{v}_{n-j} + v_j \otimes -(n-j)(\ell+(n-j)-1)\widetilde{v}_{n-j-1}\right)\\
&= \sum_j a_j(-j(k+j-1)v_{j-1} \otimes \widetilde{v}_{n-j}) + \sum_j a_j(v_j \otimes -(n-j)(\ell+n-j-1)\widetilde{v}_{n-j-1})\\
&= \sum_j -a_{j+1}(j+1)(k+j)(v_j \otimes \widetilde{v}_{n-j-1}) + \sum_j a_j(j-n)(\ell+n-j-1)(v_j \otimes \widetilde{v}_{n-j-1})\\
&= \sum_j[-a_{j+1}(j+1)(k+j) + a_j(j-n)(\ell+n-j-1)](v_j \otimes \widetilde{v}_{n-j-1})\\
&= \sum_j -[a_{j+1}(j+1)(k+j) + a_j(n-j)(\ell+n-j-1)](v_j \otimes \widetilde{v}_{n-j-1}).
\end{aligned}$$

Hence $\sum_j a_j(v_j \otimes \widetilde{v}_{n-j})$ is in the kernel of $\Delta(X_-)$ when the following recurrence relation is satisfied:

$$a_{j+1}(j+1)(k+j) + a_j(n-j)(\ell+n-j-1) = 0. \tag{4.3}$$

Solving this recurrence gives [Pev12, p. 461]:

$$a_j = (-1)^j \binom{k+n-1}{n-j}\binom{\ell+n-1}{j}. \tag{4.4}$$

We can verify (4.4) satisfies the recurrence relation (4.3) for all $j \in [0, n]$:

$$\begin{aligned}
&a_{j+1}(j+1)(k+j) = (-1)^{j+1}\binom{k+n-1}{n-(j+1)}\binom{\ell+n-1}{j+1}(j+1)(k+j) \\
&= (-1)^{j+1}\frac{(k+n-1)!}{(n-j-1)!(k+j)!}\frac{(\ell+n-1)!}{(j+1)!(\ell+n-j-2)!}(j+1)(k+j) \\
&= (-1)^{j+1}\frac{(k+n-1)!}{(n-j-1)!(k+j-1)!}\frac{(\ell+n-1)!}{j!(\ell+n-j-2)!},
\end{aligned}$$

and

$$\begin{aligned}
&a_j(n-j)(\ell+n-j-1) = (-1)^j\binom{k+n-1}{n-j}\binom{\ell+n-1}{j}(n-j)(\ell+n-j-1) \\
&= (-1)^j\frac{(k+n-1)!}{(n-j)!(k+j-1)!}\frac{(\ell+n-1)!}{j!(\ell+n-j-1)!}(n-j)(\ell+n-j-1) \\
&= (-1)^j\frac{(k+n-1)!}{(n-j-1)!(k+j-1)!}\frac{(\ell+n-1)!}{j!(\ell+n-j-2)!}.
\end{aligned}$$

So

$$a_{j+1}(j+1)(k+j) + a_j(n-j)(\ell+n-j-1) = 0.$$

Hence for each $n \in \mathbb{N}$, the lowest weight vector in $V_{k+\ell+2n}$ is given by:

$$\sum_{j=0}^{n}(-1)^j\binom{k+n-1}{n-j}\binom{\ell+n-1}{j}(v_j \otimes \widetilde{v}_{n-j}).$$

□

4.3 Relation to the Rankin–Cohen brackets

Note that the lowest weight vector in $V_{k+\ell+2n}$ is given by the action:

$$\sum_j (-1)^j\binom{k+n-1}{n-j}\binom{\ell+n-1}{j}(X_+)^j \otimes (X_+)^{n-j} \tag{4.5}$$

on $v_0 \times \widetilde{v}_0$.

In order to relate this to the Rankin–Cohen brackets, we require a number of results which we will give briefly here for the sake of completeness, but which will be worked through in detail in Chapter 5. As in Chapter 3, calculations are done in the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

A change of basis to (4.5) gives:

$$\sum_j (-1)^j \binom{k+n-1}{n-j} \binom{\ell+n-1}{j} (E_+)^j \otimes (E_+)^{n-j}. \tag{4.6}$$

Proposition 4.5. *Given a Lie group G, its Lie algebra $\mathfrak{g}$, and its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, there is an isomorphism*

$$\rho : \mathcal{U}(\mathfrak{g}) \to D_L(G),$$

where $D_L(G)$ is the algebra of left invariant differential operators on G.

This isomorphism is given by taking the representation which associates to each $X \in \mathfrak{g}$ a left invariant field, and extending this to a representation on $\mathcal{U}(\mathfrak{g})$. For a full proof, see [Hel01, Proposition 1.9].

In particular, ρ of each of the basis vectors is given by the corresponding derived action [Bum98, Proposition 2.2.5]. So by Lemma 3.13:

$$\rho(E_+) = \frac{i}{2} e^{2i\theta} \big(4y \frac{d}{dz} - \frac{d}{d\theta}\big); \tag{4.7}$$

$$\rho(E_-) = \frac{i}{2} e^{-2i\theta} \big(4y \frac{d}{d\bar{z}} - \frac{d}{d\theta}\big); \tag{4.8}$$

$$\rho(\hat{H}) = -i \frac{d}{d\theta}, \tag{4.9}$$

where $\frac{d}{dz} = \frac{1}{2}\left(\frac{d}{dx} - i\frac{d}{dy}\right)$ and $\frac{d}{d\bar{z}} = \frac{1}{2}\left(\frac{d}{dx} + i\frac{d}{dy}\right)$.

Hence, (4.6) is isomorphic to:

$$\sum_j (-1)^j \binom{k+n-1}{n-j} \binom{\ell+n-1}{j} E^j E^{n-j}, \tag{4.10}$$

where $E = \rho(E_+)$.

The lift given in Proposition 3.14 allows us to determine the corresponding function on $\mathcal{H}$. Then, up to division by a constant, (4.10) translates to:

$$\sum_j (-1)^j \binom{k+n-1}{n-j} \binom{\ell+n-1}{j} \partial_k^j \partial_\ell^{n-j},$$

which is the n^{th} Rankin–Cohen bracket, as defined in (1.12).

Thus, the Rankin–Cohen brackets arise naturally as a projection of $V_k \otimes V_\ell$ onto $V_{k+\ell+2n}$.

CHAPTER 5

APPLICATIONS OF REPRESENTATION-THEORETIC APPROACH

This chapter looks at two applications of a representation-theoretic approach to the study of modular forms and Rankin–Cohen brackets. We will use representation theory to prove the modularity of Rankin–Cohen brackets, and their uniqueness as bi-differential operators. What follows is primarily referenced from the work of El Gradechi [EG06][EG13] and Pevzner [Pev12].

Throughout, we use the notation π_k for the representation of $G = \mathrm{SL}(2,\mathbb{R})$ associated to the family $\mathcal{M}_k(\Gamma)$, as defined in Proposition 3.8. We recall that the underlying $(\mathfrak{g}, K)$-module of π_k is V_k, a lowest weight module of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$. Again, calculations will be done in $\mathcal{U}(\mathfrak{g}_\mathbb{C})$.

We note that V_k can also be considered as $\mathfrak{g}_\mathbb{C}$-module. We have $V_k \cong \mathcal{U}(\mathfrak{n}^+)$, where $\mathfrak{n}^+$ is from the triangular decomposition of $\mathfrak{g}_\mathbb{C}$:

$$\mathfrak{g}_\mathbb{C} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Here, $\mathfrak{n}^- := \mathbb{C}E_-, \mathfrak{h}^+ := \mathbb{C}\hat{H}, \mathfrak{n}^+ := \mathbb{C}E_+$. [EG06, Proposition 2.4].

5.1 Preliminary definitions

Definition 5.1. Define $\Upsilon_m^{k,\ell}$ to be the space of G-equivariant holomorphic bi-differential operators

$$[\ ,\ ]_m^{k,\ell} : \mathrm{Hol}(\mathcal{H}) \otimes \mathrm{Hol}(\mathcal{H}) \to \mathrm{Hol}(\mathcal{H}).$$

Where G-equivariance is:

$$[\pi_k(g)(f), \pi_\ell(g)(f')]_m^{k,\ell} = \pi_m(g)[f, f']_m^{k,\ell},$$

for all $f, f' \in \mathrm{Hol}(\mathcal{H})$ and $g \in G$.

Let

$$\mathcal{A}_k^- := \{F \in C^\infty(G) \mid F(g\kappa_\theta) = \exp(ik\theta)F(g) \text{ for all } g \in G, \text{ and } dE_-F = 0\}. \tag{5.1}$$

Then, for a congruence subgroup $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ we have $\mathcal{A}_k = \{F \in \mathcal{A}_k^- \mid F(\gamma g) = F(g) \text{ for } g \in G, \gamma \in \Gamma\}$.

Definition 5.2. Define $\mathcal{B}_m^{k,\ell}$ to be the space of G-equivariant holomorphic bi-differential operators

$$B : \mathcal{A}_k^- \otimes \mathcal{A}_\ell^- \to \mathcal{A}_m^-,$$

where A_k is defined in (3.9). G-equivariance is given by:

$$L_g(B(F \otimes F')) = B(L_g(F) \otimes L_g(F')),$$

for all $F \in \mathcal{A}_k^-$ and $g \in G$. Here, L_g is:

$$L_g(F)(g') = F(g^{-1}g').$$

Proposition 5.3. *Let $m \in \mathbb{N}$, $k, \ell \in \mathbb{N}_{>0}$. Then there is an isomorphism:*

$$\Upsilon_m^{k,\ell} \xrightarrow{\sim} \mathcal{B}_m^{k,\ell}.$$

Proof. We have shown in Proposition 3.14 that the map

$$\begin{aligned} \mathcal{M}_k(\Gamma) &\to \mathcal{A}_k \\ f &\mapsto (g \mapsto f|_k(g)(i)); \quad g \in G \end{aligned}$$

is an isomorphism.

The modularity condition in the domain gives invariance in the codomain (shown in (3.4)). Hence, by taking the domain to be all holomorphic functions on $\mathcal{H}$, we have an isomorphism:

$$\begin{aligned} \varphi_k : \mathrm{Hol}(\mathcal{H}) &\to \mathcal{A}_k^- \\ f &\mapsto (g \mapsto f|_k(g)(i)). \end{aligned}$$

Using this, given a map $[\ ,\ ]_m^{k,\ell} \in \Upsilon_m^{k,\ell}$, we can define a map $B \in \mathcal{B}_m^{k,\ell}$ uniquely by the commutative diagram:

$$\begin{array}{ccc} \mathrm{Hol}(\mathcal{H}) \otimes \mathrm{Hol}(\mathcal{H}) & \xrightarrow{\varphi_k \otimes \varphi_\ell} & \mathcal{A}_k^- \otimes \mathcal{A}_\ell^- \\ {\scriptstyle [\,,\,]_m^{k,\ell}} \downarrow & & \downarrow {\scriptstyle B} \\ \mathrm{Hol}(\mathcal{H}) & \xrightarrow[\varphi_m]{} & \mathcal{A}_m^- \end{array} \tag{5.2}$$

We show that B is G-equivariant. Let $g \in G$, $F \in \mathcal{A}_k^-$ and $F' \in \mathcal{A}_\ell^-$, and define $f, f' \in \mathrm{Hol}(\mathcal{H})$ such that $\varphi_k(f) = F, \varphi_\ell(f') = F'$. Then

$$\begin{aligned} L_g(B(F \otimes F'))(g') &= B(F \otimes F')(g^{-1}g') = B(\varphi_k(f) \otimes \varphi_k(f'))(g^{-1}g') \\ &= \varphi_m([f, f']_m^{k,\ell})(g^{-1}g') = [f, f']_m^{k,\ell}|_m(g^{-1}g') = \pi_m(g'^{-1}g)([f, f']_m^{k,\ell}) \\ &= [\pi_k(g'^{-1}g)f, \pi_\ell(g'^{-1}g))f']_m^{k,\ell} = B(\varphi_k(f)(g^{-1}g') \otimes \varphi_\ell(f')(g^{-1}g')) \\ &= B(F(g^{-1}g') \otimes F'(g^{-1}g')) = B(L_g(F) \otimes L_g(F'))(g') \end{aligned}$$

So B is G-equivariant, and hence in $\mathcal{B}_m^{k,\ell}$.

So there is a map:

$$\Upsilon_m^{k,\ell} \to \mathcal{B}_m^{k,\ell}.$$

Moreover, since the map φ is an isomorphism, we can define for $B \in \mathcal{B}_m^{k,\ell}$ an element of $\Upsilon_m^{k,\ell}$ by the commutative diagram:

$$\begin{array}{ccc} \mathcal{A}_k^- \otimes \mathcal{A}_\ell^- & \xrightarrow{\varphi_k^{-1} \otimes \varphi_\ell^{-1}} & \mathrm{Hol}(\mathcal{H}) \otimes \mathrm{Hol}(\mathcal{H}) \\ {\scriptstyle B} \downarrow & & \downarrow {\scriptstyle [\,,\,]_m^{k,\ell}} \\ \mathcal{A}_m^- & \xrightarrow[\varphi_m^{-1}]{} & \mathrm{Hol}(\mathcal{H}) \end{array}$$

Which gives a map

$$\mathcal{B}_m^{k,\ell} \to \Upsilon_m^{k,\ell}.$$

This is the inverse of that defined by the diagram (5.2).

Therefore $\Upsilon_m^{k,\ell}$ is isomorphic to $\mathcal{B}_m^{k,\ell}$. $\square$

Definition 5.4. Let

$$\Omega_m^{(k,\ell)} := \left\langle B \in \mathcal{U}(\mathfrak{n}^+) \otimes \mathcal{U}(\mathfrak{n}^+) \mid \Delta(E_-)B(v_0 \otimes \widetilde{v_0}) = 0, \Delta(\hat{H})B(v_0 \otimes \widetilde{v_0}) = mB(v_0 \otimes \widetilde{v_0}) \right\rangle.$$

So $\Omega_m^{(k,\ell)}$ is given by the set of actions such that $B(v_0 \otimes \widetilde{v_0})$ is annihilated by the action of $\Delta(E_-)$ and is an eigenvector of $\Delta(\hat{H})$ with eigenvalue m.

Note that the lowest weight modules V_k of $\mathfrak{g}_{\mathbb{C}}$ are given by the same actions for E_-, E_+ and $\hat{H}$ as for X_-, X_+ and H when calculating in $\mathfrak{g}$, since these are both $\mathfrak{sl}(2)$-triples.

Proposition 5.5. *There is an isomorphism:*

$$\beta : \Omega_m^{(k,\ell)} \xrightarrow{\sim} \mathcal{B}_m^{k,\ell},$$

given by

$$\beta := \mu \circ (\rho \otimes \rho). \tag{5.3}$$

Here, μ denotes mapping of the tensor product into a product, and ρ is given in Proposition 4.5.

In order to prove this, El Gradechi utilises isomorphisms from both of these spaces to the space $\mathrm{Hom}_{\mathfrak{g}}(V_m, V_k \otimes V_\ell)$. See [EG06, Lemma 3.4, Theorem 3.5] for a full proof.

5.2 Proving modularity

We have shown in Section 4 that the following map is annihilated by $\Delta(X_-)$ and gives a $\Delta(H)$ eigenvector with eigenvalue $k+\ell+2n$:

$$\sum_j (-1)^j \binom{k+n-1}{n-j}\binom{\ell+n-1}{j}(X_+)^j \otimes (X_+)^{n-j}.$$

Using a change of basis from $\mathfrak{g}$ to our alternative basis for $\mathfrak{g}_{\mathbb{C}}$, it follows that

$$B := \sum_j (-1)^j \binom{k+n-1}{n-j}\binom{\ell+n-1}{j}(E_+)^j \otimes (E_+)^{n-j}$$

is an element of $\Omega_{k+\ell+2n}^{(k,\ell)}$.

Then, using the isomorphism given in (5.3):

$$\begin{aligned}\beta(B) &= \sum_j (-1)^j \binom{k+n-1}{n-j}\binom{\ell+n-1}{j}(\rho(E_+))^j(\rho(E_+))^{n-j}\\ &= \sum_j (-1)^j \binom{k+n-1}{n-j}\binom{\ell+n-1}{j}E^jE^{n-j},\end{aligned}$$

where $E=\rho(E_+)$, as defined in (4.7).

For $k\in\mathbb{N}$, the lift φ_k gives a map $E_k:\mathrm{Hol}(\mathcal{H})\to\mathrm{Hol}(\mathcal{H})$, such that $E\circ\varphi_k=\varphi_{k+2}\circ E_k$. This is $E_k:=2i\left(\frac{d}{dz}+\frac{k}{2iy}\right)=-4\pi\partial_k$ [EG13, Proposition 3.3].

Define

$$E_k^{\overline{s}}:=E_{k+2s-2}\circ\ldots\circ E_{k+2}\circ E_k,$$

which satisfies $\varphi_{k+2s}\circ E_k^{\overline{s}}=E^s\circ\varphi_k$.

Then

$$\partial_k^s=\left(\frac{-1}{4\pi}\right)^s E_k^{\overline{s}}. \tag{5.4}$$

We can now use the lift φ_k to map $\beta(B)$ to an element in $\Upsilon^{(k,\ell)}_{k+\ell+2n}$.

$$\begin{aligned}&\sum_j(-1)^j\binom{k+n-1}{n-j}\binom{\ell+n-1}{j}E_k^{\overline{j}}E_\ell^{\overline{n-j}}\\ &=\sum_j(-1)^j\binom{k+n-1}{n-j}\binom{\ell+n-1}{j}(-1)^j(4\pi)^j\partial_k^j(-1)^{n-j}(4\pi)^{n-j}\partial_\ell^{n-j}\\ &=\sum_j(-1)^j\binom{k+n-1}{n-j}\binom{\ell+n-1}{j}(-1)^n(4\pi)^n\partial_k^j\partial_\ell^{n-j}.\end{aligned}$$

Dividing by $(-1)^n(4\pi)^n$:

$$\sum_j(-1)^j\binom{k+n-1}{n-j}\binom{\ell+n-1}{j}\partial_k^j\partial_\ell^{n-j},$$

which is the n^{th} Rankin–Cohen bracket, as defined in (1.12). So $[\ ,\ ]_n\in\Upsilon^{(k,\ell)}_{k+\ell+2n}$.

This result leads to our main theorem.

Theorem 5.6. *Given two modular forms, $f\in\mathcal{M}_k(\Gamma)$, $h\in\mathcal{M}_\ell(\Gamma)$ and $n\in\mathbb{N}$, the n^{th}*

Rankin–Cohen bracket, given by

$$[f,h]_n = \sum_j (-1)^j \binom{k+n-1}{n-j} \binom{\ell+n-1}{j} D^j(f) D^{n-j}(h)$$

is a modular form of weight $k+\ell+2n$*.*

Proof. Let $f \in \mathcal{M}_k(\Gamma)$, $h \in \mathcal{M}_\ell(\Gamma)$. Since these are holomorphic functions, we can consider the action of an element of $\Upsilon^{(k,\ell)}_{k+\ell+2n}$ on the pair.

Since $[\ ,\ ]_n \in \Upsilon^{(k,\ell)}_{k+\ell+2n}$, we have that for any $g \in G$:

$$\pi_{k+\ell+2n}(g)[f,h]_n = [\pi_k(g)(f), \pi_k(g)h]_n.$$

So for $g \in G$,

$$([f,h]_n)\,|_{k+\ell+2n}(g^{-1}) = [f|_k(g^{-1}), h|_\ell(g^{-1})]_n.$$

In particular, for $\gamma \in \Gamma$, we have

$$([f,h]_n)\,|_{k+\ell+2n}(\gamma^{-1}) = [f|_k(\gamma^{-1}), h|_\ell(\gamma^{-1})]_n.$$

And since $\gamma^{-1} \in \Gamma$ and f and h are modular, this gives

$$[f,h]_n|_{k+\ell+2n}(\gamma^{-1}) = [f,h]_n,$$

for all $\gamma \in \Gamma$. Therefore $[f,h]_n \in \mathcal{M}_{k+\ell+2n}(\Gamma)$. □

5.3 Proving uniqueness

In order to prove uniqueness of the Rankin–Cohen brackets, we will show that the space $\Upsilon^{(k,\ell)}_{k,\ell,m}$ has at most dimension one, with a basis aligning exactly with the Rankin–Cohen brackets. This proof is discussed in more detail by El Gradechi [EG06] [EG13]. For the proof that we outline here, we assume that the weights of spaces of modular forms which we use are strictly positive. Note that in [EG06], this is covered for all weights.

We have defined $W_n^{k,\ell}$ to be the subspace of $V_k \otimes V_\ell$ of vectors with $\Delta(H)$-eigenvalue $k + \ell + 2n$. So we have:

$$W_n^{k,\ell} = \langle v_j \otimes \widetilde{v}_{n-j} \mid 0 \leq j \leq n \rangle \,,$$

from Proposition 4.1.

Definition 5.7. Define $N_n^{k,\ell}$ to be the subspace of $W_n^{k,\ell}$ that is annihilated by $\Delta(X_-)$:

$$N_n^{k,\ell} := \ker\left(\Delta(X_-)|_{W_n^{k,\ell}}\right).$$

Then we have the following result.

Proposition 5.8. *Given $N_n^{k,\ell}$ as above, for all $n \in \mathbb{N}$ and $k, \ell \in \mathbb{N}_{>0}$,*

$$\dim(N_n^{k,\ell}) = 1.$$

Proof. Note that for $v_j \otimes \widetilde{v}_{n-j} \in W_n^{k,\ell}$,

$$\Delta(X_-)(v_j \otimes \widetilde{v}_{n-j}) = X_- v_j \otimes \widetilde{v}_{n-j} + v_j \otimes X_- \widetilde{v}_{n-j} = v_{j-1} \otimes \widetilde{v}_{n-j} + v_j \otimes \widetilde{v}_{n-j-1} \in W_{n-1}^{k,\ell}.$$

So

$$\Delta(X_-)|_{W_n^{k,l}} : W_n^{k,l} \to W_{n-1}^{k,l}. \tag{5.5}$$

Since $\dim(W_n^{k,l}) = n + 1$, this means that $\dim(N_n^{k,\ell}) \geq 1$.

So there is at least one non-zero element in the kernel. Let $\mathbf{v} = v_j \otimes \widetilde{v}_{n-j} \in N_n^{k,\ell}$ be non-zero.

Then $\mathbf{v}$ is an eigenvector for ΔH with eigenvalue $k + \ell + 2n$, and $\Delta(X_-)(\mathbf{v}) = 0$. Thus $\mathbf{v}$ is the lowest weight vector in a $\mathfrak{sl}(2,\mathbb{R})^\Delta$-submodule with weight $k + \ell + 2n$. We have shown in the proof of Proposition 4.2 that this is isomorphic to $V_{k+\ell+2n}$. Given a lowest weight vector of a weight module, the map $\Delta(X_+)$ then generates the entire module. So $\mathbf{v}$ generates a submodule of $\mathfrak{sl}(2,\mathbb{R})^\Delta$ that is isomorphic to $V_{k+\ell+2n}$.

Moreover, we note that

$$\Delta(X_+)|_{W_n^{k,l}} : W_n^{k,l} \to W_{n+1}^{k,l} \tag{5.6}$$

is injective since X_+ is.

Assume that there are two or more linearly independent vectors in $N_n^{k,\ell}$. These would all generate $\mathfrak{sl}(2,\mathbb{R})^\Delta$-submodules isomorphic to $V_{k+\ell+2n}$. The injectivity of $\Delta(X_+)$ ensures that these submodules would be different. However, there is only one copy of each $V_{k+\ell+2n}$ in the decomposition of $V_k \otimes V_\ell$, and hence a contradiction is reached. Therefore $\dim(N_n^{k,\ell}) = 1$. □

Proposition 5.9. *There is an isomorphism:*

$$\Omega_{k+\ell+2n}^{(k,\ell)} \xrightarrow{\sim} N_n^{(k,\ell)}.$$

Proof. This follows directly from definition. We have $\Omega_{k+\ell+2n}^{(k,\ell)} \subset W_n^{k,\ell}$, since any $B \in \Omega_{k+\ell+2n}^{(k,\ell)}$ has a $\Delta(H)$-eigenvalue of $k+\ell+2n$. Then B is also in $\ker(\Delta(X_-))$ by definition.

□

Proposition 5.10. *Let $k, \ell \in \mathbb{N}_{>0}$. If $m \notin k+\ell+2\mathbb{Z}$, then*

$$\Upsilon_m^{k,\ell} = \mathcal{B}_m^{k,\ell} = \Omega_m^{k,\ell} = \{0\}.$$

Proof. A generalised $\Delta(H)$-eigenvector is of the form $v_i \otimes \widetilde{v_j} \in V_k \otimes V_\ell$ with eigenvalue $k+\ell+2(i+j)$ from Proposition 4.1.

So for $B \in \Omega_m^{(k,\ell)}$,

$$\Delta(H)(B(v_0 \otimes \widetilde{v_0})) = mB(v_0 \otimes \widetilde{v_0}).$$

Since $B(v_0 \otimes \widetilde{v_0}) \in V_k \otimes V_\ell$, this can only be an H-eigenvector for an eigenvalue in $k+\ell+2\mathbb{Z}$.

So when $m \notin k+\ell+2\mathbb{Z}$, $\Omega_m^{(k,\ell)} = \{0\}$. The other values follow from the isomorphisms in Propositions 5.3, 5.5 and 5.9. □

This leads us to our final result.

Theorem 5.11. *For each $n \in \mathbb{N}$, the Rankin–Cohen brackets are the unique (up to scalars) bi-differential operators:*

$$[\ ,\ ] : \mathrm{Hol}(\mathcal{H}) \otimes \mathrm{Hol}(\mathcal{H}) \to \mathrm{Hol}(\mathcal{H}).$$

Proof. We know by Proposition 5.10 that if $m \notin k+\ell+2\mathbb{Z}$ then $\dim(\Upsilon_m^{k,\ell}) = \{0\}$. So we consider only $m \in k+\ell+2\mathbb{Z}$.

Then from Propositions 5.3, 5.5, and 5.9 we have isomorphisms:

$$\Upsilon^{k,\ell}_{k+\ell+2n} \cong \mathcal{B}^{k,\ell}_{k+\ell+2n} \cong \Omega^{k,\ell}_{k+\ell+2n} \cong N^{(k,\ell)}_n, \tag{5.7}$$

for all $n \in \mathbb{N}$.

By Proposition 5.8 we have $\dim(N^{k,\ell}_n) = 1$, so since $\Upsilon^{k,\ell}_n \cong N^{k,\ell}_n$, we have $\dim(\Upsilon^{k,\ell}_n) = 1$.

We have shown in Section 5.1 that the n^{th} Rankin–Cohen bracket is an element of $\Upsilon^{k,\ell}_{k+\ell+2n}$. So the Rankin–Cohen brackets are unique, up to scalars.

□

CHAPTER 6

GENERALISATIONS TO SIEGEL MODULAR FORMS

We outline here the generalisations of this study to Siegel modular forms, an area in which while much interesting research has been done, there are also a number of open questions.

The existing generalisations can be split into two main areas – first the generalisation of Rankin–Cohen brackets to Siegel modular forms, and second the association of a representation to a Siegel modular form. We note that the majority of the work in generalising these objects is for scalar-valued Siegel modular forms. To my knowledge there is no known association between the representation-theoretic approach and Rankin–Cohen brackets, as there is for classical modular forms.

6.1 Rankin–Cohen brackets for Siegel modular forms

We call bi-differential operators which map two Siegel modular forms to a third Siegel modular form Rankin–Cohen brackets, analogously to the brackets for classical modular forms. There has been much written about Rankin–Cohen brackets of Siegel modular forms. Ibukiyama uses a relation to pluri-harmonic polynomials to determine r-linear differential operators [Ibu+99]. We will focus here on the case $r = 2$ as for Rankin–Cohen brackets. Using this method, Eholzer and Ibukiyama found a recursive relation which defines the Rankin–Cohen brackets for scalar-valued Siegel modular forms [EI98].

In order to describe this method, we require the following definitions.

Definition 6.1. Let P be a polynomial in the matrix variable $X = (x_{r,s}) \in M_{n,d}$, and define

$$\Delta_{i,j}(X) = \sum_{\nu=1}^{d} \frac{\partial^2}{\partial x_{i,\nu} \partial x_{j,\nu}}, \qquad (1 \leq i, j \leq n).$$

P is *harmonic* if $\sum_{i=1}^{n} \Delta_{i,i}(X)P = 0$, and *pluri-harmonic* if $\Delta_{i,j}(X)P = 0$ for all $1 \leq i, j \leq n$.

Definition 6.2. Define $\mathcal{Q}_{n,v}(2)$ to be the space of polynomials Q of symmetric matrices $R_1, R_2 \in M_{n,n}$ such that $Q(A^t R_1 A, A^t R_2 A) = \det(A)^v Q(R_1, R_2)$ for all $A \in \mathrm{GL}(n, \mathbb{C})$.

Then define $\mathcal{H}_{n,v}(2k, 2\ell)$ to be the subspace of $\mathcal{Q}_{n,v}(2)$ consisting of elements Q such that $Q(X_1X_1^t, X_2X_2^t)$ is pluri-harmonic for $X_1 \in M_{n,2k}$, and $X_2 \in M_{n,2\ell}$.

In [EI98, Theorem 3.4], Eholzer and Ibukiyama defined a system of recursion relations for $\mathcal{H}_{n,v}(2k, 2\ell)$ which can then be solved for specific values of v and n. This is very useful in conjunction with the following proposition.

Proposition 6.3. *Let $\Gamma \subset \mathrm{Sp}(2n, \mathbb{Q})$ be commensurable with $\mathrm{Sp}(2n, \mathbb{Z})$ and let $F \in \mathcal{M}_k(\Gamma_n)$ and $G \in \mathcal{M}_\ell(\Gamma_n)$ be scalar valued Siegel modular forms. Then, let $D = Q(\partial_{Z_1}, \partial_{Z_2})$ for $Q \in \mathcal{H}_{n,v}(2k, 2\ell)$, where ∂_Z is the $n \times n$ matrix with components:*

$$\frac{1}{2}(1 + \delta_{r,s})\frac{\partial}{\partial z_{r,s}}$$

for $Z = (z_{r,s}) \in \mathcal{H}_n$.

Then $D(F(Z)G(Z))$ is a Siegel cusp form, and so D gives a Rankin–Cohen like bi-differential operator on Siegel modular forms.

Proof. See [EI98, Proposition 3.5]. □

It is noted that this method does not determine all such bi-differential operators, and may miss operators when they are not unique. Also, while it does not give a general closed form for the operators, it does give a method to evaluate the required pluri-harmonic polynomials and thus Rankin–Cohen brackets, for a given n and v. Using this method, an explicit formula for Rankin–Cohen brackets for Siegel modular forms of order 2 has been found by Choie and Eholzer [CE98, Theorem 1.4], as follows.

Proposition 6.4. *Let $F \in \mathcal{M}_k(\Gamma_2)$ and $G \in \mathcal{M}_\ell(\Gamma_2)$. Then we have the following map for all $n \in \mathbb{N}$:*

$$[\ ,\ ]_n : \mathcal{M}_k(\Gamma_2) \otimes \mathcal{M}_\ell(\Gamma_2) \to \mathcal{M}_{k+\ell+2n}(\Gamma_2),$$
$$[F, G]_n = \sum_{r+s+p=n} C_{r,s,p}(k, \ell)\mathbb{D}^p(\mathbb{D}^r(F)\mathbb{D}^s(G)).$$

Where $\mathbb{D}$ is the differential operator given by

$$\mathbb{D} := 4\frac{d}{d\tau}\frac{d}{d\tau'} - \frac{d^2}{dz^2}$$

for $Z = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in \mathcal{H}_2$, *and*

$$C_{r,s,p}(k,\ell) := \frac{(k+n-3/2)_{s+p}}{r!}\frac{(\ell+n-3/2)_{r+p}}{s!}\frac{(-(k+\ell+n-3/2))_{r+s}}{p!}.$$

Choie and Eholzer used a number of other methods to prove this proposition, including using a combinatorial proof and one using theta series. However, the proof which follows from Proposition 6.3 is perhaps most interesting as the same method can be followed to derive Rankin–Cohen operators for higher degrees.

6.2 Associating a representation to a Siegel modular form

Associating a representation to a scalar-valued Siegel modular form follows in a similar way from our association of a modular form to a representation in Chapter 3. The main resources for this are Asgari and Schmidt [AS01] and Pitale [Pit19].

We work here with representations of $\mathrm{GSp}(2n, \mathbb{A})$. As in the classical case, a representation can be associated to a scalar valued Siegel modular form by considering an isomorphism from the space of cusp forms to an automorphic function on $\mathrm{GSp}(2n, \mathbb{A})$.

Similarly to the classical case, this isomorphism is given by:

$$f \mapsto \phi_f, \tag{6.1}$$

where $\phi_f(g) = (f|_k g_\infty)(I)$ for $g \in \mathrm{GSp}(2n, \mathbb{A})$ with g_∞ the part at infinity, and $I = \mathrm{diag}(i, ..., i) \in \mathcal{H}_n$. Note that in the isomorphisms given in Chapter 3, we worked only at the point at infinity to obtain a function on $\mathrm{GL}(2, \mathbb{R})$.

Here, ϕ_f is a function in the space $L^2(Z(\mathbb{A})\,\mathrm{GSp}(2n, \mathbb{Q})\backslash \mathrm{GSp}(2n, \mathbb{A}))$, where Z is the centre of $\mathrm{GSp}(2n)$. When $f \in \mathcal{S}_k(\Gamma_n)$ is an eigenform, we let π be an irreducible component of the representation given by right translations of ϕ_f. Then π is a representation of $\mathrm{GSp}(2n, \mathbb{A})$ that is stable under $Z(\mathbb{A})$, and so is a representation of $\mathrm{PGSp}(2n, \mathbb{A})$. Let

$$\pi = \otimes_p \pi_p, \tag{6.2}$$

where π_p denotes the irreducible representation of $\mathrm{GSp}(2n, \mathbb{Q}_p)$ in the decomposition.

We can then study the representation π_p for each p. We have the following result:

Theorem 6.5. *Let $F \in S_k(\Gamma_n)$ and let π be the automorphic representation of $\mathrm{PGsp}(2n, \mathbb{A})$ associated with F. Then the local components of π are:*

1. *π_∞ is a holomorphic discrete series representation for $k > n$, and a limit of discrete series for $k = n$.*
2. *At a finite places, π_p is a spherical principal series representation of $\mathrm{PGSp}(2n, \mathbb{Q}_p)$.*

Proof. See [AS01, Theorem 2] and [Pit19, Theorem 6.14]. □

In the present research we have focused on the associated representation at infinity, since this is where Rankin–Cohen brackets arise in genus 1. We note that the part at infinity for Siegel modular forms is also a holomorphic discrete series representation, and so we may be able to follow the above method to relate the Rankin–Cohen brackets to the representation. As such, we focus again on the representation at infinity.

For any reductive linear Lie group G, the discrete series representations are determined by what is called the *Harish–Chandra parameter*, denoted λ. For the holomorphic discrete series of $\mathrm{SL}(2, \mathbb{R})$, the Harish–Chandra parameter for $\mathcal{D}^{\pm}(k)$ is given by $\pm(k+1)$.

To describe the discrete series representations associated to Siegel modular forms, we require some additional notation. For $G = \mathrm{Sp}(2n, \mathbb{R})$, a maximal compact subgroup is

$$K_\infty = \left\{ g \in \mathrm{Sp}(2n, \mathbb{R}) \mid g = \left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \right\}. \tag{6.3}$$

Define $T_i = -i\left[\begin{smallmatrix} 0 & D_i \\ -D_i & 0, \end{smallmatrix}\right]$ in the complexified Lie algebra $\mathfrak{k}_\mathbb{C}$ of K_∞. Here, D_i is the matrix that has 1 at the position (i, i) and zeros elsewhere. Then

$$h = \mathbb{R}T_1 + \mathbb{R}T_2 + ... + \mathbb{R}T_n \tag{6.4}$$

is a compact Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$.

Then define linear maps e_i on h by:

$$e_i(T_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

The holomorphic discrete series representations of $\mathrm{PGSp}(2n, \mathbb{R})$ are given by Harish–Chandra parameter

$$(k-1)e_1 + ... + (k-n)e_n. \tag{6.6}$$

When $k = n$, this is the limit of the discrete series.

When $k > n$, these representations, which we will call π_k are lowest weight representations, with lowest weight vector $k(e_1 + ... + e_n)$ and highest weight vector $-k(e_1 + ... + e_n)$.

While this is given in detail for scalar-valued Siegel modular forms, Asgari and Schmidt [AS01] also give an outline of the case for vector-valued Siegel modular forms. Here, they use a variation of the lift to associate a scalar-valued Siegel modular form to a function on $\mathrm{GSp}(2n, \mathbb{A})$. After normalising, this gives a function on $L^2(Z(\mathbb{A})\,\mathrm{GSp}(2n, \mathbb{Q})\backslash \mathrm{GSp}(2n, \mathbb{A}))$, which is the same space that was used for the scalar-valued forms. Associating a representation of $\mathrm{PGSp}(2n, \mathbb{A})$ to this function then proceeds as in the first case. At $p = \infty$, this is again a lowest weight representation of $\mathrm{GSp}(2n, \mathbb{R})$. If $f \in \mathcal{S}_\rho(\Gamma_n)$ with $\rho_{r_1, r_2, ..., r_n}$ then the associated representation π_∞ is the representation with minimal K_∞ type $\tau_{r_1, r_2, ..., r_n}$. If $r_n > n$ then π_∞ is a holomorphic discrete series with Harish–Chandra parameter

$$(r_1 - 1)e_1 + ... + (r_n - n)e_n.$$

Note that the scalar-valued version is the special case where $r_1 = r_2 = ... = r_n = k$.

6.3 Avenues for further research

This is a rich area of mathematical research, with many avenues for further studies. We have seen that the Rankin–Cohen brackets can be generalised to scalar-valued Siegel modular forms, as can the representations associated to modular forms. In particular, I am interested to see if we can generalise the way in which Rankin–Cohen brackets arise in these representations, thus connecting the two. There are a number of similarities between the representation theory perspective of classical and Siegel modular forms. These similarities suggest it may be possible to follow the methods used in this paper in the case of Siegel modular forms to derive the Rankin–Cohen brackets in this setting.

NOTATION INDEX

$\mathcal{H}$	Complex upper half plane,
$\|_k\gamma$	Slash operator of weight k; $(f\|_k\gamma)(z) = (cz+d)^{-k}f(\gamma\cdot z)$,
$j(g,z)$	The automorphy factor for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{R})$ and $z \in \mathcal{H}$; $j(g,z) = (cz+d)$,
$\mathcal{M}_k$	Space of weight k and level 1 modular forms,
$\mathcal{M}$	The graded ring of level 1 modular forms; $\mathcal{M} := \oplus_k \mathcal{M}_k$,
$\mathcal{S}_k$	Space of weight k and level 1 cusp forms,
$E_k(z)$	The Eisenstein series of weight k, for $k > 2$,
Γ	Congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$,
$\Gamma(n)$	The principal congruence subgroup of level n,
$\mathcal{M}_k(\Gamma)$	Space of weight k modular forms with respect to Γ,
$\mathcal{M}(\Gamma)$	Graded ring of modular forms with respect to Γ,
$\mathcal{S}_k(\Gamma)$	Space of weight k cusp forms with respect to Γ,
Df	The differential operator $\frac{1}{2\pi i}\frac{d}{dz}$ on $f \in \mathcal{M}_k(\Gamma)$,
$[\,,\,]_n$	n^{th} Rankin–Cohen bracket,
$\widetilde{M}_k(\Gamma)$	Space of modular but non-holomorphic functions over Γ of weight k,
∂_k	The Shimura operator for k,
$\mathcal{H}_n$	The Siegel upper half space of genus n,
I_n	The $n \times n$ identity matrix,
$J(\gamma, Z)$	The automorphy factor for $\gamma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{R})$ and $Z \in \mathcal{H}_n$; $J(\gamma, Z) = (CZ+D)$,
$\mathcal{M}_k(\Gamma_g)$	The space of all Sigel modular forms of degree $g \in \mathbb{N}$ and weight $k \in \mathbb{Z}$,
$\mathcal{M}_\rho(\Gamma_g)$	The space of all Sigel modular forms of degree $g \in \mathbb{N}$ and weight ρ,
$\mathcal{S}k(\Gamma_g)$	The space of all Sigel cusp forms of degree $g \in \mathbb{N}$ and weight $k \in \mathbb{Z}$,
$\mathcal{S}\rho(\Gamma_g)$	The space of all Sigel cusp forms of degree $g \in \mathbb{N}$ and weight ρ,
V_K	The space of all K-finite vectors of a representation (π, V); also the underlying $(\mathfrak{g}, K)$-module of (π, V) ,
V^∞	The space of all smooth vectors relative to a representation (π, V),

$d\pi$	The derived representation of (π, V) on the associated Lie algebra,
dX	The derived action of $X \in \mathfrak{g}$ on $f \in C^\infty(G)$ with respect to the right regular representation of G,
$V(\sigma)$	The sum of K-submodules of V in the equivalence class σ, where σ ranges over equivalence classes of irreducible unitary representations of K,
$\mathcal{U}(\mathfrak{g})$	The universal enveloping algebra of $\mathfrak{g}$,
C	The Casimir element of $\mathfrak{sl}(2, \mathbb{R})$,
κ_θ	The matrix $\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$,
Σ	The set of K-types of a representation V,
$\mathcal{P}(\lambda, \epsilon)$	Principal series representations of $\mathrm{SL}(2, \mathbb{R})$,
$\mathcal{D}^\pm(k)$	Discrete series representations of $\mathrm{SL}(2, \mathbb{R})$,
$\mathcal{D}^\pm(1)$	Limit of discrete series of $\mathrm{SL}(2, \mathbb{R})$,
$\mathfrak{H}$	Space of holomorphic and square-integrable functions on the upper half plane,
π_k	The representation on $\mathrm{SL}(2, \mathbb{R})$ associated to the space $\mathcal{M}_k(\Gamma)$,
V_λ	Generalised H-eigenspace of V for $\lambda \in \mathbb{C}$,
V_k	Lowest weight $\mathfrak{sl}(2, \mathbb{R})$ module with lowest weight k,
v_0	Lowest weight vector of a lowest weight module V_k for some $k \in \mathbb{C}$,
$\overline{V}_k$	Highest weight $\mathfrak{sl}(2, \mathbb{R})$ module with highest weight k,
$C^\infty(\Gamma \setminus \mathcal{H}, k)$	The space $\{f \in C^\infty(\Gamma \setminus \mathcal{H}) \mid f(z) = f\|_k \gamma(z) \text{ for all } \gamma \in \Gamma\}$,
$L^2(\Gamma \setminus \mathcal{H}, k)$	The Hilbert space completion of $C^\infty(\Gamma \setminus \mathcal{H}, k)$,
$L^2(\Gamma \setminus G, k)$	The space $\{F \in L^2(\Gamma \setminus G) \mid F(g\kappa_\theta) = \exp(ik\theta)F(g) \text{ for all } g \in G\}$,
$\mathcal{A}_k$	The space $\{F \in L^2(\Gamma \setminus G, k) \mid dE_- F = 0\}$,
Δ	The diagonal embedding of $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g} \otimes \mathcal{U}(\mathfrak{g}))$,
$\mathfrak{n}^+$	$\mathbb{C}E_+$,
$\mathcal{A}_k^-$	The space $\{F \in C^\infty(G) \| F(g\kappa_\theta) = \exp(ik\theta)F(g) \text{ for all } g \in G \text{ and } dE_- F = 0\}$,
$\Upsilon_m^{(k,\ell)}$	G-equivariant holomorphic bi-differential operators on $\mathrm{Hol}(\mathcal{H})$,
$\mathcal{B}_m^{(k,\ell)}$	G-equivariant holomorphic bi-differential operators $\mathcal{A}_k^- \otimes \mathcal{A}_\ell^- \to \mathcal{A}_m^-$.

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