Differential Equations Satisfied by Modular Forms

by

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Abstract

An important source of applications of modular forms in many branches of mathematics is the fact that if we express a modular form $f(z)$ of weight $k \geq 0$ as a function of a modular function $t(z)$, i.e., as $f(z) = \Phi(t(z))$, then Φ satisfies a $(k+1)$-st order linear differential equation with algebraic coefficients. In this thesis, we investigate what Φ is explicitly and what the differential Galois group of the linear differential equation can tell us about the nature of its solutions.

Declaration of Authorship

I, KEVIN JOHN FERGUSSON, declare that this thesis titled, 'DIFFERENTIAL EQUATIONS SATISFIED BY MODULAR FORMS' and the work presented in it are my own. I confirm that:

- The thesis comprises only my original work towards the Master of Science except where indicated in the preface;
- Due acknowledgement has been made in the text to all other material used; and
- The number of pages in this thesis is within the range of 40-60 pages in length, excluding references, appendices, figures and tables and hence complies with the assessment requirement stated in The University of Melbourne handbook.

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Preface

I declare that:

- All work carried out in this thesis is solely by me, with most of the examples and some of the proofs, which appear as exercises in the references, being worked out independently by me;
- No work towards this thesis has been submitted for other qualifications;
- No work towards this thesis has been carried out prior to enrolment in the degree;
- No third party editorial assistance has been used in the preparation of this thesis;
- As yet, no publication is forthcoming; and
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Abbreviations

DE	**D**ifferential **E**quation
LHS	**L**eft **H**and **S**ide
PV	**P**icard-**V**essiot
RHS	**R**ight **H**and **S**ide

Constants

Euler-Mascheroni constant	γ	=	$0.577215664901532860\ldots$
Euler's number or Napier's constant	e	=	$2.718281828459045\ldots$
Imaginary square root of −1	i	=	$\sqrt{-1}$
pi	π	=	$3.14159265358979323846264 33\ldots$

Symbols

$\mathrm{Aut}(K)$	set of automorphisms of the field K
$\mathrm{char}(K)$	characteristic of the field K
$\mathbb{C}$	set of complex numbers
$\mathbb{C}[x]$	ring of polynomials in x having complex coefficients
$\mathbb{C}(x)$	field of fractions of $\mathbb{C}[x]$
$\mathbb{C}[[x]]$	ring of all formal power series with coefficients in $\mathbb{C}$
$\mathbb{C}((x))$	field of fractions of $\mathbb{C}[[x]]$
$\mathbb{C}[\{x\}]$	ring of all convergent power series with coefficients in $\mathbb{C}$
$\mathbb{C}(\{x\})$	field of fractions of $\mathbb{C}[\{x\}]$
$E_k(z)$	Eisenstein series $\frac{1}{2}\sum_{(c,d)\in\mathbb{Z}^2,\,\mathrm{coprime}}(cz+d)^{-k}$
$\mathrm{Frac}(R)$	field of fractions of the integral domain R
$\mathrm{GL}_n(R)$	general linear group of n-by-n matrices having entries in R and non-zero determinant
$\mathrm{Hom}_R(A,B)$	set of R-linear homomorphisms from A to B, where A and B have R-actions, e.g., as R-modules
Id	identity element or identity operator
$\Im(z)$	imaginary part of the complex number z
$\mathrm{Isom}_R(A,B)$	set of R-linear isomorphisms from A to B, where A and B have R-actions, e.g., as R-modules
$\ker(f)$	kernel of a map f
$\mathrm{Mat}_n(k)$	set of n-by-n matrices having entries in k
$M_k(\Gamma_1)$	complex vector space of modular forms of weight $k \geq 0$

$M_*(\Gamma_1)$	algebra of modular forms $$\bigoplus_{\substack{k=0\\ k\,\text{even}}}^{\infty} M_k(\Gamma_1)$$
$\widetilde{M}_*(\Gamma_1)$	algebra of quasimodular forms $M_*(\Gamma_1)[E_2]$
$\mathrm{Mor}_R(A,B)$	set of R-linear morphisms from objects A to B, where A and B have R-actions, e.g., as R-modules
$\mathbb{N}$	set of natural numbers $\{0,1,2,\ldots\}$
$\mathrm{PSL}_n(R)$	projective special linear group given by the quotient $\mathrm{SL}_n(R)/\mathrm{SZ}_n(R)$
$\mathbb{R}$	set of real numbers
$\Re(z)$	real part of the complex number z
$\mathrm{SL}_n(R)$	special linear group of n-by-n matrices having entries in R and determinant one
$\mathrm{Span}_k\{v_1,\ldots,v_n\}$	set of elements in the k-linear span of $v_1,\ldots,v_n$
$\mathrm{Spec}(R)$	set of prime ideals of the ring R
$\mathrm{Sym}^n(L)$	differential operator whose solutions are n-th powers or n-fold products of solutions of the differential operator L
$\mathrm{SZ}_n(R)$	subgroup of $\mathrm{SL}_n(R)$ consisting of scalar transformations with unit determinant
$\mathrm{tr}(A)$	trace of the matrix A
$\mathrm{W}(y_1,\ldots,y_n)$	n-by-n matrix $\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$
$\mathrm{wr}(y_1,\ldots,y_n)$	Wronskian of $y_1,\ldots,y_n$, which is equal to $\det \mathrm{W}(y_1,\ldots,y_n)$
$\mathbb{Z}$	set of integers
Γ_1	special linear group $SL_2(\mathbb{Z})$

$\Gamma(N)$ principal congruence subgroup of level N, i.e.,

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N\right\}$$

$\Gamma_0(N)$ congruence subgroup of level N given by

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N\right\}$$

$\Gamma_1(N)$ congruence subgroup of level N given by

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N\right\}$$

μ_n multiplicative group of n-th roots of unity

$\pi_1(X, p)$ fundamental group of the topological space X with base point $p \in X$

$(a)_n$ Pochhammer symbol, equal to $a(a+1)(a+2)\ldots(a+n-1)$

Chapter 1

Introduction

1.1 Motivation of Thesis

This thesis concerns linear differential equations, having coefficients in the field of modular functions, which are satisfied by modular forms, and how differential Galois theory can shed further light on the solutions of such differential equations.

The primary motivation for this study is a remark in [Zag08]: if we express a modular form $f(z)$ as a function

$$f(z) = \Phi(t(z)) \tag{1.1}$$

of a modular function $t(z)$, i.e., t is a meromorphic modular form of weight zero, then Φ always satisfies a linear differential equation of finite order with algebraic coefficients. This fact features in many areas of mathematics, including the theory of the Picard-Fuchs differential equation and the Gauss-Manin connection.

A precise statement of this fact, given verbatim from p61 of [Zag08], is as follows.

Proposition 1.1 (Proposition 21 in Section 5.4 of [Zag08])**.** *Let $f(z)$ be a (holomorphic or meromorphic) modular form of positive weight k on some group Γ and $t(z)$ a modular function with respect to Γ. Express $f(z)$ (locally) as $\Phi(t(z))$. Then the function $\Phi(t)$ satisfies a linear differential equation of order $k+1$ with algebraic coefficients, or with polynomial coefficients if $\Gamma\backslash\mathscr{H}$ has genus 0 and $t(z)$ generates the field of modular functions on Γ.*

Remark 1.2. In [Zag18], the connection between differential equations and modular forms is described, whereby, given a family of smooth manifolds $\{Y_t\}_{t\in U}$ parametrised by some base space U, we define the period function $P(t) : t \mapsto \int_{[Y]} \omega_t$, that maps a parameter $t \in U$ to the integral of a chosen closed differential r-form ω_t, over a homology class $[Y_t] \in H_r(Y_t, \mathbb{Z})$. Here, we take U sufficiently small so that $[Y_t]$ is locally constant, i.e., equal to $[Y]$. The idea is that the set of differential r-forms $\partial^i \omega_t$, for $i \geq 0$, is a subset of the finite-dimensional r-th de Rham cohomology space $H^r(Y_t, \mathbb{C})$, and thus a finite linear combination of the differential forms is exact, and therefore the linear combination of integrals of these differential forms must vanish, i.e.,

$$\sum_{i=0}^{b} a_i(t) \frac{\partial^i \omega_t(x)}{\partial t^i} = d\eta_t \implies \sum_{i=0}^{b} a_i(t) \frac{d^i P(t)}{dt^i} = 0.$$

For further details, please see [Zag18].

Two problems under investigation are as follows:

(A) Given a modular form f and a modular function t, what is Φ explicitly, as given in (1.1)?

(B) Given a linear differential equation having coefficients in the field of modular functions, what can its differential Galois group tell us about the nature of its solutions?

1.2 Previous Work on Problem A

In [Zag08], three different proofs of Proposition 1.1 are supplied, where determining Φ explicitly is given as a matrix determinant. The idea is to express the weight-k modular form $f(z)$ as a function of a modular function $t(z)$, i.e., we write $f(z) = \Phi(t(z))$, valid for z in some open set. In this way, it can be shown that Φ satisfies a $(k+1)$-st order linear differential equation having coefficients in the field of modular functions. The first proof supplied by Zagier recasts the problem in terms of vector-valued quasimodular forms. Similar approaches are given in the papers of [BG07], [Yan04], [FM16] and [FM20]. The algorithm for generating differential equations satisfied by modular forms is supplied in Chapter 5.

In [PR21], the authors show, when given $f(z)$ and $t(z)$, how to prove whether a given holonomic DE, namely a linear DE whose coefficients are polynomials, is solved by Φ. Specifically, given a modular form f of weight $k \in \mathbb{Z}_{\geq 1}$ for the congruence subgroup Γ, and a modular function t, suppose that f has a local expansion of the form $f(z) = \Phi(t(z))$, where Φ has the series expansion $\Phi(u) = \sum_{\ell=0}^{\infty} c(\ell) u^{\ell}$. Furthermore, let $L(\Phi) = 0$ be a holonomic differential equation, i.e., where L can be written in the form

$$L = P_m(u)\partial^m + P_{m-1}(u)\partial^{m-1} + \ldots + P_0(u),$$

where $P_j(u) \in \mathbb{C}[u]$, with $P_m(u) \neq 0$. From this, the authors then supply an algorithm which proves that $\Phi(u)$ is a solution of the DE, subject to initial conditions.

The key aspect of their paper is that the authors are able to guess the DE $L(\Phi) = 0$ by computing the Puiseux series expansion of f in terms of t, guessing a recurrence relation among the coefficients $c(\ell)$ of the expansion, and constructing the holonomic DE based on the recurrence relation. It is this candidate DE that the authors subsequently prove is satisfied by the given modular form $f(z)$ parameterised in terms of $t(z)$.

Many of the examples supplied in [PR21] are examined in this thesis, employing differential Galois theory. A table summarising the differential Galois groups for the these examples is given in Chapter 5.

1.3 Previous Work on Problem B

Our second problem is that of finding solutions of linear differential equations with coefficients in the field of modular functions. Here, some light on this question is given in [Mag94], [PS03] and [Sin93], where differential Galois theory is described. Related to the differential Galois group of a linear differential equation (DE) is the monodromy group of a linear DE, which describes the behaviour of solutions to the DE when analytically continued around closed loops surrounding singularities of the DE. Such an approach is described in [Bat53], [PS03] and [Yos97]. In fact, the differential Galois group is the Zariski-closure of the monodromy group, so that computing the monodromy group is an effective way of determining the differential Galois group.

To give an idea of differential Galois (∂-Galois) theory for DEs, we use the classical Galois theory of algebraic equations as a reference. For algebraic equations there is the notion of a splitting field which contains all the roots of the equation. Similarly, for differential equations, there is the notion of a Picard-Vessiot ring which contains all solutions of a matrix differential equation $y' = Ay$ over the differential field k. Analogous to the classical case, it is the differential Galois group which gives information on whether the DE can be solved using algebraic operations, integration or exponentiation, known as Liouvillian operations.

1.4 Outline of Chapters in Thesis

In Chapter 2 we provide some background on modular forms and give several examples illustrating (1.1). In Chapter 3 we provide some background on differential Galois theory and in Chapter 4 we construct explicitly the differential Galois group of DEs for several examples. In Chapter 5 we give a formula for the DE satisfied by a given modular form, as well as giving the associated monodromy group and differential Galois group.

Chapter 2

Modular Forms

2.1 Background on Modular Forms

Because our problems concern differential equations satisfied by modular forms, we give some requisite background on modular forms; for a more detailed treatment see, e.g., [Zag08] or [Kob00] for more details.

Definition 2.1. *Modular forms* are holomorphic complex-valued functions on the upper half plane

$$\mathcal{H} = \{z \in \mathbb{C} \,|\, \Im(z) > 0\}$$

and at infinity, satisfying the functional equation

$$f(\gamma z) = (cz+d)^k f(z), \tag{2.1}$$

where

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$$

and γ acts on $z \in \mathbb{C}$ via $\gamma z = (az+b)/(cz+d)$. Here, $k \in \mathbb{Z}$ denotes the weight of the modular form; see, e.g., Chapter 1 of [Zag08]. We denote by $M_k(\Gamma_1)$ the $\mathbb{C}$-vector space of modular forms of weight k.

Inserting $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in (2.1), we see that $f(z) = (-1)^k f(z)$, and hence modular forms having odd weight are zero-valued. Furthermore, for $k < 0$, we have $M_k(\Gamma_1) = \{0\}$; see Proposition 3 on p12 of [Zag08]. In fact, $M_0(\Gamma_1) = \{f : \mathscr{H} \to \{c\} \mid c \in \mathbb{C}\} \cong \mathbb{C}$ and $M_2(\Gamma_1) = \{0\}$. Thus, we denote by $M_*(\Gamma_1)$ the algebra

$$\bigoplus_{\substack{k=0 \\ k \,\text{even}}}^{\infty} M_k(\Gamma_1).$$

Examples of modular forms include the Eisenstein series

$$E_k(z) := \frac{1}{2} \sum_{\text{coprime}\,(c,d)\in\mathbb{Z}\times\mathbb{Z}} \frac{1}{(cz+d)^k} \in M_k(\Gamma_1), \tag{2.2}$$

where $k \geq 4$ is an even integer. In fact, $M_*(\Gamma_1)$ is freely generated as a $\mathbb{C}$-algebra by E_4 and E_6.

Definition 2.2. A *modular function* is a meromorphic complex-valued function on the upper half plane and at infinity satisfying

$$f(\gamma z) = f(z), \tag{2.3}$$

for all $z \in \mathscr{H}$ and $\gamma \in \Gamma_1$.

Here, we extend the set $M_0(\Gamma_1)$ of modular forms having weight zero to the larger set $M(\Gamma_1)$ of modular functions which contains meromorphic functions in addition to holomorphic functions. By virtue of (2.3), we can view modular functions as functions on the compact quotient space $X(\Gamma_1) = \Gamma_1 \backslash (\mathscr{H} \bigcup \mathbb{Q} \bigcup \{\infty\})$, so that by including meromorphic functions in our definition of modular functions we are guaranteed to have non-constant modular functions on $X(\Gamma_1)$.

A fundamental domain for the Γ_1-action on $\mathscr{H}$ is

$$\mathscr{F}(\Gamma_1) = \{z \in \mathbb{C} \mid |z| \geq 1 \,\text{and}\, \Re(z) \in [-1/2, 1/2]\}$$

and is depicted in Figure 2.1. Two generators for Γ_1 are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

An important example of a modular function is Klein's j-invariant

$$j(z) = 1728 \frac{E_4(z)^3}{(E_4(z)^3 - E_6(z)^2)}, \tag{2.4}$$

which gives a bijection from $\mathscr{F}(\Gamma_1)$ to the Riemann sphere $\mathbb{P}^1 \cong \mathbb{C}\bigcup\{\infty\}$. Furthermore, all modular functions are simply rational functions of j, so that $M(\Gamma_1) \cong \mathbb{C}(j)$; e.g., see Proposition 12 on p119 of [Kob00].

Definition 2.3. A *hauptmodul*, or *principal modular function*, is a modular function which generates the field of modular functions with respect to Γ_1. The hauptmodul is unique up to a Möbius transformation, i.e., a transformation in $\text{Möb}(\mathbb{C}\bigcup\{\infty\}) = \text{PSL}_2(\mathbb{C})$.

In fact, we can extend our definitions of modular forms and modular functions to other discrete subgroups of $\text{SL}_2(\mathbb{R})$.

Definition 2.4. Let N be a positive integer. We define the *principal congruence subgroup of level N* to be

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

A subgroup Γ of Γ_1 is called a *congruence subgroup of level N* if $\Gamma \supseteq \Gamma(N)$. In particular, we define the following congruence subgroups of level N:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

For comparison with Γ_1, a fundamental domain for the $\Gamma(2)$-action on $\mathscr{H}$ is given by

$$\mathscr{F}(\Gamma(2)) = \bigcup_{\ell=1}^{6} \gamma_i^{-1} \mathscr{F}(\Gamma_1),$$

where $\gamma_1 = I$, $\gamma_2 = T$, $\gamma_3 = S$, $\gamma_4 = TS$, $\gamma_5 = ST$ and $\gamma_6 = T^{-1}ST$, and is depicted in Figure 2.2.

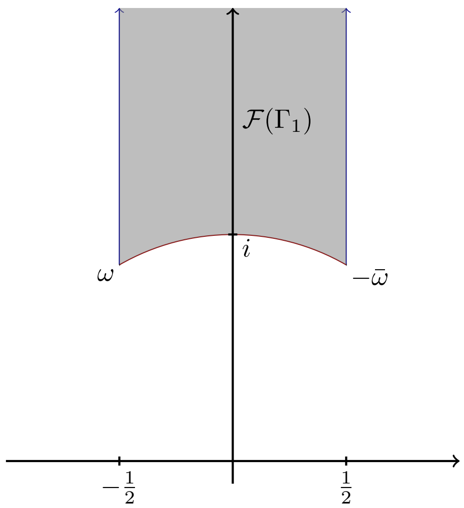

FIGURE 2.1: A fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathscr{H}$.

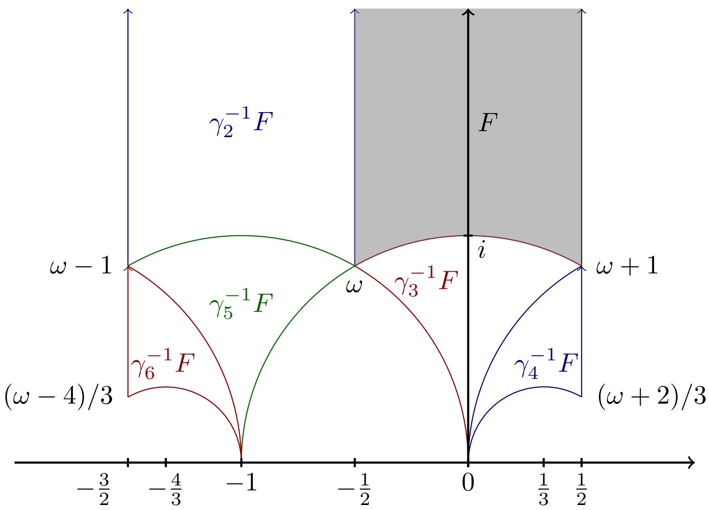

FIGURE 2.2: A fundamental domain $\mathscr{F} = \bigcup \gamma_i^{-1} F$ for the action of $\Gamma(2)$ on $\mathscr{H}$.

2.2 Application to Differential Equations

We describe notions of derivations, differential rings and algebras that are applied to differential equations (DEs) involving modular forms. These notions will be clarified in more detail in Chapter 3.

As described on p48 of [Zag08], a derivation ∂, sometimes called the Serre derivative, can be defined on $M_*(\Gamma_1)$ as follows,

$$\partial f := \frac{1}{2\pi i}\frac{df}{dz} - \frac{k}{12}E_2 f, \tag{2.5}$$

where $f \in M_k(\Gamma_1)$, $k \geq 0$ is an even integer and

$$E_2(z) := \frac{6}{\pi^2}\left(\frac{1}{2}\sum_{n\in\mathbb{Z}\, n\neq 0}\frac{1}{n^2} + \frac{1}{2}\sum_{c\in\mathbb{Z}\, c\neq 0}\sum_{d\in\mathbb{Z}}\frac{1}{(cz+d)^2}\right).$$

This derivation satisfies the Leibniz rule and has the property that $\partial M_k(\Gamma_1) \subseteq M_{k+2}(\Gamma_1)$ for even integers $k \geq 0$.

For example, using the Ramanujan equations

$$\frac{1}{2\pi i}\frac{d}{dz}E_2(z) = \frac{E_2^2 - E_4}{12}, \quad \frac{1}{2\pi i}\frac{d}{dz}E_4(z) = \frac{E_2E_4 - E_6}{3}, \quad \frac{1}{2\pi i}\frac{d}{dz}E_6(z) = \frac{E_2E_6 - E_4^2}{2}, \tag{2.6}$$

we see that

$$\partial E_2 = -\frac{1}{12}(E_2^2 + E_4), \quad \partial E_4 = -E_6/3 \in M_6(\Gamma_1), \quad \partial E_6 = -E_4^2/2 \in M_8(\Gamma_1), \tag{2.7}$$

also given in Section 5.1, pp48-49 of [Zag18].

As indicated previously, any modular form $f \in M_k(\Gamma_1)$ can be written as a polynomial in E_4 and E_6, e.g., as demonstrated in Proposition 10 on p118 of [Kob00]. However, terms involving E_2 appear when we compute ∂f or $\frac{1}{2\pi i}\frac{df}{dz}$. Hence, introducing the ring of quasimodular forms

$$\widetilde{M}_*(\Gamma_1) := M_*(\Gamma_1)[E_2] = \mathbb{C}[E_2, E_4, E_6], \tag{2.8}$$

we find that $\partial\widetilde{M}_*(\Gamma_1) \subset \widetilde{M}_*(\Gamma_1)$ and, therefore, the pair $(\widetilde{M}_*(\Gamma_1), \partial)$ forms a differential ring; see, e.g., Chapter 1 of [PS03].

Now, suppose we have a linear differential operator $L \in \widetilde{M}_*(\Gamma_1)[\partial]$. Write $R = \widetilde{M}_*(\Gamma_1)$ and let $k = \mathrm{Frac}(R)$. We wish to solve the DE $L(y) = 0$ over k. We seek a field extension K of k such that the solution space

$$\mathrm{Sol}_K(L) := \{f \in K \mid Lf = 0\}$$

has dimension, as a $\mathbb{C}$-vector space, equal to $\deg L$. Choosing a basis $\{f_1, \ldots, f_n\}$ for $\mathrm{Sol}_K(L)$, where $n = \deg L = \dim \mathrm{Sol}(L)$, we can view the differential Galois group $\mathrm{Gal}^\partial(L)$ as the subgroup of $\mathrm{GL}_n(\mathbb{C})$ composed of all linear transformations which preserve all algebraic relations among

$$f_1, \ldots, f_n, \partial f_1, \ldots, \partial f_n, \ldots, \partial^{n-1} f_1, \ldots, \partial^{n-1} f_n,$$

as described in Section 1.4 in Chapter 1 of [PS03]. We will revisit this example involving the Serre derivative in Example 4.3 in Chapter 4.

Since any quasimodular form $f \in \widetilde{M}_*(\Gamma_1)$ and its higher order differentials $\partial^m f$ are expressible as $\mathbb{C}$-linear combinations of elements in $\{E_2^i E_4^j E_6^k \mid i, j, k \in \mathbb{N}\}$, we see that the set

$$\{f, \partial f, \partial^2 f, \partial^3 f\}$$

is algebraically dependent, i.e., f satisfies a third-order nonlinear differential equation; see Proposition 16 on Page 49 of [Zag08]. In particular, using the Ramanujan equations (2.6) we see that $f = E_2$ satisfies the DE $f^{(3)} - f f'' + \frac{3}{2}(f')^2 = 0$.

Despite this being a nonlinear differential equation, it is possible for the equation to be recast in terms of vector-valued quasimodular forms. Along these lines of research are the papers of [BG07], [Yan04], [FM16] and [FM20].

However, in this thesis we focus on linear DEs satisfied by modular forms.

2.3 Concrete Illustrations of Proposition 1.1

The following examples from [PR21] and [Zag18] illustrate Proposition 1.1. In each of the following examples we explicitly give $f(z)$, k, $t(z)$, $\Phi(t)$ and the $(k+1)$-st order differential equation (DE) satisfied by Φ.

Our first example is given on p63 of [Zag08].

Example 2.1. We let $\Gamma = \Gamma(2)$, where

$$\Gamma(2) = \left\{ \gamma \in \Gamma_1 | \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2 \right\}.$$

Remark 2.5. We have the exceptional isomorphism $\Gamma(2) \cong \Gamma_0(4)$, whereby conjugation by $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ gives the bijection $M_k(\Gamma_0(4)) \xrightarrow{\sim} M_k(\Gamma(2))$, $f(z) \mapsto f(z/2)$.

Also, we let $f(z) = \theta_3(z)^2$, where $\theta_3(z) = \theta(z/2) = \sum_{n\in\mathbb{Z}} q^{n^2/2}$ and $\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$ is Jacobi's theta function; see, e.g., [Bel61]. Finally, let $t(z) = \lambda(z)$, where $\lambda(z) = (\theta_2(z)/\theta_3(z))^4$ and $\theta_2(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{n^2}$. Observe that $\theta_2(z)$ and $\theta_3(z)$ are modular forms of weight 1/2 on Γ, and the fourth power of their quotient, which is $t(z)$, is a meromorphic modular form of weight zero, i.e., a modular function. Also, we observe that $f(z)$ is a modular form of weight 1 on Γ, so that $k = 1$. Then we have $f(z) = \Phi(t(z))$, where

$$\Phi(t) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right)$$

and

$${}_2F_1(a, b; c; t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} t^n$$

is the hypergeometric function that satisfies the hypergeometric differential equation

$$t(t-1)y'' + ((a+b+1)t - c)y' + ab\,y = 0. \tag{2.9}$$

In particular, we have that Φ satisfies the second order DE

$$t(t-1)\Phi'' + (2t-1)\Phi' + \frac{1}{4}\Phi = 0.$$

The following example is given on p728, Section 5.1 of [PR21].

Example 2.2. We let $\Gamma = \Gamma_1$, $f(z) = E_4(z)$, where $E_4(z)$ is given in (2.2). Also, let $t(z) = 1728/j(z)$, where $j(z)$ is given in (2.4). Since $j(z)$ is a modular function, so is $t(z)$. We note that $f(z)$ is a modular form of weight 4 on Γ, so that $k = 4$. Then we have $f(z) = \Phi(t(z))$, where

$$\Phi(t) = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; t\right)^4$$

satisfies the fifth order DE

$$(t^4 - 3456t^5 + 2985984t^6)\Phi^{(5)} + 10(t^3 - 4320t^4 + 4478976t^5)\Phi^{(4)} + 5(5t^2 - 29616t^3 + 38486016t^4)\Phi^{(3)}$$

$$+15(t - 9864t^2 + 17252352t^3)\Phi'' + (1 - 30000t + 81100800t^2)\Phi' + 240(-1 + 6144t)\Phi = 0,$$

as shown explicitly in Output Line 25 in [PR21].

The following example is supplied on p728, Section 5.1 of [PR21], as well as in Equation (74) on p64 of [Zag08].

Example 2.3. We let $\Gamma = \Gamma_1$, $f(z) = E_4(z)^{1/4}$ and $t(z) = 1728/j(z)$. We note that $f(z)$, subject to a choice of branch cut, behaves as a modular form of weight 1 with respect to Γ, so that

$k = 1$. Then we have $f(z) = \Phi(t(z))$, where

$$\Phi(t) = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; t\right)$$

satisfies the second order DE (hypergeometric differential equation)

$$t(t-1)\Phi'' + \left(\frac{3}{2}t - 1\right)\Phi' + \frac{5}{144}\Phi = 0.$$

Remark 2.6. We can deduce the DE in Example 2.2 from the one in Example 2.3 by computing the fifth-order differential operator $\mathrm{Sym}^4(L)$, whose solutions are fourth powers or four-fold products of the differential operator $L = t(t-1)\partial^2 + \left(\frac{3}{2}t - 1\right)\partial + \frac{5}{144}$. A formula for $\mathrm{Sym}^n(L)$ for general n is supplied in Lemma A.1.

Here is another example that is supplied on p728, Section 5.1 of [PR21].

Example 2.4. We let $\Gamma = \Gamma_1$, $f(z) = \sqrt{E_4(z)}$ and $t(z) = 1728/j(z)$. We note that $f(z)$, subject to a choice of branch cut, behaves as a modular form of weight 2 with respect to Γ, so that $k = 2$. Then we have $f(z) = \Phi(t(z))$, where

$$\Phi(t) = {}_3F_2\left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; t\right).$$

Here, Φ satisfies the third order DE

$$t\frac{d}{dt}\left(t\frac{d}{dt}\right)\left(t\frac{d}{dt}\right)\Phi = t\left(t\frac{d}{dt} + \frac{1}{6}\right)\left(t\frac{d}{dt} + \frac{1}{2}\right)\left(t\frac{d}{dt} + \frac{5}{6}\right)\Phi, \tag{2.10}$$

i.e.,

$$t^2(t-1)\Phi^{(3)} + \frac{9}{2}t\left(t - \frac{2}{3}\right)\Phi'' + \left(\frac{113}{36}t - 1\right)\Phi' + \frac{5}{72}\Phi = 0.$$

This DE is a particular case of the hypergeometric DE; see, e.g., [Yos97].

As remarked in [PR21], Clausen's identity, given in Proposition A.2 in Appendix A, connects the expressions for Φ in Examples 2.3 and 2.4.

The following example is supplied on p730, Section 5.2 of [PR21].

Example 2.5. We let $\Gamma = \Gamma_1(6)$,

$$f(z) = \frac{\eta(z)^7\eta(6z)^7}{\eta(2z)^5\eta(3z)^5} \quad \text{and} \quad t(z) = \frac{\eta(z)^{12}\eta(6z)^{12}}{\eta(2z)^{12}\eta(3z)^{12}}.$$

In fact, t is a Hauptmodul, or principal modular function, for Γ. We note that $f(z)$ is a modular form of weight 2 on Γ, so that $k=2$. Then we have $f(z)=\Phi(t(z))$, where Φ satisfies the third-order DE

$$t^2(t^2-34t+1)\Phi^{(3)}+3t(2t^2-51t+1)\Phi''+(7t^2-112t-1)\Phi'+(t-5)\Phi=0,$$

with initial conditions $\Phi(0)=1$, $\Phi'(0)=5$ and $\Phi''(0)=146$. Also, by taking the square root of f, with an appropriate branch cut, the authors show that $\sqrt{f(z)}=\Phi(t(z))$, where Φ satisfies the second-order DE

$$4t(t^2-34t+1)\Phi''+4(2t^2-51t+1)\Phi'+(t-10)\Phi=0,$$

with the initial conditions $\Phi(0)=1$ and $\Phi'(0)=5/2$.

In Chapter 5, given a congruence subgroup Γ, and $f\in M_k(\Gamma)$ and $t\in M(\Gamma)$, we provide an explicit expression for Φ, thus addressing Problem A.

In regards to Problem B, we observe that the hypergeometric DE features in several of the preceding examples. Thus, we investigate its associated ∂-Galois group, and monodromy group, in Chapter 4. However, it is necessary to firstly provide some background on differential Galois theory in the next chapter.

Chapter 3

Background on Differential Galois Theory

3.1 Overview of Differential Galois Theory

Differential Galois theory was pioneered in the late nineteenth century by Picard and Vessiot; see [Pic08] and [Ves92]; and subsequently developed in the twentieth century by Kolchin; see [Kol73] and [PS03]. It extends classical Galois theory of algebraic equations to linear differential equations and partial differential equations.

For example, with classical Galois theory, given a field k and an irreducible polynomial

$$f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_1 X + a_0 \in k[X],$$

with distinct roots $\alpha_1, \ldots, \alpha_n$ lying in the splitting field K of f, the Galois group $\mathrm{Gal}(K/k)$ can be viewed as the subgroup of the symmetric group on n elements consisting of $\sigma \in S_n$ such that for every $F(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$ with $F(\alpha_1, \ldots, \alpha_n) = 0$, it is also the case that

$$F(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}) = 0.$$

By analogy, given a differential field k, namely a field k with a derivation map $\partial : k \to k$, and a linear differential operator $L = \partial^n + a_{n-1}\partial^{n-1} + \ldots + a_0\partial^0$, where $a_j \in k$, we associate a differential equation $L(y) = 0$, whose solutions lie in a differential field extension K of k, called the Picard-Vessiot extension. The differential Galois group $\mathrm{Gal}^{\partial}(K/k)$ is the subgroup of $\mathrm{GL}_n(C)$ consisting of $\sigma \in \mathrm{GL}_n(C)$ such that for every

$$F(\partial^j X_i) \in K[\partial^j X_i | i \in \{1,2,\ldots,n\},\ j \in \{0,1,\ldots,n-1\}]$$

with $F(\partial^j \alpha_i) = 0$, it is also the case that $F(\partial^j \sigma(\alpha_i)) = 0$. Here, C denotes the field of constants, which is the subfield of k equal to the kernel of the derivation.

In our setting of differential Galois theory, we only deal with fields k containing $\mathbb{Q}$, i.e., of characteristic zero, and having an algebraically closed field of constants C. All rings considered here are commutative, have a unit, and contain $\mathbb{Q}$.

We introduce the following definitions, as given in [PS03].

Definition 3.1. A *differential ring* is a ring R equipped with a *derivation* $(\cdot)' : R \to R$ satisfying, $\forall\, a, b \in R$,

(i) $(a+b)' = a' + b'$; and

(ii) $(ab)' = a'b + a'b$.

A *differential field* is a field equipped with a derivation. A differential ring S containing R is a *differential extension* of R or a *differential ring over* R if the derivation on S when restricted to R is equal to the derivation on R.

Example 3.1. *Let* $R = \mathbb{C}[x]$*, with* $f' := \frac{df}{dx}$*, for* $f \in R$*. Then* $(R, (\cdot)')$ *is a differential ring. Now let* $K = \mathrm{Frac}(R) = \mathbb{C}(x)$*, where the derivation is extended to* K *via* $(f/g)' = f'/g - f g'/g^2$*, for* $f, g \in R$*. Then* K *is a differential extension of* R*.*

For the following two definitions, let k be a differential field.

Definition 3.2. We define the set of *constants* of k by $C_k = \{x \in k | x' = 0\}$, abbreviating it as C when the context is clear.

Definition 3.3. A differential module (M, ∂) of dimension n over k is a k-vector space of dimension n where the map $\partial : M \to M$ is additive, satisfying $\partial(a\,m) = a'm + a\partial m$ for all $a \in k$ and $m \in M$.

Thus, a differential k-module M of dimension n is a free module, having a k-basis $\{e_1, \ldots, e_n\}$.

Remark 3.4. Note that we have the derivation $(\cdot)' : k \to k$ for a differential ring k and the corresponding map $\partial : M \to M$ for a differential k-module M. Throughout this thesis, for $f \in R$, where R is a differential ring, f' and ∂f will have various meanings which will be clear from the context, however, in all cases they are derivations and thus satisfy the rules given in Definition 3.1.

The rules for differentiation in M lead to a differential equation as follows. Let $\{e_1, \ldots, e_n\}$ be a k-basis for M. Since, for each $i \in \{1, 2, \ldots, n\}$ we have $\partial e_i \in M$, we may write it as $\partial e_i = -\sum_{j=1}^n a_{j,i} e_j$, for some $a_{j,i} \in k$. We arrive at the following condition.

Condition A

$$m = \sum_{i=1}^n y_i e_i \in \ker(\partial) \tag{3.1}$$

if and only if

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}' = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n,1} & a_{n,2} & \ldots & a_{n,n} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}. \tag{3.2}$$

We employ obvious notation to rewrite this matrix differential equation as $y' = Ay$, where $y \in k^n$ and $A \in \mathrm{Mat}_n(k)$.

Remark 3.5. Note that we may choose another k-basis $\{f_1, \ldots, f_n\}$ of M, with $\partial f_i = -\sum_{j=1}^n b_{j,i} f_j$, so that $m = \sum_{i=1}^n z_i f_i \in \ker(\partial)$ iff $z' = Bz$. Thus, we have another DE, $z' = Bz$, associated to the k-module M. Note that by writing $f_i = \sum_{j=1}^n \lambda_{j,i} e_j$, we see

$$m = \sum_i z_i f_i = \sum_i z_i \sum_j \lambda_{j,i} e_j = \sum_j \left(\sum_i \lambda_{j,i} z_i \right) e_j.$$

Comparing with (3.1), we see that $y_j = \sum_i \lambda_{j,i} z_i$, or $y = \Lambda z$, where $\Lambda = [\lambda_{i,j}] \in \mathrm{GL}_n(k)$. Hence,

$$y' = \Lambda' z + \Lambda z' = \Lambda' \Lambda^{-1} y + \Lambda B \Lambda^{-1} y,$$

and we can see that if $A = \Lambda' \Lambda^{-1} + \Lambda B \Lambda^{-1}$ then the two differential equations $y' = Ay$ and $z' = Bz$ emanate from the same module M.

Remark 3.6. Given a matrix DE $y' = Ay$ of dimension n over k, we can construct a module $M = k^n$ and prescribe an additive map $\partial : M \to M$ satisfying $\partial e_i := -\sum_j a_{j,i} e_j$ and

$$\partial(ae_i) := a' e_i + a\partial(e_i), \tag{3.3}$$

for each $j \in \{1,2,\dots,n\}$ and for any $a \in k$. As a consequence of this construction, for any $a \in k$ and $m \in M$ we have, using additivity of ∂,

$$\partial(am) = \partial(\sum_i a y_i e_i) = \sum_i \partial(a y_i e_i),$$

and, by applying the rule (3.3), we have

$$\sum_i \partial(a y_i e_i) = \sum_i a' y_i e_i + \sum_i a\partial(y_i e_i) = a' m + a\partial m.$$

Hence, we have explicitly constructed a module M and map ∂ associated to the DE.

Now, given a scalar differential operator $L \in k[\partial]$,

$$L = \partial^n + a_{n-1}\partial^{(n-1)} + \ldots + a_0, \tag{3.4}$$

we associate the homogeneous scalar differential equation over k

$$L(f) = f^{(n)} + a_{n-1} f^{(n-1)} + \ldots + a_0 f = 0, \tag{3.5}$$

where f lies in some field K extension of k, yet to be specified. We can represent (3.5) as a matrix differential equation using the following definition.

Definition 3.7. Given the differential operator L in (3.4), we associate the *companion matrix*

$$A_L = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix}.$$

Observe that $L(f) = 0$ if and only if

$$\begin{pmatrix} f \\ f' \\ \vdots \\ f^{(n-1)} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} \begin{pmatrix} f \\ f' \\ \vdots \\ f^{(n-1)} \end{pmatrix}.$$

Remark 3.8. Thus far, we have three ways of expressing a DE. Firstly, there is the homogeneous scalar DE given in (3.5). Secondly, there is the construction of a differential module (M, ∂) described in Remark 3.6, whereby the kernel of ∂ consists of solutions of the matrix DE in (3.2). Lastly, we have the companion matrix given in Definition 3.7 for the matrix DE $f' = Af$ over k.

Lemma 3.9 (Lemma 1.8 in [PS03])**.** *Consider the matrix equation $y' = Ay$ over k of dimension n. The solution space V of $y' = Ay$ in k is defined as $\{v \in k^n | v' = Av\}$. Then V is a vector space over C of dimension $\leq n$.*

Proof. See p8 of [PS03]. □

Our aim is to extend k to some larger differential field K which contains all n solutions of (3.5). This larger field K is the Picard-Vessiot (PV) field extension of the DE $y' = Ay$ over k and we will need the following definition.

Definition 3.10. A *Picard-Vessiot ring* R over the differential field k for the equation $y' = Ay$, with $A \in \mathrm{Mat}_n(k)$, is a differential ring satisfying:

(A1) R is a simple differential ring, i.e., the only differential ideals are (0) and R;

(A2) There is a fundamental matrix $F \in \mathrm{GL}_n(R)$ such that $F' = AF$; and

(A3) $R = k[F_{i,j}, 1/\det(F) \,|\, i, j \in \{1, 2, \dots, n\}]$.

Importantly, the following proposition guarantees the existence of a PV ring for any given DE over a base field.

Proposition 3.11. *(See Proposition 1.20 on p15 of [PS03]) Let $y' = Ay$ be a matrix DE of dimension n over the differential field $(k, (\cdot)')$. Then:*

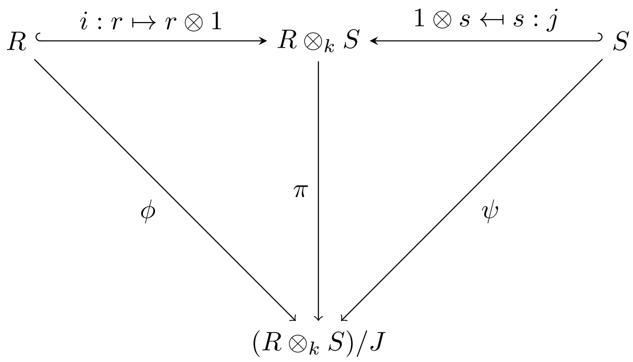

FIGURE 3.1: Commutative diagram of PV rings R and S for the same DE.

(1) There exists a PV ring for the DE; and

(2) Any two PV rings for the DE are isomorphic.

Proof. The proof of (1) is constructive. Let $X_{i,j}$, for $i, j \in \{1, 2, \ldots, n\}$, be indeterminates, and extend the derivation on k to the ring $R_0 = S^{-1}k[X_{i,j}]$, where S is the multiplicative set in $k[X_{i,j}]$ generated by $1/\det X$, in such a way that $X' = AX$. Let I be a maximal differential ideal of R_0. Then, we define the ring $R = R_0/I$, which is seen to be a PV ring since it satisfies Conditions (A1), (A2) and (A3) in Definition 3.10.

The proof of (2) constructs an isomorphism between two PV rings employing tensor products. Let R and S be two PV rings, and define the differential ring $R \otimes_k S$, having the derivation $(r \otimes s)' = r' \otimes s + r \otimes s'$. Let $i : R \to R \otimes_k S$ and $j : S \to R \otimes_k S$ be the inclusion maps. Further, let J be a maximal differential ideal of $R \otimes_k S$ and let $\pi : R \otimes_k S \to (R \otimes_k S)/J$ be the projection map. Then we have the commutative diagram in Figure 3.1.

Observe that the homomorphisms $\phi = \pi \circ i : R \mapsto (R \otimes_k S)/J$ and $\psi = \pi \circ j : R \mapsto (R \otimes_k S)/J$ are nonzero. Since each of R and S is ∂-simple, each of $\ker \phi$ and $\ker \psi$ is trivial, and we must have that ϕ and ψ are isomorphisms. Hence, $\psi^{-1} \circ \phi : R \to S$ gives us our isomorphism from R to S. □

Remark 3.12. As a byproduct of the construction in the proof of (2) above, we obtain a relation between the fundamental matrices of PV rings. Let $X \in \mathrm{GL}_n(R)$ and $Y \in \mathrm{GL}_n(S)$ be fundamental matrices for the respective PV rings R and S. Then consider $U = \psi(Y)^{-1}\phi(X) \in$

$\mathrm{GL}_n(R \otimes_k S)$. Observe that

$$\begin{aligned} U' &= -\psi(Y^{-1})\psi(Y')\psi(Y^{-1})\phi(X) + \psi(Y)^{-1}\phi(X') \\ &= -\psi(Y^{-1})\psi(AY)\psi(Y^{-1})\phi(X) + \psi(Y)^{-1}\phi(AX) \\ &= -\psi(Y^{-1})\psi(AY)\psi(Y^{-1})\phi(X) + \psi(Y)^{-1}\phi(AX) \\ &= 0, \end{aligned}$$

demonstrating that $U \in C_{R\otimes_k S}$. We will see later in this chapter that this construction will aid in characterising automorphisms of PV rings, and hence the differential Galois group.

Analogous to a PV ring for a DE, we associate a meaning for a PV ring for a module in the next section.

3.2 Associating Picard-Vessiot Modules with Linear DEs

Recall from Condition A that we associated a module to a DE. Now we give the notion of a PV ring over a module. The advantage of this approach is that we have a coordinate-free way of describing a PV ring.

Definition 3.13. A PV ring for a differential module M over k is defined as the PV ring of some DE $y' = Ay$ associated to M. As per Remark 3.5, there may be more than one DE associated to M.

By analogy to Definition 3.10 for a PV ring over a DE, we have the following definition of a PV ring for the module M.

Definition 3.14. Let M be a differential module over k of dimension n associated with the DE $y' = Ay$. Then R is a PV ring for the module M (equivalently, R is a PV ring for the DE $y' = Ay$ associated with M) if:

(B1) R is a simple differential ring;

(B2) $V = \ker(\partial, R \otimes_k M)$ has dimension n over C; and

(B3) Let $\{e_1, \ldots, e_n\}$ be a basis for M over k. Then $R = k[v_i \in R | \exists v = \sum_{i=1}^n v_i (1 \otimes_k e_i) \in V]$.

Lemma 3.15. *(See Exercise 1.16(2) of [PS03].) Definitions 3.10 and 3.14 are equivalent.*

Proof. (Definition 3.10 implies Definition 3.14) Suppose that R is a PV ring associated to the differential module M satisfying (A1), (A2) and (A3). Being the same as (A1), Condition (B1) holds.

We now show that (B2) holds. Since M is a differential module over k of dimension n, we may choose a basis $\{e_1, \dots, e_n\}$. Since R is a PV ring for the DE $y' = Ay$ associated to M, we may write $\partial e_i = -\sum_{j=1}^n a_{j,i} e_j$, where $a_{j,i} \in k$ for $i, j \in \{1, 2, \dots, n\}$. Hence, for all $i \in \{1, 2, \dots n\}$,

$$\begin{aligned}
\partial\left(\sum_{i=1}^n r_i e_i\right) = 0 &\iff \sum_{i=1}^n r_i' e_i + \sum_{j=1}^n r_j \partial e_j = 0 \\
&\iff \sum_{i=1}^n r_i' e_i - \sum_{j=1}^n r_j \sum_{i=1}^n a_{i,j} e_i = 0 \\
&\iff \sum_{i=1}^n \left(r_i' - \sum_{j=1}^n r_j a_{i,j}\right) e_i = 0 \\
&\iff r_i' = \sum_{j=1}^n r_j a_{i,j}.
\end{aligned}$$

We employ the notation

$$\underset{\sim}{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \in R^n,$$

which allows us to write succinctly

$$\partial\left(\sum_{i=1}^n r_i \otimes_k e_i\right) = 0 \iff \underset{\sim}{r}' = A\underset{\sim}{r}, \tag{3.6}$$

where $A = [a_{i,j}]_{i,j=1,\dots,n}$. As in (A2), let F be a fundamental matrix satisfying $F' = AF$, and denote the columns of F by $\underset{\sim}{f}_1, \dots, \underset{\sim}{f}_n \in R^n$. Then, $\underset{\sim}{f}_i' = A\underset{\sim}{f}_i$ and, by virtue of (3.6),

$$0 = \partial\left(\sum_{j=1}^n (\underset{\sim}{f}_i)_j \otimes_k e_j\right) = \partial\left(\sum_{j=1}^n F_{j,i} \otimes_k e_j\right).$$

Now, for any $v = \sum_{j=1}^n v_j \otimes_k e_j \in V$, where $v_j \in R$, we have $\underset{\sim}{v}' = A\underset{\sim}{v}$. Letting $\underset{\sim}{\alpha} = F^{-1}\underset{\sim}{v} \in R^n$, we see that

$$\begin{aligned}
\underset{\sim}{\alpha}' &= -F^{-1}F'F^{-1}\underset{\sim}{v} + F^{-1}\underset{\sim}{v}' \\
&= -F^{-1}AFF^{-1}\underset{\sim}{v} + F^{-1}A\underset{\sim}{v} = 0,
\end{aligned}$$

so that $\underset{\sim}{\alpha} \in C^n$. Hence,

$$\underset{\sim}{v} = \sum_{i=1}^{n} \alpha_i \underset{\sim}{f}_i \in \mathrm{Span}_C\{\underset{\sim}{f}_1, \ldots, \underset{\sim}{f}_n\},$$

and we arrive at (B2), namely that V has dimension n over C, with basis $\{g_1, \ldots, g_n\}$, where $g_i = \sum_{j=1}^{n} F_{j,i} \otimes_k e_j$.

Finally, we show that (B3) holds. Let

$$\tilde{R} = k[v_i \in R | v = \sum_{i=1}^{n} v_i \otimes_k e_i, \exists v \in V].$$

Since $\tilde{R}$ is generated by elements v_i of R, we certainly have $\tilde{R} \subseteq R$.

Now we show that $R \subseteq \tilde{R}$. Since for all $i \in \{1, 2, \ldots, n\}$, $\underset{\sim}{f}_i = \sum_{j=1}^{n} F_{j,i} e_j$, it follows that $\sum_{j=1}^{n} F_{j,i} \otimes_k e_j \in V$, so that $F_{j,i} \in \tilde{R}$. Also, if $v \in V$, then $v = \sum_i v_i (1 \otimes_k e_i) \in R \otimes_k M$, giving the corresponding vector equation $\underset{\sim}{v} = \sum_{i=1}^{n} \alpha_i \underset{\sim}{f}_i \in R^n$. Thus, $v_j = \sum_i \alpha_i (\underset{\sim}{f}_i)_j = \sum_i \alpha_i F_{j,i}$. That is to say, $\forall v_j \in \tilde{R}$, v_j is expressible as a C-linear combination of $F_{j,1}, F_{j,2}, \ldots, F_{j,n}$. Hence, $\widetilde{R} \supseteq k[F_{i,j} | i, j \in \{1, 2, \ldots, n\}]$. If we can show that $1/\det F \in \widetilde{R}$ then it would follow that $\widetilde{R} \supseteq k[F_{i,j}, 1/\det F | i, j \in \{1, 2, \ldots, n\}] = R$. Rewriting $F_{j,i} (1 \otimes_k e_j)$ as $(1/\det F)((F_{j,i} \det F) \otimes_k e_j)$, we see that $1/\det F \in \widetilde{R}$.

(Definition 3.14 implies Definition 3.10) Suppose R satisfies (B1), (B2) and (B3). Clearly (A1) holds, being identical to (B1).

Now we show (A2). From (B2), V has dimension n over C, so let $\{f_1, \ldots, f_n\}$ be a C-basis for V, where $f_i = \sum_{j=1}^{n} F_{j,i} \otimes e_j$, for $i \in \{1, 2, \ldots, n\}$. Now write

$$\underset{\sim}{f}_i = \begin{pmatrix} F_{1,i} \\ F_{2,i} \\ \vdots \\ F_{n,i} \end{pmatrix} \in R^n,$$

and we see that $\underset{\sim}{f}'_i = A \underset{\sim}{f}_i$, $\forall i \in \{1, \ldots, n\}$. Further, letting $F = [F_{i,j}]_{i,j=1,\ldots,n} = [\underset{\sim}{f}_1, \ldots, \underset{\sim}{f}_n]$, we also see that $F' = AF$. Since $\{\underset{\sim}{f}_1, \ldots, \underset{\sim}{f}_n\}$ is a linearly independent set over R, $F \in \mathrm{GL}_n(R)$, which demonstrates that F is a fundamental matrix for R. Hence, (A2) holds.

Finally we show (A3). From (B3), $R = k[v_i | v = \sum_i v_i 1 \otimes_k e_i, \exists v \in V]$. Now let $\widetilde{R} = k[F_{i,j}, 1/\det(F)]$. Showing that $\widetilde{R} = R$ will demonstrate (A3). We know that $F_{i,j} \in R$, since $\forall i \in \{1, 2, \ldots, n\}$,

$f_i = \sum_{j=1}^{n} F_{j,i} 1 \otimes_k e_j \in V$. Also, rewriting f_i as

$$f_i = \sum_{j=1}^{n} \frac{1}{\det F}\Big(((\det F)\, F_{j,i}) \otimes_k e_j\Big),$$

we see that $1/\det F \in R$, giving $\widetilde{R} \subseteq R$.

Now we show $R \subseteq \widetilde{R}$. Let $v = \sum v_i(1 \otimes e_i)$ be such that $\partial v = 0$. So, $v_i \in R$. Then $\underset{\sim}{v} = \sum_i \alpha_i \underset{\sim}{f}_i \in R^n$, i.e., $v_j = \sum_i \alpha_i F_{j,i}$, which implies that $v_j \in \tilde{R}$. □

We now proceed to show that there are no new constants introduced by the field of fractions of a PV ring. We require the following preliminary lemma.

Lemma 3.16. *(Lemma 1.17 in [PS03]) Let R be a simple ∂-ring containing the algebraically closed (differential) field k. Then R is an integral domain.*

Proof. Arguing by contradiction, we suppose that there are non-zero elements x and y in R such that $xy = 0$. Now let $I = \{r \in R | x^n r = 0, \exists n \in \mathbb{N}\}$. We observe that $\forall r \in R$, $\forall z \in I$, with $x^n z = 0$, we have $x^n(rz) = 0$, so that $rz \in I$. Also, $\forall z_1, z_2 \in I$, with $x^{n_j} z_j = 0$ for $j = 1,2$, we see that $x^{n_1+n_2}(z_1 + z_2) = x^{n_2}(x^{n_1} z_1) + x^{n_1}(x^{n_2} z_2) = 0$, so that $z_1 + z_2 \in I$. Further, $\forall z \in I$, with $x^n z = 0$, we see that $0 = (x^{n+1} z)' = (n+1)x^n x' z + x^{n+1} z' = x^{n+1} z'$, so that $z' \in I$. Hence, I is a ∂-ideal. Furthermore, $y \in I$ implies that $I \neq (0)$. Since R is ∂-simple, we have $I = R$, and hence $1 \in I$ and x is nilpotent.

Thus far we have shown that if $xy = 0$, with x and y being nonzero, then x is nilpotent. By symmetry, y must also be nilpotent.

Now let $J = \{r \in R | r^n = 0 \exists n \in \mathbb{N}\}$, which contains $x \neq 0$. We observe that $\forall r \in R$, $\forall x \in J$, with $x^n = 0$, we have $(rx)^n = r^n x^n = 0$, so that $rx \in J$. Also, $\forall x_1, x_2 \in J$, with $x_i^{n_j} = 0$ for $j = 1,2$, we see that

$$(x_1 + x_2)^{n_1+n_2} = \sum_{j=0}^{n_1+n_2} \binom{n_1+n_2}{j} x_1^j x_2^{n_1+n_2-j} = 0,$$

so that $x_1 + x_2 \in J$. Further, $\forall x \in J$, with $x^n = 0$, we see that $0 = (x^n)' = n x^{n-1} x'$. Since $R \supseteq k \supseteq \mathbb{Q}$, $n \in R^\times$ and $x^{n-1} x' = 0$. Differentiating $x^{n-1} x'$ gives $(n-1)x^{n-2}(x')^2 + x^{n-1} x'' = 0$. Multiplying by x' gives $(n-1)x^{n-2}(x')^3 = 0$, which, for $n > 1$, simplifies to $x^{n-2}(x')^3 = 0$. If $n = 2$ we have $(x')^3 = 0$ and $x' \in J$. Otherwise, if $n > 2$, we continue in this vein, eventually arriving at $(x')^{2n-1} = 0$, so that $x' \in J$. Hence, J is a ∂-ideal. Since $1 \notin J$ and R is ∂-simple, we have $J = (0)$. This contradicts $0 \neq x \in J$.

Thus, R is an integral domain. □

Proposition 3.17. *(Lemma 1.17 in [PS03]) Let R be a simple ∂-ring containing the algebraically closed (differential) field k. Further, suppose that R is finitely generated over k. Then $C_{\mathrm{Frac}(R)} = C$.*

Proof. By virtue of Lemma 3.16, R is an integral domain, so that $R-\{0\}$ is a multiplicative set, which allows us to define the field of fractions $K = \mathrm{Frac}(R) = (R-\{0\})^{-1}R$. Our aim is to show $C_K = C_k$.

Firstly, we show that $C_K \subseteq R$. Suppose $x \in C_K$. Then let $I = \{r \in R | xr \in R\}$. Now, $\forall r \in R$, $\forall z \in I$, $x(rz) = r(xz) \in R$, so that $rz \in I$. Also, $\forall z_1, z_2 \in I$, $xz_j \in R$ for $j = 1,2$ and so $x(z_1+z_2) = xz_1 + xz_2 \in R$, i.e., $z_1 + z_2 \in I$. Further, $\forall z \in I$, $0 = (xz)' = x'z + xz' = xz'$, since $x' = 0$. Thus, $xz' = 0$ and $z' \in I$. Hence, I is a ∂-ideal. Writing $x = a/b$, for some $a, b \in R$, with $b \neq 0$, we see that $xb = a \in R$, showing that $0 \neq b \in I$, i.e., $I \neq (0)$. Since R is ∂-simple, $I = R$, i.e., $1 \in I$ and, hence, $x \in R$. This shows that $C_K \subseteq R$.

Now suppose that $x \notin C$. Indeed, $\forall y \in C$ we have $y \in R$ and the ideal $I = R(x-y) \subseteq R$. Also, I is a ∂-ideal, since:

(1) $\forall r_1, r_2 \in R$, $(x-y)r_1 + (x-y)r_2 = (x-y)(r_1+r_2) \in I$;

(2) $\forall r \in R$, $\forall r_1(x-y) \in I$, $rr_1(x-y) \in I$;

(3) $\forall (x-y)r \in I$, let $y = ((x-y)r)'$. Then $y = (x-y)'r + (x-y)r' = (x-y)r'$, since $x - y \in C_K$, and we have $y \in I$.

Also, $R(x-y) \neq (0)$, since if $R(x-y) = (0)$, then $\forall r \in R$, $r(x-y) = 0$, i.e., $1(x-y) = 0$, i.e., $x = y \in C$, contradicting $x \notin C$. Therefore $I = R(x-y) \neq (0)$, and R being ∂-simple gives us $I = R$, and thus $x - y \in R^\times$. Lemma A.4 of [PS03] implies that x must be algebraic over k. Since being algebraic over k implies being algebraic over C, and since C is algebraically closed, we have $x \in C$, which is a contradiction. Thus, $C_K = C$. □

We are in a position to give the following definition.

Definition 3.18. A *PV field* for the DE $y' = Ay$ over k (or for a differntial module M over k) is the field of fractions of a PV ring for this equation.

An equivalent definition of a PV field, proven in Proposition 1.22 of [PS03], is the following, which will be convenient when proving finite Galois extensions are ∂-Galois extensions.

Definition 3.19. The field K is a *PV field* for the DE $y' = Ay$ over k iff the following conditions are fulfilled:

(a) $C_K = C$;

(b) $\exists F \in \mathrm{GL}_n(K)$ such that $F' = AF$; and

(c) $K = k[F_{1,1}, F_{1,2}, \dots, F_{n,n}]$.

From Proposition 3.17 we see that no new constants are introduced by the field of fractions of a PV ring. Hence, Definition 3.18 implies Part (a) in Definition 3.19, while Parts (b) and (c) are immediate consequences of $K = \mathrm{Frac}(R)$, where R is a PV ring for the DE over k.

3.3 Wronskians

We consider the Wronskian matrix of a set of elements of a differential field k, as it will be of use in showing agreement of the classical Galois group with the differential Galois group for algebraic extensions in the next section, as well as providing an explicit construction of a DE having differential Galois group $\mathrm{GL}_n(C)$ in the next chapter.

Definition 3.20. The *Wronskian matrix* of $y_1, \dots, y_n \in k$ is the $n \times n$ matrix

$$\mathrm{W}(y_1, \dots, y_n) = \begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \dots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix}.$$

The *Wronskian* $wr(y_1, \dots, y_n)$ of $y_1, \dots, y_n \in k$ is $\det \mathrm{W}(y_1, \dots, y_n)$.

Lemma 3.21. *(See Exercise 1.14(5)(a) of [PS03]) If Z is a fundamental matrix for $y' = Ay$, then* $(\det Z)' = \mathrm{tr}(A)\,(\det Z)$.

Proof. We let $z_1, \dots, z_n$ denote the columns of Z, so that $z_i' = Az_i$ and observe that

$$\det(z_1, \dots, z_n)' = \sum_{i=1}^{n} \det(z_1, \dots, z_i', \dots, z_n)$$

$$= \sum_{i=1}^{n} \det(z_1, \ldots, Az_i, \ldots, z_n)$$

$$= \sum_{i=1}^{n} \sum_{\sigma \in S_n} [z_1]_{\sigma(1)} \cdot \ldots [Az_i]_{\sigma(i)} \ldots [z_n]_{\sigma(n)}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{\sigma \in S_n : \sigma(i)=k} [Az_i]_k . [z_1]_{\sigma(1)} \cdot \ldots . [z_n]_{\sigma(n)}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{\sigma \in S_n : \sigma(i)=k} \sum_{\ell=1}^{n} a_{k,\ell} [z_i]_\ell . \prod_{j=1 : j \neq i}^{n} [z_j]_{\sigma(j)}$$

Now, let us denote the (i,k)-th cofactor by

$$\det(Z^{(k,i)}) = \sum_{\sigma \in S_n : \sigma(i)=k} \prod_{j=1 : j \neq i}^{n} [z_j]_{\sigma(j)}.$$

Then

$$\det(z_1, \ldots, z_n)' = \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} a_{k,\ell} [z_i]_\ell det(Z^{(k,i)}).$$

Note that

$$\sum_{i=1}^{n} [z_i]_\ell \det(Z^{(k,i)}) = \det(Z) \delta_{k,\ell}$$

and so

$$\det(z_1, \ldots, z_n)' = \sum_{k=1}^{n} \sum_{\ell=1}^{n} a_{k,\ell} \det(Z) \delta_{k,\ell}$$

$$= \sum_{k=1}^{n} a_{k,k} \det(Z) = \mathrm{tr}(A) \det(Z).$$

□

Lemma 3.22. *(See Exercise 1.14(5)(b) of [PS03]) Let $\{y_1, \ldots, y_n\} \subset k$ be a fundamental set of solutions of $L(y) = 0$, where $L = \partial^n + a_{n-1}\partial^{n-1} + \ldots + a_0$. Then $w = \mathrm{wr}(y_1, \ldots, y_n)$ satisfies*

$$w' = -a_{n-1} w.$$

Proof. We consider the companion matrix of L, i.e.,

$$A_L = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix}.$$

Let $W = \mathrm{W}(y_1,\ldots,y_n)$. Then $W' = A_L W$ and applying Lemma 3.21 we obtain $w' = \mathrm{tr}(A_L)w$. Since $\mathrm{tr}(A_L) = -a_{n-1}$, we have the result. □

3.4 Differential Galois Group

We commence with the following definition.

Definition 3.23. (See Definition 1.25 on p18 of [PS03]) The differential Galois group of an equation $y' = Ay$ over the differential field k, or of a differential module over k, where $A \in \mathrm{Mat}_n(k)$, is defined as the group $\mathrm{Gal}^{\partial}(R/k)$ of differential k-algebra automorphisms of a Picard-Vessiot ring R for the equation. More precisely, $\mathrm{Gal}^{\partial}(R/k)$ consists of the k-algebra automorphisms σ of R satisfying $\sigma(f') = \sigma(f)'$ for all $f \in R$, i.e.,

$$\mathrm{Gal}^{\partial}(R/k) = \{\sigma \in \mathrm{Aut}(R) | \forall x \in k, \sigma(x) = x, \& \forall f \in R, \sigma(f)' = \sigma(f')\}.$$

We provide here an independently obtained proof of the following proposition, given as Exercise 1.24 in [PS03], which verifies that the classical Galois group of a finite Galois field extension of a base field k agrees with the differential Galois group of a Picard-Vessiot field of a particular matrix differential equation over a differential field k.

Proposition 3.24. *Let $(k,(\cdot)')$ be a differential field, having an algebraically closed field of constants C. Let K be a finite Galois extension of k with Galois group G. Then K is a PV extension of k and the differential Galois group is also G.*

Proof. Let $n = [K : k]$. Using Definition 3.19, we show that K is a PV extension of k by showing that:

(a) The field of constants of K is C;

(b) There exists a fundamental matrix $F \in \mathrm{GL}_n(K)$ for some differential equation to be specified later in this proof; and

(c) $K = k[F_{i,j} | 1 \le i, j \le n]$, where $F_{i,j}$ is the (i, j)-th entry of F.

Now K is finitely generated over k, since K is a finite extension of k, and K is ∂-simple. Thus, applying Proposition 3.17, we obtain (a).

Since K is a finite extension of k, by the Primitive Element Theorem (e.g., see Proposition 27.12 of [Ree15]), there exists $w \in K$ such that $K = k[w]$. Let $f \in k[y]$ be the minimum polynomial of w, and hence $n = \deg f$. Let $w_1 = w, w_2, \ldots, w_n$ be the n roots of f in K. We may write $K = k[w_1, \ldots, w_n]$. Let $V = \mathrm{Span}_C\{w_1, \ldots w_n\}$. Note that $\dim V = n$ since $K = \mathrm{Span}_k\{1, w, w^2, \ldots, w^{n-1}\}$ and, by Lemma 1.7 of [PS03], linear dependence over k implies linear dependence over C. Since G acts on K by permuting the roots w_i of f and fixing elements in k, we have that $\forall \sigma \in G$, $\sigma(V) = V$. Now, let us choose a C-basis $\{v_1, \ldots, v_n\}$ for V.

Claim 1: For each $\sigma \in G$, $\exists A^\sigma \in \mathrm{GL}_n(C)$ such that $\sigma(\mathrm{W}(v_1, \ldots, v_n)) = \mathrm{W}(v_1, \ldots, v_n) A^\sigma$.

Fix $\sigma \in \mathrm{Gal}(K/k)$. Since $\sigma(V) = V$, we have $\sigma(v_i) \in V$, allowing us to write

$$\sigma(v_i) = \sum_{j=1}^{n} v_j \, a^\sigma_{j,i}$$

for some $a^\sigma_{j,i} \in C$. Hence, we may write

$$\sigma(v_1, \ldots, v_n) = (v_1, \ldots, v_n)\, A^\sigma,$$

where $A^\sigma = [a^\sigma_{j,i}]_{1 \le i,j \le n} \in \mathrm{GL}_n(C)$. Furthermore,

$$\partial^m \sigma(v_i) = \sum_{j=1}^{n} a^\sigma_{j,i} \partial^m v_j,$$

and thus we have

$$\sigma(\mathrm{W}(v_1, \ldots, v_n)) = \mathrm{W}(v_1, \ldots, v_n) A^\sigma.$$

Claim 2: For each $\sigma \in G = \mathrm{Gal}(K/k)$, we have $\sigma \in \mathrm{Gal}^\partial(K/k)$.

It remains to show that any automorphism commutes with the derivation. Fix $\sigma \in G$ and consider the map $\psi : K \to K$, $f \mapsto \sigma^{-1}(\sigma(f)')$. Observe that ψ is a derivation on K which extends $(\cdot)'$ on k. Since the derivation extended to K from k is unique, e.g., as shown when doing Exercise 1.5(3)(a) of [PS03], we must have $\psi(\cdot) = (\cdot)'$. Hence, $\psi(v) = v'$, and we see that $\sigma(v') = \sigma(v)'$, for all $\sigma \in \mathrm{Gal}(K/k)$ and $v \in V$. Hence, $\sigma \in \mathrm{Gal}^\partial(K/k)$.

Claim 3: $\mathrm{wr}(v_1, \ldots, v_n) \neq 0$.

Arguing by contradiction, if $\mathrm{wr}(v_1,\ldots,v_n)=0$, then $\exists\beta_1,\ldots,\beta_m \in K\backslash\{0\}$, where $m\in\{1,2,\ldots,n\}$, such that

$$\mathrm{W}(v_1,\ldots,v_n)\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \vdots\\ 0\end{pmatrix}. \tag{3.7}$$

Choose m minimal with this property. If $m=1$ then $\beta_1 v_1=0$, giving the contradiction that $v_1=0$. Hence, $m\geq 2$, and we may assume $\beta_1=1$, since we can divide all coefficients by β_1 if necessary. From (3.7) we have the equations

$$v_1+\beta_2 v_2+\ldots+\beta_m v_m=0 \tag{3.8}$$

and

$$\partial v_1+\beta_2\partial v_2+\ldots+\beta_m\partial v_m=0. \tag{3.9}$$

Further, differentiating (3.8) gives

$$\partial v_1+\beta_2\partial v_2+\ldots+\beta_m\partial v_m+\partial\beta_2 v_2+\ldots+\partial\beta_m v_m=0$$

and subtracting (3.9) from this gives

$$\partial\beta_2 v_2+\ldots+\partial\beta_m v_m=0.$$

Since m is minimal, we must have $\partial\beta_2=\ldots=\partial\beta_m=0$, i.e., $\beta_2,\ldots,\beta_m\in C_K$. From Proposition 3.17, we must have $\beta_2,\ldots,\beta_m\in C$, and hence (3.8) contradicts $\{v_1,\ldots,v_n\}$ being a C-basis for V. Therefore, $\mathrm{wr}(v_1,\ldots,v_n)\neq 0$, as claimed.

Claim 4: There exists a fundamental matrix $F=\mathrm{W}(v_1,\ldots v_n)$ for the DE $y'=By$, where B is given in (3.10).

Let $F=\mathrm{W}(v_1,\ldots v_n)$. By virtue of $\mathrm{wr}(v_1,\ldots,v_n)\neq 0$, we have $F\in\mathrm{GL}_n(K)$. Now let

$$B=F'F^{-1}. \tag{3.10}$$

Observe that

$$\sigma(B)=\sigma(F')\sigma(F^{-1})=\sigma(F)'\sigma(F)^{-1}=(FA^{\sigma})'(FA^{\sigma})^{-1}$$

$$= F' A^\sigma (A^\sigma)^{-1} F^{-1} = F' F^{-1} = B,$$

so $B \in \mathrm{Mat}_n(k)$. Also, from (3.10) we have $F' = BF$, so that $F = \mathrm{W}(v_1, \dots v_n)$ is a fundamental matrix for the DE $y' = By$, which verifies (b).

It follows from K being equal to $k[w_1, \dots, w_n]$ that

$$K \supseteq k[v_1, v_1', v_1'', \dots, v_1^{(n-1)}, v_2, v_2', \dots, v_n^{(n-1)}] \supseteq k[w_1, \dots, w_n] = K,$$

so that (c) holds.

Hence, K is a PV extension of k.

Certainly, $G \supseteq \mathrm{Gal}^\partial(K/k)$, since $\sigma \in \mathrm{Gal}^\partial(K/k)$ means that $\sigma \in G$, with the additional condition $\sigma(f') = \sigma(f)'$. Also, Claim 2 shows that $G \subseteq \mathrm{Gal}^\partial(K/k)$. Hence, $\mathrm{Gal}^\partial(K/k) = G$, as required. □

3.5 The Differential Galois Group is a Linear Algebraic Group

The proof supplied for Proposition 3.24 indicates that any $\sigma \in \mathrm{Gal}^\partial(K/k)$ can be associated with a matrix $A^\sigma \in \mathrm{GL}_n(C)$. In this section we provide an explicit association of the differential Galois group with a linear algebraic group.

First we supply some essential definitions and notions. For more details, please see [Spr09] or Chapter 15 of [DF04].

Definition 3.25. (See Section 1.4.3 on p7 of [Spr09]) An affine k-variety is a ringed space $(X, \mathcal{O}_X)$, where X is an algebraic set, i.e., for some $n \in \mathbb{N}$ and for some $f_1, \dots, f_r \in k[X_1, \dots, X_n]$,

$$X = \{x \in k^n | f_1(x) = 0, \dots, f_r(x) = 0\}.$$

A *morphism* between two affine k-varieties $X \subseteq k^n$ and $Y \subseteq k^\ell$ is a map $\phi : X \to Y$ such that there are polynomials $g_1, \dots, g_\ell \in k[X_1, \dots, X_n]$ whereby

$$\forall x \in X, \phi(x) = (g_1(x), \dots, g_\ell(x)).$$

We denote the category of affine k-varieties by $\mathbf{Var}_k$, having affine k-varieties as objects and morphisms as given above. In particular, we denote by $\mathbb{A}_k^n$ the affine k-variety given by $\{x \in k^n\}$. The *coordinate ring* of X is the set $\mathcal{O}(X) := \mathrm{Mor}(X, \mathbb{A}_k^1)$, and is a commutative k-algebra.

We denote the category of commutative k-algebras by $\mathbf{CAlg}_k$, and the subcategory of those that are finitely generated and reduced by $\mathbf{CAlg}_k^{fg,red}$.

It is a fact that the functor $\mathcal{O} : \mathbf{Var}_k \to \mathbf{CAlg}_k^{fg,red}$ defines an equivalence of categories.

Definition 3.26. (cf Section 2.1.1, p21 of [Spr09]) Let $\mathscr{C}$ be a category having binary products and a terminal object $\mathbb{1}$. Then a *group object* in $\mathscr{C}$ is an object G of $\mathscr{C}$ such that there are:

(1) A unit map $e : \mathbb{1} \to G$;

(2) A multiplication map $m : G \times G \to G$; and

(3) An inverse map $i : G \to G$,

such that the diagrams in Figures 3.2, 3.3 and 3.4 commute.

Definition 3.27. A *linear algebraic group* is a group object in $\mathbf{Var}_k$.

For example, in $\mathbf{Var}_k$ we have $\mathbb{1} = \mathbb{A}_k^0$, and we can see that $\mathrm{GL}_n(k)$ is a linear algebraic group with the:

(1) Unit map $e : \mathbb{A}_k^0 \to \mathrm{GL}_n(k)$, $* \mapsto \mathrm{Id}_n$;

(2) Multiplication map $m : \mathrm{GL}_n(k) \times \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$,

$$\big((A_{i,j}, 1/\det A), (B_{i,j}, 1/\det B)\big) \mapsto ((AB)_{i,j}, 1/\det(AB));$$

and

(3) Inverse map $i : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$, $(A_{i,j}, 1/\det A) \mapsto ((A^{-1})_{i,j}, 1/\det(A^{-1}))$.

Proposition 3.28. *(See Proposition 5.1 of [Dyc20]) Let R be a PV ring for $y' = Ay$ over k, having field of constants C, and let S be a ∂-ring extension of k satisfying:*

- *S is an integral domain;*
- *$C_{\mathrm{Frac}(S)} = C$; and*
- *$\exists Y \in \mathrm{GL}_n(S)$ such that $Y' = AY$.*

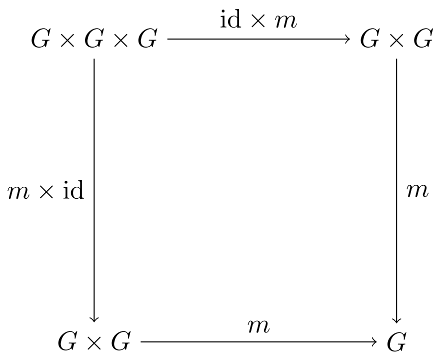

FIGURE 3.2: Commutative diagram showing associativity of multiplication.

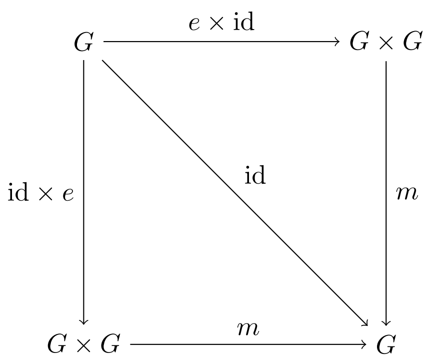

FIGURE 3.3: Commutative diagram showing the unit map with respect to multiplication.

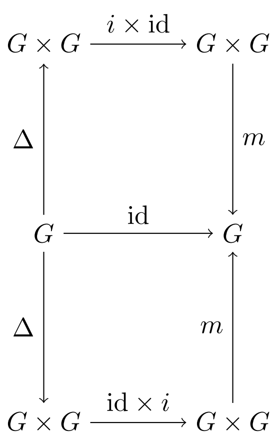

FIGURE 3.4: Commutative diagram showing the inverse map with respect to multiplication.

Let $U = \{u \in S \otimes_k R | u' = 0\}$, i.e., the C-algebra of constants in $S \otimes_k R$. Then:

(1) The map $\Theta : S \otimes_k U \to S \otimes_k R$, given by $s \otimes u \mapsto (s \otimes 1).u$, is an S-linear isomorphism of ∂-rings.

(2) Let $X \in \mathrm{GL}_n(R)$ be a fundamental matrix for R. Then $U = C[Z, \frac{1}{\det Z}]$, where $Z = Y^{-1} \otimes X \in \mathrm{GL}_n(U)$.

(3) U is a reduced C-algebra, i.e., U has no nonzero nilpotent elements.

Proof. For (1), to show that Θ is surjective, it suffices to show that for any element of the form $1 \otimes X_{i,j} \in S \otimes_k R$, there is some $t_{i,j} \in S \otimes U$ such that $\Theta(t_{i,j}) = 1 \otimes X_{i,j}$. Then, it follows by R- and S-linearity of Θ that for any given $r = \sum_{i,j} r_{i,j} X_{i,j} \in R$ and $s \in S$, we have $\Theta(s \sum_{i,j} r_{i,j} t_{i,j}) = s \otimes r$. Writing $1 \otimes X_{i,j}$ as

$$1 \otimes Y_{i,j} = e_j^T \otimes e_j^T I \otimes Y e_i \otimes e_i = e_j^T \otimes e_j^T (Y \otimes I)(Y^{-1} \otimes X) e_i \otimes e_i,$$

we see that $e_j^T \otimes e_j^T (Y \otimes I) \in S^n$ and $(Y^{-1} \otimes X) e_i \otimes e_i \in U^n$.

We now show that Θ is injective. Let $\{e_1, \ldots, e_n\}$ be a C-basis of U and let $f = \sum_i s_i \otimes e_i \in \ker\Theta - \{0\}$, with $s_1 \neq 0$, be of minimal length. Then, the element $g := s_1 \partial(f) - f \partial(s_1) \in \ker\Theta$, and the length of g is strictly less than that of f. Therefore, by minimality of the length of f, we have $g = 0$. Hence $s_1 \partial(f) = f \partial(s_1)$ and, decomposing into basis elements, we see that

$$s_1 \partial(\sum_i s_i \otimes e_i) = \partial(s_1) \sum_i s_i \otimes e_i,$$

i.e.,

$$\sum_i (s_1 \partial s_i) \otimes e_i + \sum_i (s_1 s_i) \otimes \partial e_i = \sum_i \partial(s_1) s_i \otimes e_i.$$

But $\partial e_i = 0$ because U is composed of constants of $S \otimes_k R$. Therefore,

$$\sum_i (s_1 \partial s_i) \otimes e_i = \sum_i \partial(s_1) s_i \otimes e_i,$$

and we have that $\partial(s_i / s_1) = 0$, i.e., $s_i / s_1 \in C$. Thus,

$$f = \sum_i s_i \otimes e_i = \sum_i s_1 (s_1^{-1} s_i \otimes e_i) = \sum_i (s_1 \otimes s_1^{-1} s_i e_i) = s_1 \otimes u,$$

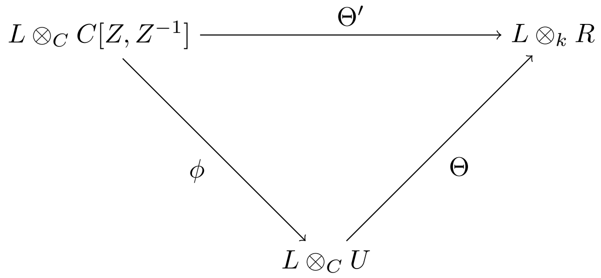

FIGURE 3.5: Commutative diagram used to show that $U \cong C[Z, Z^{-1}]$.

where $u == \sum_i s_1^{-1} s_i e_i \in U$. Since $f \in \ker \Theta$, we have that

$$0 = \Theta(f) = \Theta(s_1 \otimes u) = (s_1 \otimes 1).u,$$

which gives us that $u = 0$, since $s_1 \neq 0$. This means that $f = s_1 \otimes u = 0$, which is a contradiction. Hence, $\ker \Theta = \{0\}$.

To show (2), we consider the inclusion $i : C[Z, Z^{-1}] \hookrightarrow U$. Let $L = \mathrm{Frac}(S)$, so that $Y' = AY$ and $Y \in \mathrm{GL}_n(S) \subseteq \mathrm{GL}_n(L)$. We have the commutative diagram in Figure 3.5, where the map Θ' is the restriction of Θ to $L \otimes_C C[Z, Z^{-1}]$. We see that Θ and Θ' are isomorphisms, and therefore $\phi : L \otimes_C C[Z, Z^{-1}] \to E \otimes_C U$ also must be an isomorphism. Thus, the inclusion i is an isomorphism, and so $U \cong C[Z, Z^{-1}]$.

To show (3), we note that R and S are reduced. By Lemma A.16 in [PS03] we see that $S \otimes_k R$ is reduced. Since $U \subseteq S \otimes_C U \cong S \otimes_k R$, we have that U is also reduced. □

Corollary 3.29. *(See Corollary 5.3 of [Dyc20]) Let K be a PV extension for the DE $y' = Ay$ over k, having a fundamental matrix $Y \in \mathrm{GL}_n(K)$. Then $S = k[Y, \frac{1}{\det Y}]$ is a PV ring for the DE.*

We note that we have proven Corollary 3.29 previously in Proposition 3.11.

Corollary 3.30. *(See Corollary 5.4 of [Dyc20]) Let R and S be PV rings for the DE $y' = Ay$ over k. Define the functor*

$$F_{R,S} : \mathbf{CAlg}_C \to \mathbf{Set},$$

$$T \mapsto \mathrm{Isom}^{\partial}_{k \otimes_C T}(R \otimes_C T, S \otimes_C T).$$

Then $F_{R,S}$ is represented by the C-algebra U, i.e., $\mathrm{Hom}_{\mathbf{CAlg}_C}(U, \_) \cong F_{R,S}(\_)$.

Proof. As given in [Dyc20], let T be any C-algebra. Immediately we have

$$\mathrm{Hom}_{\mathbf{CAlg}_C}(U,T) \cong \mathrm{Hom}_S^{\partial}(S\otimes_C U, S\otimes_C T),$$

and, applying Proposition 3.28, gives

$$\mathrm{Hom}_{\mathbf{CAlg}_C}(U,T) \cong \mathrm{Hom}_S^{\partial}(S\otimes_k R, S\otimes_C T).$$

Since, S-linear homomorphisms out of $S\otimes_k R$ are in bijection with k-linear homomorphisms out of R, we have

$$\mathrm{Hom}_{\mathbf{CAlg}_C}(U,T) \cong \mathrm{Hom}_k^{\partial}(R, S\otimes_C T),$$

which is $\cong \mathrm{Hom}^{\partial}_{k\otimes_C T}(R\otimes_C T, S\otimes_C T)$, by the same reasoning given just above. Now, given any $\phi : R \to S\otimes_C T$, it must be that $\ker\phi = (0)$, since R is ∂-simple. Therefore, extending ϕ to $\tilde{\phi} : R\otimes_C T \to S\otimes_C T$ must have $\ker\tilde{\phi} = (0)$. Furthermore, since $R = k[X, 1/\det X]$ and $S = k[Y, 1/\det Y]$, for respective fundamental matrices X and Y of the PV rings R and S, we have that $Y^{-1}\phi(X) \in \mathrm{GL}_n(C_{S\otimes_C T})$, where $C_{S\otimes_C T}$ is the field of constants of $S\otimes_C T$. Since S admits no new constants, $C_{S\otimes_C T} = T$ and, therefore, $Y^{-1}\phi(X) \in \mathrm{GL}_n(T)$. So, $Y\otimes_C T = \phi(X)\phi(X)^{-1}Y\otimes_C T = \phi(X)\otimes_C(\phi(X)^{-1}YT)$, gives us that ϕ and its extension $\tilde{\phi}$ are surjective. Hence,

$$\mathrm{Hom}^{\partial}_{k\otimes_C T}(R\otimes_C T, S\otimes_C T) \cong \mathrm{Isom}^{\partial}_{k\otimes_C T}(R\otimes_C T, S\otimes_C T),$$

and we have shown that $\mathrm{Hom}_{\mathbf{CAlg}_C}(U,T) \cong \mathrm{Isom}^{\partial}_{k\otimes_C T}(R\otimes_C T, S\otimes_C T)$. □

Corollary 3.31. *(See Corollary 5.5 of [Dyc20]) For all PV rings R and S for the DE $y' = Ay$ over k, there exists a k-linear ∂-isomorphism $R \cong S$.*

We have shown the above corollary in Proposition 3.11, however it can also be shown by applying Corollary 3.30 directly to C, i.e.,

$$\mathrm{Hom}_{\mathbf{CAlg}_C}(U,C) \cong F_{R,S}(C) = \mathrm{Isom}^{\partial}(R,S)(C) = \mathrm{Isom}_k^{\partial}(R,S).$$

Before giving our theorem stating the Gal^{∂} is a linear algebraic group, we a definition.

Definition 3.32. Let $G \in \mathbf{Var}_k$ and $R \in \mathbf{CAlg}_k$. An *R-valued point* of G is a homomorphism $\varphi : \mathcal{O}(G) \to R$ of k-algebras. The set of R-valued points of G is denoted by $G(R)$.

Corollary 3.33. *(See Corollary 5.6 of [Dyc20]) Let R be a PV ring for the DE $y' = Ay$ over k and let $K = \mathrm{Frac}(R)$. Then the ∂-Galois group $\mathrm{Gal}^{\partial}(K/k)$ can be identified with the group of C-valued points $\mathrm{Hom}_C(U, C)$ of a linear algebraic group $G(C)$ with coordinate ring $U := C_{R\otimes_k R}$.*

Proof. Note that if $U = \mathscr{O}(G)$, i.e., U is the coordinate ring of G, then $G(C) = \mathrm{Hom}_C(U, C)$. Corollary 3.30, with $S = R$, gives $\mathrm{Hom}_C(U, C) \cong \mathrm{Isom}_k^{\partial}(R, R)$. Hence, $\mathrm{Gal}^{\partial}(K/k) = \mathrm{Isom}_k^{\partial}(R, R) \cong G(C)$. As stated earlier, the functor $\mathscr{O} : \mathbf{Var}_C \to \mathbf{CAlg}_C^{fg,red}$ defines an equivalence of categories, so that $G = \mathrm{Gal}^{\partial}(K/k)$ is mapped to $\mathscr{O}(G) = U$. □

3.6 The ∂-Galois Correspondence

By analogy with classical Galois theory, we have the following theorem.

Theorem 3.34. *(See Proposition 1.34 on p26 of [PS03]) Let $y' = Ay$ be a DE over k with PV field K and write $G := \mathrm{Gal}(K/k)$. Let*

$$\mathscr{S} = \left\{ H \le \mathrm{Gal}^{\partial}(K/k) | H = \overline{H} \right\}$$

be the set of closed subgroups of G and let

$$\mathscr{K} = \{k \subseteq L \subseteq K | L = \partial - \text{subfield}\}$$

be the set of differential subfields of K that contain k. Define $\alpha : \mathscr{S} \to \mathscr{K}$ by $\alpha(H) = K^H$, the subfield of K consisting of the H-invariant elements. Define $\beta : \mathscr{K} \to \mathscr{S}$ by $\beta(L) = \mathrm{Gal}^{\partial}(K/L)$, which is the subgroup of G consisting of the L-linear differential automorphisms. Then:

(1) The maps α and β are inverses of each other.

(2) The subgroup $H \in \mathscr{S}$ is a normal subgroup of G iff $L = K^H$ is, as a set, invariant under G. If $H \in \mathscr{S}$ is normal then the canonical map $G \to \mathrm{Gal}(L/k)$ is surjective and has kernel H. Moreover L is a PV field for some linear DE over k.

(3) Let G° denote the identity component of G. Then $K^{G^{\circ}} \supset k$ is a finite Galois extension with Galois group G/G° and is the algebraic closure of k in K.

Proof. See p26 of [PS03]. □

3.7 The Differential Galois Group is the Zariski Closure of the Monodromy Group

The monodromy of a DE describes how its solutions change when analytically continuous around a loop encircling a singularity of the DE. Consider the DE, $y' = Ay$ over k, where $k = \mathbb{C}(z)$. Our field of constants is $C = \mathbb{C}$. As mentioned in Chapter 5 of [PS03], we can take as domain $\mathbb{C}\bigcup\{\infty\}\} = \mathbb{P}^1$ for the functions in k. Let $K = \mathbb{C}(\{z\})$ be our field extension. Then solutions of our DE, with $y \in K$, can be found by series substitution. Let $V_0 = \mathrm{Sol}_K(\partial - A)$. Then for U an open subset of $\mathbb{P}^1$ containing zero, and for a given path $\lambda : [0,1] \to U$, such that $\lambda(0) = \lambda(1) = 0$, there exists a fundamental solution matrix $F_{\lambda(t)}$ for each open subset $O_{\lambda(t)} \subset U$ containing $\lambda(t)$, for all $t \in [0,1]$. We obtain a $\mathbb{C}$-linear bijection $M(\lambda) : V_0 \to V_0$, depending only on the homotopy class of λ. Thus, we have $\forall \lambda \in \pi_1(U,0)$, $M(\lambda) \in \mathrm{GL}(V_0)$.

Definition 3.35. We say that $M(\lambda)$ is the *monodromy map* and that the image, under M, of $\pi_1(U,0)$ in $\mathrm{GL}(V_0)$ is the *monodromy group*.

It is fact that the Zariski-closure of the monodromy group equals the ∂-Galois group, and this is of use when computing ∂-Galois groups of DEs; e.g., see p326 of [BH89].

3.8 Liouvillian Extensions

Beyond algebraic expressions for solutions of DEs, we are seeking solutions which are expressible in terms of integrals and exponentials. Thus, we have the following definition.

Definition 3.36. Let the differential field k have an algebraically closed field of constants C. An extension $K \supset k$ of differential fields is called a *Liouvillian* extension of k if the field of constants of K is C and if there exists a tower of fields

$$k = K_0 \subset K_1 \subset \ldots \subset K_n = K$$

such that $K_i = K_{i-1}(t_i)$ for $i = 1, \ldots, n$, where either:

1. $t_i' \in K_{i-1}$, that is t_i is an integral (of an element of K_{i-1});
2. $t_i \neq 0$ and $t_i'/t_i \in K_{i-1}$, that is t_i is an exponential (of an integral of an element of K_{i-1}); or

3. t_i is algebraic over K_{i-1}.

Proposition 3.37. *(Theorem 1.43 on Page 33 of [PS03]) Let K be a Picard-Vessiot extension of k with differential Galois group G. The following are equivalent:*

(1) G is a solvable group.

(2) K is a Liouvillian extension of k.

(3) K is contained in a Liouvillian extension of k.

3.9 Classification of Differential Galois Groups

The following two lemmas emanate from Exercise 1.35(5) in [PS03].

Lemma 3.38. *Let K be a PV extension for $y' = Ay$ of dimension n over k. Then,* $\mathrm{Gal}^{\partial}(K/k) \leq \mathrm{SL}_n(C)$ *iff the DE* $u' = \mathrm{tr}(A)u$ *has a nonzero solution in k.*

Proof. ($\Rightarrow$) Let $W = \mathrm{W}(y_1, \ldots, y_n)$, where $\{y_1, \ldots, y_n\} \subset K$ is a fundamental set of solutions for the DE. If $G \leq \mathrm{SL}_n(C)$, then for $\sigma \in G$, $\sigma(W) = W A_\sigma$, where $A_\sigma \in \mathrm{SL}_n(C)$. Let $u = \det W$. Since $W \in \mathrm{GL}_n(K)$, $u \neq 0$. Also,

$$\sigma(u) = \sigma(\det W) = \det(\sigma(W)) = \det(W A_\sigma) = \det(W)\det(A_\sigma) = \det(W) = u,$$

since $\det(A_\sigma) = 1$. Hence u is fixed by all $\sigma \in G$, which means that $u \in k$. From Lemma 3.22, $u' = \mathrm{tr}(A)u$ and hence the DE has a nonzero solution in k.

($\Leftarrow$) Suppose $u' = \mathrm{tr}(A)u$ has a nonzero solution $u \in k$. Let $v = \det W \neq 0$, which we know, from Lemma 3.22, that it also satisfies the DE. Then

$$\left(\frac{v}{u}\right)' = \frac{v'u - vu'}{u^2} = \frac{\mathrm{tr}(A)vu - v\,\mathrm{tr}(A)u}{u^2} = 0.$$

Hence, $v/u \in C$. Thus, $\sigma(v/u) = v/u$, giving us $\sigma(v) = \sigma(u)v/u = v$, since σ fixes u. From $\sigma(W) = W A_\sigma$, for $A_\sigma \in G$, we see that

$$v = \det \sigma(W) = \det W \det A_\sigma = v \det A_\sigma,$$

namely $\det A_\sigma = 1$, i.e., $A_\sigma \in \mathrm{SL}_n(C)$. □

Lemma 3.39. *Let K be a PV extension for $y' = Ay$ over k of dimension n, and let W be a fundamental matrix for the DE. Then,* $\mathrm{Gal}^{\partial}(K/k) \le \mathrm{SL}_n(C)$ *iff* $\det W \in k$.

Proof. Let $L = \partial^n + f_{n-1}\partial^{n-1} + \ldots + f_0 \in k[\partial]$ be a differential operator. Form the companion matrix A_L for this operator and let K be a PV extension for the DE $y' = A_L y$ over k. Let $G = \mathrm{Gal}^{\partial}(K/k)$ and let W be a fundamental matrix for the DE.

($\Rightarrow$) If $G \le \mathrm{SL}_n(C)$, then

$$\sigma(\det W) = \det \sigma(W) = \det(W A_\sigma) = \det W \det A_\sigma = \det W,$$

since $\det A_\sigma = 1$. Hence $\det W$ is fixed by any $\sigma \in G$, and we have $\det W \in k$.

($\Leftarrow$) From Lemma 3.21, $(\det W)' = \mathrm{tr}(A_L) \det W = -f_{n-1} \det W$. Since $\det W \in k$, the DE $u' = \mathrm{tr}(A)\, u$ has a nonzero solution in k and, from Lemma 3.38, $G \le \mathrm{SL}_n(C)$. □

For the case $n = 2$, once we have established that the $\mathrm{Gal}^{\partial}(K/k) \le \mathrm{SL}_2(C)$, such as via the above lemma, we can employ the following classification of algebraic subgroups of $\mathrm{SL}_2(C)$.

Theorem 3.40. *(See Theorem 4.29 on p133 of [PS03]) Let G be an algebraic subgroup of* $\mathrm{SL}_2(C)$. *Then, up to conjugation, one of the following cases occurs:*

(1) G is the subgroup of the Borel group

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in C^\times, b \in C \right\};$$

(2) G is not contained in a Borel group and is a subgroup of the infinite dihedral group

$$D_\infty = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c \in C^\times \right\} \bigcup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \middle| c \in C^\times \right\};$$

(3) G is one of the groups $A_4^{\mathrm{SL}_2}$ (the tetrahedral group), $S_4^{\mathrm{SL}_2}$ (the octahedral group), or $A_5^{\mathrm{SL}_2}$ (the icosahedral group). These groups are the preimages in $\mathrm{SL}_2(C)$ *of the subgroups A_4, S_4, A_5 contained in* $\mathrm{PSL}_2(C)$;

(4) $G = \mathrm{SL}_2(C)$.

Chapter 4

Examples of Differential Galois Groups of DEs

4.1 Examples of Differential Galois Groups

In this chapter we supply various concrete examples of differential Galois groups. In particular, we supply examples of DEs whose differential Galois groups are the full general linear group, the special linear group and finite subgroups of the general linear group.

4.2 Explicit Construction of PV Extension Having $\mathrm{GL}_n(C)$ as ∂-Galois Group

We give an explicit construction of a PV extension for which $\mathrm{GL}_n(C)$ is its ∂-Galois group. This is an example of an inverse Galois problem, whereby given a group G, one is required to find a field extension K whose Galois group is G.

Let k be a ∂-field, $\mathrm{char}(k) = 0$, with $C = C_k$ algebraically closed.

Let

$$R = k[y_i^{(j)}],$$

where $i \in \{1,2,\ldots,n\}$ and $j \in \mathbb{N}$, such that $\partial y_i^{(j)} = y_i^{(j+1)}$. Let $K = \mathrm{Frac}(R)$. Define the differential operator $L \in K[\partial]$ as

$$L(y) = \frac{\mathrm{wr}(y, y_1, y_2, \ldots, y_n)}{\mathrm{wr}(y_1, y_2, \ldots, y_n)} = \partial^n + a_{n-1}\partial^{n-1} + \ldots + a_0,$$

where $a_j \in K$.

Automatically we have $L(y_i) = 0$, for all $i \in \{1,2,\ldots,n\}$.

Now let $M = k(a_0, a_1, \ldots, a_{n-1})$. Let $A_L \in \mathrm{Mat}_n(M)$ be the companion matrix associated with the operator L.

For each $A \in GL_n(K)$, define $\sigma_A : K \mapsto K$ via

$$\sigma_A(y_1, \ldots, y_n) = (y_1, \ldots, y_n)\,A.$$

Note that $K = k(y_1, \ldots, y_n)$.

Claim: K is a PV extension of M.

Firstly, we have that $Y = \mathrm{W}(y_1, \ldots, y_n)$ is a fundamental matrix for the DE $y' = A_L y$ over M.

Secondly, K is generated as an M-algebra by $y_i^{(j)}$, $i \in \{1,2,\ldots,n\}$ and $j \in \{0,1,2,\ldots,n-1\}$.

Thirdly, we now show that $C_K = C$. Let $r(y_i^{(j)})/s(y_i^{(j)}) \in C_K$, i.e., $r(y_i^{(j)})/s(y_i^{(j)}) \in K$ satisfying $r(y_i^{(j)}), s(y_i^{(j)}) \in R - \{0\}$, $(r(y_i^{(j)})/s(y_i^{(j)}))' = 0$ and $\gcd(r(y_i^{(j)}), s(y_i^{(j)})) = 1$. Then

$$\frac{r's - r\,s'}{s^2} = 0,$$

so that

$$r's = r\,s'. \tag{4.1}$$

Consider the case when r and s are monomials of the form

$$r = \prod_{i,j} (y_i^{(j)})^{e_{i,j}}, \quad s = \prod_{i,j} (y_i^{(j)})^{f_{i,j}},$$

where $i \in \{1,2,\ldots,n\}$, $j \in \mathbb{N}$, $e_{i,j}, f_{i,j} \in \mathbb{N}$, and all but finitely many are nonzero. Then

$$r' = r\sum_{i,j} e_{i,j}\frac{y_i^{(j+1)}}{y_i^{(j)}}, \quad s = \sum_{i,j} f_{i,j}\frac{y_i^{(j+1)}}{y_i^{(j)}}.$$

Hence, (4.1) becomes

$$rs\sum_{i,j} e_{i,j}\frac{y_i^{(j+1)}}{y_i^{(j)}} = rs\sum_{i,j} f_{i,j}\frac{y_i^{(j+1)}}{y_i^{(j)}}.$$

Equating like terms gives $e_{i,j} = f_{i,j}$, for all i, j, which implies $r = s$. Now $\gcd(r,s) = 1$ and we have $s = 1$. It follows that $r' = 0$ and, thus, $r/s \in C$.

The argument can be extended for several terms, so that we have $C_K = C$.

Hence K is a PV extension of M.

Claim: $\mathrm{Gal}^\partial(K/M) = \mathrm{GL}_n(C)$.

Certainly, $\mathrm{Gal}^\partial(K/M) \subseteq \mathrm{GL}_n(C)$. We now show the reverse inclusion.

For all $A \in \mathrm{GL}_n(C)$ define $\sigma_A : K \mapsto K$ via

$$\sigma_A(y_1,\ldots,y_n) = (y_1,\ldots,y_n)A.$$

Then, since $\sigma_A(y_1',\ldots,y_n') = (\sigma_A(y_1,\ldots,y_n))'$, we have

$$\sigma_A Y = YA.$$

Note that σ_A fixes $a_0, a_1, \ldots, a_{n-1} \in M$. This follows from

$$\sigma_A(A_L) = \sigma_A(Y'Y^{-1}) = (\sigma_A Y)'\sigma_A(Y)^{-1}$$

$$= (YA)'(YA)^{-1} = Y'AA^{-1}Y^{-1} = Y'Y^{-1} = A_L.$$

Thus, for each $A \in \mathrm{GL}_n(C)$, $\sigma_A \in \mathrm{Gal}^\partial(K/M)$, i.e., $\mathrm{GL}_n(C) \subseteq \mathrm{Gal}^\partial(K/M)$.

4.3 Further Examples

The following examples, some of which appear as exercises in various references, illustrate the computation of Gal^∂ for several DEs.

Example 4.1. *(See Exercise 1.2 in [PS03]).* Fix $t \in \mathbb{R}$ and let $L = \partial - \frac{t}{x}$ and $k = \mathbb{C}(x)$. We have $C = \mathbb{C}$ as our field of constants. Let K be a Picard-Vessiot extension for $L(y) = 0$ over k. Then we see that

$$\begin{aligned}
\mathrm{Sol}_K(L) &= \{y \in K | L(y) = 0\} \\
&= \{y \in K | \partial y - \frac{t}{x} y = 0\} \\
&= \{y \in K | \frac{1}{y} \partial y = \frac{t}{x}\} \\
&= \{y \in K | \log y = t \log x + c,\ c \in \mathbb{C}\} \\
&= < x^t >_{\mathbb{C}} .
\end{aligned}$$

Let $R = \mathbb{C}(x)[x^t]$. Then (R, ∂) is a ∂-ring and $F = [x^t]$ is a fundamental matrix. Thus, $K = \mathrm{Frac}(R)$ is a PV extension. We have $\mathrm{Gal}^{\partial}(K/k) \leq \mathrm{GL}(\mathrm{Sol}_K(L)) \cong \mathrm{GL}_1(\mathbb{C})$. Now, for any $\sigma \in \mathrm{Gal}^{\partial}(K/k)$, $\sigma(x^t) = c_\sigma x^t$, for some $c_\sigma \in \mathbb{C}^\times$. We want all $c_\sigma \in \mathrm{GL}_1(\mathbb{C})$ that preserve algebraic relations among $S = \{x^{tj} | j \in \{0, 1, 2, \ldots\}\}$.

We have two cases to consider.

Case 1: $t \notin \mathbb{Q}$

Now, if there are

$$a_0(x), \ldots, a_n(x) \in \mathbb{C}(x)$$

such that

$$\sum_{j=0}^{n} a_j(x)(x^t)^j = 0,$$

then, clearing denominators, we have

$$\sum_{j=0}^{n} b_j(x)(x^t)^j = 0,$$

for some $b_j(x) \in \mathbb{C}[x]$, not all zero. Thus

$$b_0(x) = -\sum_{j=1}^{n} b_j(x) x^{tj}.$$

But the ℓ-th derivative of the LHS is zero, for $\ell \in \mathbb{N}$ sufficiently large, and this is not the case for the RHS. Hence, there is no algebraic relation among the elements of S and therefore $\mathrm{Gal}^{\partial}(K/k) \cong \mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$.

Case 2: $t \in \mathbb{Q}$

Write $t = p/q$, where $p, q \in \mathbb{Z}$ and $\gcd(p,q) = 1$. Then there is an algebraic relation, with coefficients in $\mathbb{C}(x)$, satisfied by elements of $S = \{x^{jp/q} | j \in \mathbb{N}\}$, namely

$$(y_j)^q - x^{jp} = 0,$$

where $y_j = x^{jp/q} \in \mathrm{Sol}_K(L)$.

We note that for $\sigma \in \mathrm{Gal}^{\partial}(K/k)$ we must have $\sigma(y_j^q) = \sigma(x^{jp}) = x^{jp}$, since $x^{jp} \in k$. Thus, $\sigma(y_j^q) = y_j^q = x^{tq}$, and we have the relation $\sigma(y_j^q) = (c_\sigma x^t)^q = x^{tq}$, i.e., $c_\sigma^q = 1$. Thus, $c_\sigma \in \mu_q$. This gives us $\mathrm{Gal}^{\partial}(K/k) \le \mu_q$ Since any $c \in \mu_q$ preserves algebraic relations among elements of S, we have $\mathrm{Gal}^{\partial}(L) \cong \mu_q = \{z \in \mathbb{C} | z^q = 1\}$.

Hence

$$\mathrm{Gal}^{\partial}(L) = \begin{cases} \mu_q, & \text{if } t = p/q \in \mathbb{Q},\, gcd(p,q) = 1; \\ \mathbb{C}^\times, & \text{if } t \notin \mathbb{Q} \end{cases}.$$

Remark 4.1. Alternatively, we can compute the ∂-Galois group of the DE as in Example 4.1 by computing the monodromy group, as described in Section 3.7. We take $k = \mathbb{C}(x)$, $K = \mathbb{C}(\{z\})$. Then solutions of $L(y) = 0$, with $y \in K$ can be found by series substitution. Let $V_0 = \mathrm{Sol}_K(L)$. Then, V_0 has basis $\{x^t\}$ and $M(\lambda)(x^t) = \exp(2\pi i t).x^t$. When $t \notin \mathbb{Q}$ the monodromy group is $GL_1(\mathbb{C})$ and when $t = p/q \in \mathbb{Q}$, with $\gcd(p,q) = 1$, the monodromy group is μ_q, confirming the calculation of $\mathrm{Gal}^{\partial}(K/k)$ in the above example.

Example 4.2. *(See Example 0.1 in [Sin90])* Consider the DE

$$L(y) := y^{(3)} + \frac{2}{x}y'' - \frac{1}{4x^2}y' + \frac{1}{4x^3}y = 0$$

over the differential field $k = \mathbb{C}(x)$. We have $\mathrm{Sol}(L) = \mathrm{Span}_{\mathbb{C}}\{x, \sqrt{x}, 1/\sqrt{x}\}$. Let $R = k[y, 1/y]$, where $y = x^{1/2}$. Observe that, with

$$F = \begin{pmatrix} x & y & 1/y \\ 0 & 1/(2y) & -1/(2xy) \\ 0 & -1/(4xy) & 3/(4x^2y) \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/(4x^3) & 1/(4x^2) & -2/x \end{pmatrix},$$

we have $F' = AF$. Since $\det F = (2x+1)/(2xy) = 1/\sqrt{x} + 1/(2x^{3/2}) \neq 0$, we have $F \in GL_3(R)$. However, R is not equal to the ring generated over $\mathbb{C}(x)$ by the elements x, y, $1/y$ and $1/\det F$, which is to say that Condition (A3) in Definition 3.10 is not fulfilled.

Instead, we choose

$$F = \begin{pmatrix} 1 & y \\ 0 & 1/(2y) \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -1/(2x) \end{pmatrix},$$

so that $\det F = 1/(2y) \neq 0$, i.e., $F \in \mathrm{GL}_2(R)$. Further, we have $F' = AF$, so that F is a fundamental matrix for the DE. Also, R is equal to the ring generated over $\mathbb{C}(x)$ by the elements x, y, $1/y$ and $1/\det F$. Thus, R is a PV ring for the DE and $K = \mathrm{Frac}(R) = \mathbb{C}\left(\sqrt{x}\right)$ is a PV extension. We see that K is a degree 2 extension over k, which is a Galois extension. Thus, from Proposition 3.24, we have $\mathrm{Gal}^{\partial}(K/k) \cong \mu_2$.

Example 4.3. As given in (2.5), consider the Serre derivative $\partial f := \frac{1}{2\pi i}\frac{d}{dz} - \frac{k}{12}E_2 f$. Then, with $R = \widetilde{M}_*(\Gamma_1)$, as defined in (2.8), we have a differential ring (R, ∂), and we can form the differential field $k = \mathrm{Frac}(R)$. Let $L = \partial^2 + \frac{1}{4}\frac{E_4^2}{E_6}\partial - \frac{1}{12}E_4 \in k[\partial]$, and consider the DE, $L(y) = 0$, over k. Write $L = \frac{1}{2}L_1 + \frac{1}{2}L_2$, where $L_1 = \partial^2 - \frac{1}{6}E_4$ and $L_2 = \partial^2 + \frac{E_4^2}{2E_6}\partial$. Then we observe that $L_1(E_4) = \partial^2 E_4 - \frac{1}{6}E_4^2 = 0$ and $L_2(E_4) = \partial^2 E_4 + \frac{E_4^2}{2E_6}\partial E_4 = 0$, so that $L(E_4) = 0$. Thus, we have one solution to our DE that lies in k. Let y be another solution. Then the Wronskian satisfies a first-order DE whose solution is $\mathrm{wr}(E_4, y) = \exp\left\{-\int \frac{E_4^2}{4E_6}\right\}$, up to multiplication by a constant. Note that the integral of $\frac{E_4^2}{4E_6}$ is its antiderivative with respect to the Serre derivative. This, in turn, gives us a first-order DE for y. Let $S = R[y]$. Then S is a PV ring for our DE. Indeed, let

$$F = \begin{pmatrix} E_4 & y \\ -E_6/3 & \partial y \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ E_4/12 & -E_4^2/(4E_6) \end{pmatrix},$$

so that $\det F = E_4 \partial y + y E_6/3 = \mathrm{wr}(E_4, y) \neq 0$, i.e., $F \in \mathrm{GL}_2(R)$. Further, we have $F' = AF$, so that F is a fundamental matrix for the DE. Also, S is equal to the ring generated over k by the elements E_4, E_6, y, ∂y and $1/\det F$. Thus, S is a PV ring for the DE and $K = \mathrm{Frac}(S)$ is a PV extension. We also have $\mathrm{Gal}^{\partial}(K/k) \cong \mathrm{GL}_2(\mathbb{C})$, since by Lemma 3.39, it is not the case that $\mathrm{Gal}^{\partial}(K/k) \leq \mathrm{SL}_2(\mathbb{C})$.

Example 4.4. *(See Exercise 1.35(1) on p27 of [PS03]).* Consider the DE $y' = a$ over the differential field k. We determine a PV ring R for the DE and compute $\mathrm{Gal}^{\partial}(K/k)$, where $K = \mathrm{Frac}(R)$ is the PV extension of k for the DE. There are two cases to consider.

Case A: $\exists t \in k$ such that $t' = a$. If $a = 0$ then $R = C$ is a PV ring, $K = \mathrm{Frac}(R) = R = C$ and we take $k = C$, so that $\mathrm{Gal}^{\partial}(K/k) = \{\mathrm{Id}\}$. Hence, we assume $a \neq 0$. Then $R = k$ is a PV ring, because it satisfies the three properties: it is simple (since any field is ∂-simple), it has fundamental matrix $F = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(R)$ satisfying $F' = AF$, where $A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, and $R = k[t] = k$. Further, $K = \mathrm{Frac}(R) = k$ is the trivial PV extension, so that $\mathrm{Gal}^{\partial}(K/k) = \{\mathrm{Id}\}$.

Case B: $\nexists t \in k$ such that $t' = a$. Since $1 \in k$ and $1' = 0$, we have $a \neq 0$. Let T be an indeterminant satisfying $T' = a$ and let $R = k[T]$. Then R is a PV ring for the DE. Indeed, it has fundamental matrix $F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(R)$ satisfying $F' = AF$, where $A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, and, by definition, $R = k[T]$. It remains to show that R is ∂-simple. If $\exists I \leq R$, $I \neq (0), R$, then let $x = a_0 + a_1 T + \ldots + a_n T^n \in I$ be of minimal degree, with $a_i \in k$ and $a_n \neq 0$. Without loss of generality, we may assume $a_n = 1$ since $y = x/a_n$ has leading coefficient 1. Also, we may assume that $n \geq 1$, since otherwise $a_0 \in I$ implies $I = R$. Hence,

$$x' = (n\,a + a'_{n-1})\,T^{n-1} + ((n-1)\,a + a'_{n-2})\,T^{n-2} + \ldots + (a + a'_0) \in I$$

and is of lower degree than n, hence is zero. In particular, the coefficient of T^{n-1} vanishes, so that $(-\frac{1}{n}a_{n-1})' = a$ with $-\frac{1}{n}a_{n-1} \in k$ contradicts our hypothesis. Hence, R is ∂-simple and thus is a PV ring.

We have that $K = \mathrm{Frac}(R) = k(T)$ is a PV extension of k. Let $\sigma \in \mathrm{Gal}^{\partial}(K/k)$. Observe that

$$\partial(\sigma(T)) = \sigma(\partial T) = \sigma(a) = a = \partial T,$$

so that $\partial(\sigma(T) - T) = 0$, i.e., $\sigma(T) - T \in C$. Define c_σ to be $\sigma(T) - T$. We claim that $\Phi : \left(\mathrm{Gal}^{\partial}(K/k), \circ\right) \to (C, +)$, $\sigma \mapsto c_\sigma$, is an isomorphism of Abelian groups. Define $\Psi : C \to \mathrm{Gal}^{\partial}(K/k)$, via $c \mapsto (T \mapsto T + c)$. Clearly, Ψ is inverse to Φ, and $\Phi(\sigma_1 \circ \sigma_2) = c_{\sigma_1} + c_{\sigma_2} = \Phi(\sigma_1) + \Phi(\sigma_2)$ verifies that it is a homomorphism.

Example 4.5. *(See Exercise 1.35(2) on Page 27 of [PS03])*. Let $L = \partial - a\,\mathrm{Id} \in k[\partial]$, where $a \in k^\times$. Let K be a PV extension for the DE $L(y) = 0$ over k.

To compute $\mathrm{Gal}^{\partial}(K/k)$ we have two cases to consider.

Case A: $\nexists n \in \mathbb{Z} - \{0\}$ such that $\exists y \in k^\times$ satisfying $y' = n\,a\,y$. Then, let $R = k[T, T^{-1}]$, where T is an indeterminate satisfying $\partial T = a\,T$. We show that R is a PV ring for $y' = a\,y$ over k.

Firstly, R is ∂-simple. Otherwise, if $\exists I \subset R$, where I is an ideal and $I \neq (0), R$, then let $x = \sum_{i=\ell}^{m} x_i T^i \in I$. We can choose x to be such that $\ell = 0$ and m is minimal, with $x_0 \neq 0$ and

$x_m = 1$. Note that $m > 0$, since otherwise $1 \in I$, forcing the contradiction $I = R$. Then

$$\partial x = x_0' + (a\,x_1 + x_1')T + \ldots + (m\,a\,x_m + x_m')T^m \in I$$

and, therefore,

$$\begin{aligned}&\partial x - m\,a\,x\\ &= (x_0' - m\,a\,x_0) + (a\,x_1 + x_1' - m\,a\,x_1)T + \ldots + ((m-1)\,a\,x_{m-1} + x_{m-1}' - m\,a\,x_{m-1})T^{m-1} \in I.\end{aligned}$$

Minimality of m gives $\partial x - m\,a\,x = 0$, i.e., $x_0' - m\,a\,x_0 = 0$, contradicting our supposition.

Secondly, $\exists Y \in \mathrm{GL}_1(R)$, namely $Y = [T]$, such that $Y' = A\,Y$, where $A = [a] \in \mathrm{Mat}_1(k)$.

Thirdly, $R = k[T, T^{-1}]$ by definition. Therefore, all three conditions for R being a PV ring are fulfilled.

Let, $K = \mathrm{Frac}(R)$ be the PV extension of k. Now, fix $\sigma \in \mathrm{Gal}^{\partial}(K/k)$. Then, $\sigma(T'\,T^{-1}) = \sigma(a) = a = T'\,T^{-1}$, and we have $\sigma(T)'\,T - \sigma(T)\,T' = 0$, i.e.,

$$\left(\frac{\sigma(T)}{T}\right)' = 0.$$

Hence, $\frac{\sigma(T)}{T} \in C$, and cannot be zero since $T \notin k$. Thus, $\sigma(T) = T\,c_\sigma$, for some $c_\sigma \in C^\times$. Indeed, $\Phi : (\mathrm{Gal}^{\partial}(K/k), \circ) \to (C^\times, .)$ is an isomorphism of groups, so that $\mathrm{Gal}^{\partial}(K/k) \cong (C^\times, .)$.

In the particular case $k = \mathbb{C}(x)$, we have $K = \mathbb{C}(x, \exp(x))$ is the Picard-Vessiot extension for L over k. Then $\mathrm{Sol}_E(L) = \{y \in E | y' = y\} = <\exp(x)>_{\mathbb{C}}$ and $\dim \mathrm{Sol}_E(L) = 1$. We have $\mathrm{Gal}^{\partial}(L) \le \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. Now $\exp(x)$ is transcendental over $\mathbb{C}(x)$, and hence $\mathrm{Gal}^{\partial}(L) = \mathbb{C}^\times$.

Case B: $\exists n \in \mathbb{Z} - \{0\}$ such that $\exists y \in k^\times$ satisfying $y' = n\,a\,y$. We may restrict our consideration to postive n since if $\exists y$ such that $y' = -n\,a\,y$, then $\partial(1/y) = -y'/y^2 = n\,a(1/y)$. Now, let n be minimal among all positive n. Let, $S = k[T, T^{-1}]/(T^n - y) = k[t]$, where t is an indeterminate satisfying $t^n = y$. Then S is a PV ring for the DE $y' = a\,y$ over k. Arguing in a similar manner as in Case A, we see that S is ∂-simple. Also, $t \in \mathrm{GL}_1(S)$ is a fundamental matrix for the DE $y' = a\,y$. Finally, $S = k[t]$ by definition. Hence, S is a PV ring for the DE and $K = \mathrm{Frac}(S)$ is a PV extension.

Now we determine $\mathrm{Gal}^{\partial}(K/k)$. Let $\sigma \in \mathrm{Gal}^{\partial}(K/k)$. Then σ is determined by the value of $\sigma(t)$. Since $y \in k^\times$, σ fixes y, and therefore $\sigma(t)^n = \sigma(y) = y = t^n$. Hence, $\sigma(t) = t\,\exp(2\pi i\,m/n)$, for some $m \in \{0,1,2,\ldots,n-1\}$, and we have $\text{Gal}^{\partial}(K/k) \cong (\mu_n,.)$, the multiplicative group of n-th roots of unity.

Example 4.6. *(See Exercise 1.35(3) on Page 27 of [PS03])*. Let $L = \partial^2 - c^2\text{Id} \in k[\partial]$, where $c \in C^{\times}$. Let $R = C(z)[T, T^{-1}]$, where z and T are indeterminates satisfying $z' = 1$ and $T' = cT$. Then R is a PV ring for the DE $L(y) = 0$ over $k = C(z)$. Indeed, R is ∂-simple. Otherwise, if it contains an ideal $I \neq (0), R$, then let $x = x_0 + x_1 T + \ldots + x_m T^m \in I$, $x \neq 0$, such that $m \geq 0$ is minimal. If $m = 0$ then $1 \in I$ forces the contradiction $I = R$. Hence, $m > 0$, and $x_0 \neq 0$. We may assume that $x_m = 1$ since we can divide x by the leading coefficient to effect this. Then $\partial x - c\,m\,x \in I$ and has lower degree than m, which means it must be zero. The constant term in this expression $x_0' - c\,m\,x_0$ vanishes. However, $\nexists f(z) \in C(z)$ such that $f'(z) = c\,f(z)$, giving a contradiction.

Also, $F = \begin{pmatrix} T & T^{-1} \\ T' & (T^{-1})' \end{pmatrix} \in GL_2(R)$ is a fundamental matrix satisfying $F' = AF$, where $A = \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix} \in \text{Mat}_2(k)$. Indeed, $\det F = -2c \neq 0$.

Finally, $R = C(z)[T, T^{-1}] = k[F_{i,j}, 1/\det F]$ by definition. Hence, we may take $K = \text{Frac}(R)$. Now, for $\sigma \in \text{Gal}^{\partial}(K/k)$, σ is determined by the value of $\sigma(T)$. Observe that

$$\sigma(T)'\sigma(T)^{-1} = \sigma(T'T^{-1}) = \sigma(c) = c = T'T^{-1},$$

so that $\sigma(T)'T - \sigma(T)\,T' = 0$, i.e., $(\sigma(T)/T)' = 0$, so that $\sigma(T)/T = c_\sigma$, for some $c_\sigma \in C^{\times}$. Thus, $\text{Gal}^{\partial}(K/k) \cong (C^{\times},.)$.

4.4 Hypergeometric DE

4.4.1 Particular Cases

The calculation of Gal^{∂} for the hypergeometric DE in the following examples provides some insight into the DEs satisfied by modular forms shown in Examples 2.1, 2.2, 2.3 2.4 and 2.5. The generalised hypergeometric DE appears in (2.10) within Example 2.4 and its monodromy group and ∂-Galois group are given in [BH89].

Example 4.7. *(See Example 5.10 in [PS03])*. Let $k = \mathbb{C}(z)$ and consider the hypergeometric DE

$$y'' + \frac{(a+b+1)z - c}{z(z-1)}y' + \frac{ab}{z(z-1)}y = 0, \tag{4.2}$$

where $a, b, c \in \mathbb{R}$ and each of $c-1$, $a-b$ and $a+b-c$ is not an integer. Let $y_1(z) = {}_2F_1(a, b; c; z)$ and $y_2(z) = z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z)$ be linearly independent solutions, valid for $z \in U$, where U is some open neighbourhood of zero in $\mathbb{C}$; see pp315-321, Section 4.4 of [Ahl79]. Then

$$F = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \in \mathrm{GL}_2(R)$$

is a fundamental matrix for the matrix DE

$$y' = \begin{pmatrix} 0 & 1 \\ -\frac{ab}{z(z-1)} & -\frac{(a+b+1)z-c}{z(z-1)} \end{pmatrix} y,$$

with $\det F = \mathrm{wr}(y_1, y_2) = z^{-c}(1-z)^{c-a-b-1}$. Here, $R = \mathbb{C}(z)[y_1, y_2, z^c(1-z)^{a+b+1-c}]$ is a PV ring for the DE, having a fundamental matrix F.

We obtain the following generators of $\mathrm{Gal}^{\partial}(K/k)$,

$$A^{\sigma_0} = \begin{bmatrix} 1 & 0 \\ 0 & \exp(-2\pi i c) \end{bmatrix} \quad \text{and} \quad A^{\sigma_1} = \begin{bmatrix} B_{1,1} & B_{2,1} \\ B_{1,2} & B_{2,2} \end{bmatrix},$$

where

$$\begin{aligned}
B_{1,1} &= 1 - 2i\exp(\pi i(c-a-b))\frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi c)} \\
B_{1,2} &= -2\pi i\exp(\pi i(c-a-b))\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \\
B_{2,1} &= -2\pi i\exp(\pi i(c-a-b))\frac{\Gamma(2-c)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a+1-c)\Gamma(b+1-c)} \\
B_{2,2} &= 1 + 2i\exp(\pi i(c-a-b))\frac{\sin\pi(c-a)\sin\pi(c-b)}{\sin(\pi c)}.
\end{aligned}$$

Details of the calculations are shown in Appendix B.

Example 4.8. *(See Exercise 5.11 in [PS03]).* Let $k = \mathbb{C}(z)$ and consider the hypergeometric DE

$$y'' + \frac{2z-1}{z(z-1)}y' + \frac{1}{4z(z-1)} = 0,$$

which is the particular case of (4.2) with $a = b = 1/2$ and $c = 1$. Let $y_1(z) = {}_2F_1(1/2, 1/2; 1; z)$. We observe that when $c = 1$, $y_2(z) = z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z) = y_1(z)$ and therefore seek an independent solution of the form $y_2(z) = \log(z)y_1(z) + h(z)$, for some function $h(z)$

analytic in a neighbourhood of zero. Details are supplied in Appendix B, where the generators

$$A^{\sigma_0} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{\sigma_1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

of $\mathrm{Gal}^{\partial}(K/k)$ are computed, from which it is proven that the monodromy group is $\Gamma(2)$. The differential Galois group is then the Zariski closure of $\Gamma(2)$, which is $\mathrm{SL}_2(\mathbb{C})$.

Example 4.9. Let $k = \mathbb{C}(z)$ and consider the hypergeometric DE

$$y'' + \frac{\frac{3}{2}z - 1}{z(z-1)}y' + \frac{5}{144z(z-1)} = 0,$$

which is the particular case of (4.2) with $a = 1/12$, $b = 5/12$ and $c = 1$. Let $y_1(z) = {}_2F_1(1/12, 5/12; 1; z)$. As in Example 4.8, we observe that when $c = 1$, $y_2(z) = z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z) = y_1(z)$ and therefore seek an independent solution of the form $y_2(z) = \log(z)y_1(z) + h(z)$, for some function $h(z)$ analytic in a neighbourhood of zero. As referenced in Appendix B, the monodromy group is $\mathrm{PSL}_2(\mathbb{Z})$ and $\mathrm{Gal}^{\partial}(K/k) \cong \mathrm{PSL}_2(\mathbb{C})$.

4.4.2 Classifying Gal^{∂} with Schwarz Triangles

As described in [Yos97], we restrict our consideration to the case where a, b and c are real numbers such that neither of $1-c$, $c-a-b$ and $a-b$ is a nonzero integer. Since the hypergeometric DE has singularities in the set $\{0, 1, \infty\}$, at around each of the singularities we have the linearly independent sets of solutions

$$y_{0,1}(z) = {}_2F_1(a, b; c; z) \tag{4.3}$$

$$y_{0,2}(z) = \begin{cases} z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z) & \text{, if } c \neq 1; \\ \lim_{c\to 1} \frac{z^{1-c}{}_2F_1(a+1-c,b+1-c;2-c;z) - {}_2F_1(a,b;1;z)}{c-1} & \text{, if } c = 1, \end{cases}$$

$$y_{1,1}(z) = {}_2F_1(a,b;1+a+b-c;1-z) \tag{4.4}$$

$$y_{1,2}(z) = \begin{cases} (1-z)^{c-a-b}{}_2F_1(c-a,c-b;1+c-a-b;1-z) & \text{, if } c-a-b \neq 0; \\ \lim_{c-a-b\to 0} \frac{z^{c-a-b}{}_2F_1(c-a,c-b;1+c-a-b;1-z) - {}_2F_1(a,b;1;1-z)}{c-a-b} & \text{, if } c-a-b = 0, \end{cases}$$

and

$$y_{\infty,1}(z) = \left(-\frac{1}{z}\right)^a {}_2F_1(a,1+a-c;1+a-b;1/z) \tag{4.5}$$

$$y_{\infty,2}(z) = \begin{cases} \left(-\frac{1}{z}\right)^b {}_2F_1(b,1+b-c;1+b-a;1/z) & \text{, if } a-b \neq 0; \\ \lim_{a-b\to 0} \frac{\left(-\frac{1}{z}\right)^b {}_2F_1(b,1+b-c;1+b-a;1/z) - \left(-\frac{1}{z}\right)^a {}_2F_1(a,1+a-c;1;1/z)}{a-b} & \text{, if } a-b = 0, \end{cases}$$

respectively.

Definition 4.2. A *Schwarz map* is a map such as $f_0 : \mathscr{H} \to \mathbb{P}^1$, given by

$$z \mapsto [y_{0,1}(z) : y_{0,2}(z)], \tag{4.6}$$

which is well defined, since the case that $y_{0,1}(z)$ and $y_{0,2}(z)$ are simultaneously zero is precluded by linear independence of these two functions. Similarly, the maps $f_1 : z \mapsto [y_{1,1}(z) : y_{1,2}(z)]$ and $f_\infty : z \mapsto [y_{\infty,1}(z) : y_{\infty,2}(z)]$ are Schwarz maps. Alternatively, the Schwarz map can be defined via

$$z \mapsto y_{0,2}(z)/y_{0,1}(z),$$

where the range is also the upper-half plane.

The Schwarz map

$$f_0(z) = z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z)/{}_2F_1(a,b;c;z)$$

conformally maps the upper-half plane to a triangle in the upper half plane, which has as edges lines or circular arcs. This is to be expected, since the Schwarz maps f_0, f_1 and f_∞ are related by Möbius transformations $\mathrm{PSL}_2(\mathbb{C})$, e.g., $z \mapsto 1-z$ induces the map $y_{0,1} \mapsto y_{1,1}$, and $z \mapsto 1/z$ induces the map $y_{0,1} \mapsto y_{\infty,1}$.

We use the following notation

$$\lambda = |1-c|, \quad \nu = |c-a-b|, \quad \mu = |a-b| < 1$$

which corresponds to the angle sizes $\pi\lambda$, $\pi\mu$ and $\pi\nu$ in the target triangle. This allows us to categorise the target triangle as being either a spherical triangle, a Euclidean triangle or a hyperbolic triangle, corresponding to the sign of the $A = \pi(\lambda + \mu + \nu - 1)$ being either 1, 0 or -1.

A key fact concerning Schwarz maps is that when the target triangle is non-overlapping, the inverse of the Schwarz map is a modular function for the target triangle's triangle group. Thus, the Schwarz map connects the monodromy group of the hypergeometric DE with the subgroup Γ for the modular form equal to the inverse of the Schwarz map. It is this fact that allows us to infer the monodromy group for the hypergeometric DE when we know the subgroup Γ for the modular form.

4.4.3 Spherical Triangles

Here we have the case that $A = \pi(\lambda + \mu + \nu - 1) > 0$, where A denotes the area of a spherical triangle, on a sphere of radius one, with angles $\pi\lambda$, $\pi\mu$ and $\pi\nu$.

To permit the target triangle to tessellate, it is necessary for the parameters λ, ν and μ to be rational numbers, and this leads us to the following definition.

Definition 4.3. A *Schwarz triangle* is a triangle in $\mathbb{P}^1$ bounded by circular arcs, resulting from the image of the three arcs $(-\infty, 0) \bigcup_{\ell=1} (0,1) \bigcup (1,\infty)$ under the Schwarz map f_0 given in (4.6).

We have the following lemma.

Proposition 4.4. *(See Proposition 6.1 on Page 68 of [Yos97].) If*

$$\lambda = |1-c|, \quad \nu = |c-a-b|, \quad \mu = |a-b| < 1,$$

then the Schwarz map f_0 is a bijection from $\mathcal{H}$ to a Schwarz triangle having the angles $\pi\lambda$, $\pi\nu$ and $\pi\mu$ at the points $f_0(0)$, $f_0(1)$ and $f_0(\infty)$, respectively.

In [Sch73], the triples (λ, μ, ν) giving rise to finite ∂-Galois groups are supplied, which are shown in Table 4.1. These correspond to triples of rational numbers (λ, μ, ν) where the target triangles are permitted to overlap.

No.	λ	μ	ν	$\frac{\text{Area}}{\pi}$	Polyhedron
I.	$\frac{1}{2}$	$\frac{1}{2}$	ν	ν	Regular double pyramid
II.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}=A$	Tetrahedron
III.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}=2A$	
IV.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{12}=B$	Cube and octahedron
V.	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}=2B$	
VI.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{30}=C$	Dodecahedron and Icosahedron
VII.	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{15}=2C$	
VIII.	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{15}=2C$	
IX.	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{10}=3C$	
X.	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{15}=4C$	
XI.	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}=6C$	
XII.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{5}=6C$	
XIII.	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}=6C$	
XIV.	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{7}{30}=7C$	
XV.	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}=10C$	

TABLE 4.1: Schwarz's table of triples (λ,μ,ν) giving rise to finite differential Galois groups.

p	q	r
2	2	n
2	3	3
2	3	4
2	3	5

TABLE 4.2: Table showing the possible values of p, q and r giving rise to tessellations of the Sphere plane by triangles.

A subset of Schwarz triples where the triangles tessellate are categorised by triples of positive integers (p,q,r) satisfying

$$\lambda = |1-c| = 1/p, \quad \nu = |c-a-b| = 1/q, \quad \mu = |a-b| = 1/r.$$

These are shown in Table 4.2.

Remark 4.5. Because the triangle groups on the sphere are finite, the monodromy group and differential Galois group of the corresponding hypergeometric DE are finite. As we have proven in Proposition 3.24, finite ∂-Galois groups correspond to finite Galois groups, and hence the solutions to the hypergeometric DE, for these cases, are algebraic functions.

p	q	r
2	2	∞
2	3	6
2	4	4
3	3	3

TABLE 4.3: Table showing the possible values of p, q and r giving rise to tessellations of the Euclidean plane by triangles.

p	q	r	Monodromy Group
∞	3	2	$PSL_2(\mathbb{Z})$
∞	∞	∞	$\Gamma(2)$

TABLE 4.4: Table showing some of the possible values of p, q and r giving rise to tessellations of hyperbolic space by triangles.

As an example, the particular case $(p,q,r) = (2,2,3)$ corresponds to $\lambda = |1-c| = 1/2$, $\nu = |c-b-a| = 1/2$ and $\mu = |a-b| = 1/3$ in Row I of Table 4.1, which gives $(a,b,c) = (1/3,2/3,3/2)$. We have

$$ {}_2F_1(1/3,2/3;3/2;-27z^2/4) = \frac{1}{z\sqrt{3}}\left(\sqrt[3]{(3z\sqrt{3}+\sqrt{27z^2+4})/2} - \sqrt[3]{2/(3z\sqrt{3}+\sqrt{27z^2+4})}\right). $$

As another example, the particular case $(p,q,r) = (2,2,3)$ corresponds to $\lambda = |1-c| = 1/3$, $\nu = |c-b-a| = 1/2$ and $\mu = |a-b| = 1/6$ in Row I of Table 4.1, which gives $(a,b,c) = (1/2,1/3,1/3)$. We have ${}_2F_1(1/2,2/3;2/3;z) = 1/\sqrt{1-z}$.

4.4.4 Euclidean Triangles

Here we have the case that $\lambda+\mu+\nu = 1$. All possible cases where the triangles tessellate are shown in Table 4.3. These correspond to the crystallographic groups; e.g., see Section 8.2, p75 of [Yos97].

4.4.5 Hyperbolic Triangles

Here we have the case that $A = \pi(1-(\lambda+\mu+\nu)) > 0$, where A denotes the area of a hyperbolic triangle, on a hyperboloid of one sheet, with angles $\pi\lambda$, $\pi\mu$ and $\pi\nu$. There are infinitely many solutions to λ, μ, ν in reciprocals of positive integers, so we show only some in Table 4.4.

Remark 4.6. The particular case $(p,q,r) = (\infty,\infty,\infty)$ in Table 4.4 corresponds to $|1-c| = 0$, $|c-b-a| = 0$ and $|a-b| = 0$, which gives $(a,b,c) = (1/2,1/2,1)$. The map $z \mapsto (z-i)/(z+i)$ sends the upper-half plane to the Poincaré disk. The Schwarz maps sends the upper-half plane to a hyperbolic triangle in the upper-half plane, corresponding to the fundamental domain F shown in Figure 2.2. This shows that the monodromy group of the DE in Example 4.8 is $\Gamma(2)$, as given in Proposition 8.1 on p78 of [Yos97]

Furthermore, the particular case $(p,q,r) = (\infty,2,3)$ in Table 4.4 corresponds to $|1-c| = 0$, $|c-b-a| = 1/2$ and $|a-b| = 1/3$, which gives $(a,b,c) = (1/12,5/12,1)$. Once again, we see that the fundamental domain F shown in Figure 2.1 is a hyperbolic triangle, showing that the monodromy group of the DE in Example 4.9 is $\mathrm{PSL}_2(\mathbb{Z})$, as given in Proposition 8.2 on p78 of [Yos97].

Chapter 5

Differential Equations Satisfied by Modular Forms

In our final chapter we give the proof of Proposition 1.1, where we we exhibit the differential equation satisfied by a modular form as a differential equation involving Φ, as given in (1.1). For some examples, we give Φ explicitly, which partly addresses our first research question, as stated in Chapter 1.

An important component of the proof requires the following construction of higher weight modular forms, as given on p53 in Section 5.2 of [Zag08].

Definition 5.1. Let $f(z)$ and $g(z)$ be modular forms of weights k and ℓ respectively. Then their n-th Rankin-Cohen bracket is defined as

$$[f,g]_n = \frac{1}{(2\pi i)^n} \sum_{r+s=n} (-1)^r \binom{k+n-1}{s} \binom{\ell+n-1}{r} \frac{d^r f}{dz^r} \frac{d^s g}{dz^s},$$

which is a modular form of weight $k+\ell+2n$.

Proof. (of Proposition 1.1, the third proof supplied on p62 of [Zag08]) Consider the case $k=1$. Then $t' = D(t)$ is a meromorphic modular form of weight 2. Indeed, since $t(\gamma z) = t(z)$, for $\gamma \in \Gamma$, we have $\frac{d}{dz} t(\gamma z) = \frac{d}{dz} t(z)$, i.e., $t'(\gamma z) \frac{d}{dz} \gamma z = t'(z)$, which simplifies to $t'(\gamma z) = (cz+d)^2 t'(z)$. Observe that the Rankin-Cohen brackets $[f,t']_1$ and $[f,f]_2$ are modular forms of weights 5 and 6, respectively. Therefore, the quotients

$$A = \frac{[f,t']_1}{f(t')^2} = \frac{f t'' - 2f' t'}{f(t')^2}$$

and

$$B = -\frac{[f,f]_2}{2f^2(t')^2} = -\frac{2f f'' - 4(f')^2}{2f^2(t')^2}$$

have weight zero. Observe that

$$\frac{df}{dt} = \left(\frac{dt}{dz}\right)^{-1} \frac{df}{dz} = \frac{f'}{t'}$$

and

$$\frac{d^2f}{dt^2} = \frac{1}{t'}\frac{d}{dz}\left(\frac{f'}{t'}\right) = \frac{1}{t'}\left(-\frac{t''}{(t')^2}f' + \frac{1}{t'}f''\right) = \frac{f''t' - f't''}{(t')^3}.$$

We have

$$\begin{aligned}
&\frac{d^2f}{dt^2} + A\frac{df}{dt} + Bf \\
&= \frac{f''t' - f't''}{(t')^3} + \frac{f t'' - 2f'(t')^2}{f(t')^2}\frac{f'}{t'} - \frac{2ff'' - 4(f')^2}{2f^2(t')^2}f \\
&= \frac{f''}{(t')^2} - \frac{f't''}{(t')^3} + \frac{t''}{(t')^2}\frac{f'}{t'} - \frac{2f'}{f t'}\frac{f'}{t'} - \frac{f''}{(t')^2} + \frac{2(f')^2}{f(t')^2} \\
&= \frac{f''}{(t')^2} - \frac{f't''}{(t')^3} + \frac{f' t''}{(t')^3} - \frac{2(f')^2}{f(t')^2} - \frac{f''}{(t')^2} + \frac{2(f')^2}{f(t')^2} \\
&= 0.
\end{aligned}$$

We have, upon letting $f(z) = \Phi(t(z))$, $A(z) = a(t(z))$ and $B(z) = b(t(z))$, that Φ satisfies

$$\Phi'' + a(t)\Phi' + b(t)\Phi = 0,$$

namely a second-order DE.

For the case when f has weight $k > 0$, we let $h = f^{1/k}$. Even though the function h is not a modular form, since it is not single-valued around zeroes of f in $\mathscr{H}$, the quotients

$$A = \frac{[h,t']_1}{h\,t'} = \frac{[f,t']_1}{k f t'} = \frac{f t'' - 2f' t'}{k f t'}$$

and

$$B = \frac{[h,h]_2}{2h^2(t')^2} = \frac{[f,f]_2}{k^2(k+1)f^2(t')^2} = 4\frac{f f'' - (f')^2}{k^2(k+1)f^2(t')^2}$$

are well defined. As before, but with h in place of f, we have

$$\frac{d^2h}{dt^2} + A\frac{dh}{dt} + Bh = 0,$$

Example	Φ	DE	Monodromy Group	Gal^{∂}
2.1	${}_2F_1(1/2,1/2;1;z)$	$z(z-1)\Phi''+(2z-1)\Phi'+\frac{1}{4}\Phi=0$	$\Gamma(2)$	$\mathrm{SL}_2(\mathbb{C})$
2.2	$({}_2F_1(1/12,5/12;1;z))^4$	$(z^4-3456z^5+2985984z^6)\Phi^{(5)}$ $+10(z^3-4320z^4+4478976z^5)\Phi^{(4)}$ $+5(5z^2-29616z^3+38486016z^4)\Phi^{(3)}$ $+15(z-9864z^2+17252352z^3)\Phi''$ $+(1-30000z+81100800z^2)\Phi'$ $+240(-1+6144z)\Phi=0$		
2.3	${}_2F_1(1/12,5/12;1;z)$	$z(z-1)\Phi''+(3z/2-1)\Phi'+\frac{5}{144}\Phi=0$	$\mathrm{PSL}_2(\mathbb{Z})$	$\mathrm{PSL}_2(\mathbb{C})$
2.4	$({}_2F_1(1/12,5/12;1;z))^2$ $={}_3F_2(1/6,1/2,5/6;1,1;z)$	$z^2(z-1)\Phi^{(3)}+\frac{9}{2}z\left(t-\frac{2}{3}\right)\Phi''$ $+\left(\frac{113}{36}z-1\right)\Phi'+\frac{5}{72}\Phi=0$		$\mathrm{SL}_3(\mathbb{C})$
2.5		$4z(z^2-34z+1)\Phi''$ $+4(2z^2-51z+1)\Phi'+(z-10)\Phi=0$	$\Gamma_1(6)$	$\mathrm{SL}_2(\mathbb{C})$

TABLE 5.1: Shown is Φ, the DE satisfied by Φ, the monodromy group and the ∂-Galois group, in respect of each of the examples in Chapter 2.

leading to $L(\Phi^{1/k})=0$, where L is the differential operator

$$L=\partial^2+a(t)\partial+b(t).$$

Hence, Φ satisfies the DE $\mathrm{Sym}^k(L)(\Phi)=0$, where $\mathrm{Sym}^k(L)$ is the $(k+1)$-st order differential operator whose solutions are k-th powers or k-fold products of solutions of the second-order differential operator L. The expression for $\mathrm{Sym}^k(L)$ is given in Lemma A.1. □

As mentioned previously, on pp61-61 in [Zag18], three proofs are supplied and the first of these also provides an explicit DE satisfied by a given modular form, which necessarily agrees with that given above.

A summary of information pertaining to Φ, the DEs and the associated ∂-Galois groups, in respect of Examples 2.1 - 2.5 given in Chapter 2, is shown in Table 5.1. Some of the entries have been left blank as further work is needed to determine these. Comparing with the cases given in Theorem 3.40, we have confirmation with our computations where $\mathrm{Gal}^{\partial}(K/k)=\mathrm{SL}_2(\mathbb{C})$. As mentioned at the end of Chapter III in [Yos97], is that under certain circumstances the monodromy group of the hypergeometric DE is congruent to the group associated to the modular form that results from taking the inverse of the Schwarz map. This fact is used to infer monodromy groups in Examples 2.3 and 2.5. In Example 2.1, i.e., for the case ${}_2F_1(1/2,1/2;1;z)$, the monodromy group is computed explicitly in Appendix B. In Example 2.4, using Case II of Theorem 6.5 in [BH89], we have $\mathrm{Gal}^{\partial}\cong\mathrm{SL}_3(\mathbb{C})$.

Appendix A

Miscellaneous Theorems

A.1 Differential Equation Satisfied by n-th Powers

We state the following lemma and give an independent proof. An alternative proof is supplied in the proof of Proposition 4.26 on pp131-132 of [PS03].

Lemma A.1. *Let* (k,∂) *be a differential field and let* $L=\partial^2+a\partial+b$ *be a second-order differential operator, with* $a,b\in k$*. Then, for* $n\in\mathbb{Z}^+$*,* $\mathrm{Sym}^n(L)$ *is an* $(n+1)$*-st order differential operator over* k*.*

Proof. The case $n=2$ is proven as follows. Let C be the field of constants of k and let y_1, y_2 be C-linearly independent solutions of $Ly=0$, lying in some PV extension K of k. Let $z=y_i y_j$, where $i,j\in\{1,2\}$. Then we have

$$\begin{aligned}
z' &= y_i' y_j + y_i y_j' \\
z'' &= y_i'' y_j + 2y_i' y_j' + y_i y_j'' \\
&= (-a y_i' - b y_i)\, y_j + 2y_i' y_j' + y_i\,(-a y_j' - b y_j) \\
&= -a z' - 2b z + 2y_i' y_j' \\
z^{(3)} &= -a' z' - a z'' - 2b' z - 2b z' + 2y_i'' y_j' + 2y_i' y_j'' \\
&= -a' z' - a z'' - 2b' z - 2b z' - 4a y_i' y_j' - 2b z' \\
&= -(a' + 4b)\, z' - a z'' - 2b' z - 4a y_i' y_j',
\end{aligned}$$

so that

$$z^{(3)} + 2a\,z'' = -(2a^2 + a' + 4b)\,z' - a\,z'' - 2(2a\,b + b')\,z.$$

Hence,

$$\mathrm{Sym}^2(L) = \partial^3 + 3a\partial^2 + (2a^2 + a' + 4b)\,\partial + 2(2a\,b + b').$$

The case for arbitrary n is proven by considering the $(n+1)$-st order DE

$$M(y) := \frac{\mathrm{wr}(y, y_1^n, y_1^{n-1}y_2, y_1^{n-2}y_2^2, \ldots, y_2^n)}{\mathrm{wr}(y_1^n, y_1^{n-1}y_2, y_1^{n-2}y_2^2, \ldots, y_2^n)} = 0.$$

Observe that $M(y_1^i y_2^{n-i}) = 0$, for $i \in \{0, 1, 2, \ldots, n\}$, so that all n-fold products of y_1 and y_2 are solutions of the DE. Furthermore, observe that, since $\mathrm{Gal}^{\partial}(K/k) \cong \mathrm{GL}_2(C)$ is generated as a C-algebra by the matrices

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\},$$

each of which leaves M invariant, we have $M \in k[\partial]$. □

A.2 Clausen's Hypergeometric Identity

In [Cla28] the following identity is supplied, which we independently prove using the Picard-Lindelöf theorem, a theorem stating the uniqueness of a solution satisfying initial conditions.

Proposition A.2.

$${}_2F_1(a, b; a + b + \frac{1}{2}; z)^2 = {}_3F_2(2a, 2b, a + b; a + b + \frac{1}{2}, 2a + 2b; z).$$

Proof. We know that ${}_2F_1(a, b; c; z)$ is a solution to the second-order DE

$$z(z-1)\,y'' + ((a + b + 1)z - c)\,y' + a\,b\,y = 0,$$

which we rewrite in the form

$$y'' + A(z)\,y' + B(z)\,y = 0,$$

where $A(z) = \frac{(a+b+1)z-c}{z(z-1)}$ and $B(z) = \frac{ab}{z(z-1)}$. Also, ${}_3F_2(p,q,r;s,t;z)$ is a solution to the third-order DE

$$z^2(z-1)\,y^{(3)} + ((p+q+r+3)z-(s+t+1))\,z\,y'' + ((pq+pr+qr+p+q+r+1)z-st)\,y' + pqr\,y = 0, \tag{A.1}$$

which we rewrite in the form

$$y^{(3)} + P(z)\,y'' + Q(z)\,y' + R(z)\,y = 0,$$

where $P(z) = \frac{(p+q+r+3)z-(s+t+1)}{z(z-1)}$, $Q(z) = \frac{(pq+pr+qr+p+q+r+1)z-st}{z^2(z-1)}$ and $R(z) = \frac{pqr}{z^2(z-1)}$. As a corollary of Lemma A.1, we have that $({}_2F_1(a,b;c;z))^2$ satisfies the third-order DE

$$\begin{aligned} &\mathrm{Sym}^2(\partial^2 + A(z)\,\partial + B(z))\,y \\ &= y^{(3)} + 3A(z)\,y'' + (2A(z)^2 + A'(z) + 4B(z))\,y' + 2(2A(z)\,B(z) + B'(z)) = 0. \end{aligned} \tag{A.2}$$

We determine relations among a, b, c, p, q, r, s and t such that (A.2) agrees with (A.1). This will demonstrate that $({}_2F_1(a,b;c;z))^2$ and ${}_3F_2(p,q,r;s,t;z)$ lie in the solution space of the same DE. Then showing that these two expressions satisfy the same initial conditions will prove equality, by virtue of the Picard-Lindel"of theorem.

Equating coefficients of like orders of derivatives in (A.1) and (A.2) gives

$$\begin{aligned} P(z) &= 3A(z) \\ Q(z) &= 2A(z)^2 + A'(z) + 4B(z) \\ R(z) &= 4A(z)\,B(z) + 2B'(z), \end{aligned} \tag{A.3}$$

leading to the relations

$$\begin{aligned} p+q+r+3 &= 3(a+b+1) \\ s+t+1 &= 3c \\ 0 &= \{2(a+b+1-c)-1\}\,(a+b+1-c) \\ pq+qr+pr+p+q+r+1 &= 4(ab+bc+ca)+3c-2c^2 \end{aligned} \tag{A.4}$$

$$st = 2c^2 - c$$
$$0 = ab\{2(a+b+1-c)-1\}$$
$$pqr = 2ab(2c-1).$$

We have the two cases $a+b-c+1/2=0$ and $a+b+1-c=0$ arising from the third equation in (A.4).

Case: $a+b-c+1/2=0$

The relations in (A.4) simplify to

$$\begin{aligned} p+q+r &= 3(a+b) \\ pq+qr+pr &= 2a^2+2b^2+8ab \\ pqr &= 4ab(a+b) \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} s+t &= 3c-1 \\ st &= 2c^2-c. \end{aligned} \tag{A.6}$$

Thus, p, q and r are the roots of the cubic equation

$$x^3 - 3(a+b)x^2 + (2a^2+2b^2+8ab)x - 4ab(a+b) = 0,$$

which factorises as

$$(x-2a)(x-2b)(x-a-b) = 0,$$

so that $p, q, r \in \{2a, 2b, a+b\}$. Similarly, s and t are the roots of the quadratic equation

$$x^2 - (3c-1)x + (2c^2-c) = 0,$$

which factorises as

$$(x-(2c-1))(x-c) = 0,$$

so that $s, t \in \{c, 2c-1\}$.

To show that

$${}_2F_1(a,b;c;z)^2 = {}_3F_2(p,q,r;s,t;z),$$

when $c = a+b+1/2$, $p,q,r \in \{2a,2b,a+b\}$ and $s,t \in \{c,2c-1\}$, we check that the LHS and RHS, and corresponding first and second-order derivatives, agree at $z=0$. For the LHS, we have ${}_2F_1(a,b;c;0)^2 = 1$,

$$\frac{d}{dz}\left({}_2F_1(a,b;c;z)^2\right)\Big|_{z=0} = 2\,{}_2F_1(a,b;c;0)\,{}_2F_1'(a,b;c;0) = 2\frac{ab}{c} = \frac{2ab}{a+b+1/2}$$

and

$$\begin{aligned}
\frac{d^2}{dz^2}\left({}_2F_1(a,b;c;z)^2\right)\Big|_{z=0} &= 2\,{}_2F_1(a,b;c;0)\,{}_2F_1''(a,b;c;0) + 2\left({}_2F_1'(a,b;c;0)\right)^2 \\
&= 2a(a+1)b(b+1)/\{c(c+1)\} + 2\{ab/c\}^2 \\
&= 2ab\frac{(a+1)(b+1)c + ab(c+1)}{c^2(c+1)} \\
&= \frac{2ab}{c^2(c+1)}\left[(a+1)(b+1)(c+1/2) - \frac{1}{2}(a+1)(b+1) + ab(c+1/2) + \frac{1}{2}ab\right] \\
&= \frac{2ab}{c^2(c+1)}\left[(c+1/2)(2ab+a+b+1) - \frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}\right] \\
&= \frac{2ab}{c^2(c+1)}(c+1/2)(a+1/2)(2b+1) \\
&= \frac{ab(a+b+1)(2a+1)(2b+1)}{c^2(c+1)}.
\end{aligned}$$

For the RHS, we have ${}_3F_2(p,q,r;s,t;0) = 1$,

$$\begin{aligned}
\frac{d}{dz}\left({}_3F_2(p,q,r;s,t;z)\right)\Big|_{z=0} &= pqr/(st) \\
&= \frac{4ab(a+b)}{c(2c-1)} \\
&= \frac{2ab}{a+b+1/2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2}{dz^2}\left({}_3F_2(p,q,r;s,t;z)\right)\Big|_{z=0} &= p(p+1)q(q+1)r(r+1)/(s(s+1)t(t+1)) \\
&= \frac{2a(2a+1)2b(2b+1)(a+b)(a+b+1)}{c(c+1)(2c-1)2c} \\
&= \frac{ab(a+b+1)(2a+1)(2b+1)}{c^2(c+1)}.
\end{aligned}$$

This proves our theorem.

Case: $a+b+1-c=0$

The fifth equation (A.4) forces $ab=0$. It is clear that we must have a trivial solution because in this case ${}_2F_1(a,b;c;z)=0$. Without loss of generality, let $a=0$, so that $c=b+1$. Then the relations in (A.4) simplify to

$$\begin{aligned} p+q+r &= 3b \\ pq+qr+pr &= 2b^2 \\ pqr &= 0 \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} s+t &= 3b+2 \\ st &= 2b^2+3b+1. \end{aligned} \tag{A.8}$$

Thus, p, q and r are the roots of the cubic equation $x^3-3bx^2+2b^2x=0$, which factorises as $x(x-b)(x-2b)=0$, so that $p,q,r\in\{0,b,2b\}$. Similarly, s and t are the roots of the quadratic equation $x^2-(3b+2)x+(2b^2+3b+1)=0$, which factorises as $(x-(2b+1))(x-(b+1))=0$, so that $s,t\in\{b+1,2b+1\}$.

Now, $a=0$ implies that ${}_2F_1(a,b;c;z)=0$ and, since one of p,q,r is zero, ${}_3F_1(p,q,r;s,t;z)=0$. □

Appendix B

Monodromy Group of the Hypergeometric DE

B.1 The Case $c \notin \mathbb{Z}$

Let

$$y_1 = y_{0,1} = {}_2F_1(a, b; c; z) \tag{B.1}$$

and

$$y_2 = y_{0,2} = z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z) \tag{B.2}$$

be the analytic solutions to the hypergeometric DE (2.9) around $z = 0$. We see that the solution (B.1) is valid when $c \notin \{0, -1, -2, \ldots\}$ and that the second solution (B.2) is valid when $2 - c \notin \{0, -1, -2, \ldots\}$. Furthermore, when $c = 1$ the two solutions y_1 and y_2 are identical and their Wronskian $\mathrm{wr}(y_1, y_2) = z^{-c}(1-z)^{c-a-b-1}$ vanishes. Hence, when $c \notin \mathbb{Z}$ the two solutions are linearly independent and well defined in a neighbourhood of $z = 0$.

In this section we explicitly compute the monodromy group of (2.9) when $c \notin \mathbb{Z}$. Generators of the group are given in in Chapter 5 of [PS03], and most of the details of the calculation are shown in Chapter 2 of [Bat53]. However, there is a discrepancy in sign of one of the matrix entries, so we deliberately work through all details here.

Let $U \subseteq \mathbb{P}^1$ be an open pathwise-connected subset containing the singularities 0 and 1 of the hypergeometric DE. For each $p \in \{0, 1\}$, let $V_p = \mathrm{Span}_{\mathbb{C}}\{y_{p,1}(z), y_{p,2}(z)\}$ denote the solution

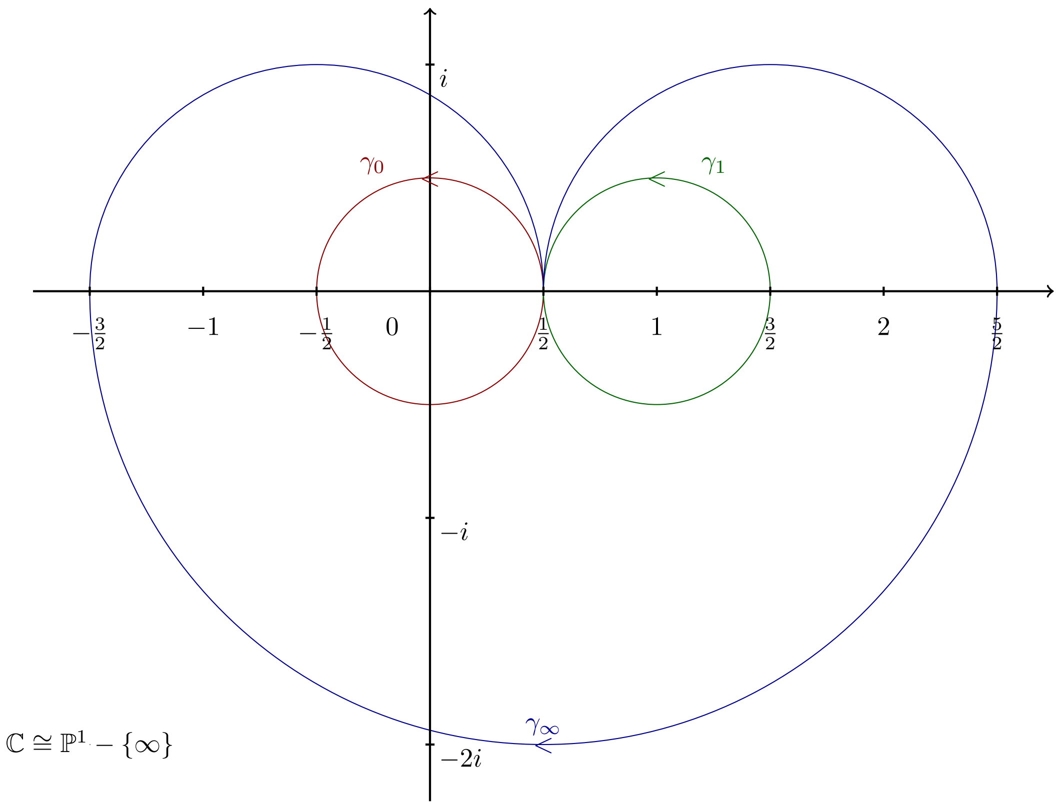

FIGURE B.1: Loops γ_0, γ_1 and γ_∞ which generate the fundamental group of $\mathbb{P}^1 - \{0,1,\infty\}$ with base point $1/2$.

space of the DE around p. Let $R = \mathbb{C}[y_{0,1}(z), y_{0,2}(z)]$ be a PV ring and $K = \mathrm{Frac}(R)$ be the corresponding PV field extension. Let $M(\gamma_p) : V_p \to V_p$ denote the induced isomorphism resulting from analytically continuing a solution of the DE near p around the loop and returning to p. Hence, the monodromy representation $M : \pi_1(U, 1/2) \to \mathrm{GL}_2(V_p)$ maps the homotopy class of a loop γ_p, based at $1/2$, to a linear automorphism of V_p.

In particular, we know that $\mathrm{Gal}^\partial(K/k) =< \sigma_0, \sigma_1 >$, where $\sigma_p = M(\gamma_p)$ and $\gamma_p : [0,1] \to \mathbb{C}$, for $p \in \{0,1,\infty\}$, are loops defined by $\gamma_0(\theta) = \frac{1}{2}\exp(2\pi i\theta)$, $\gamma_1(\theta) = 1 - \frac{1}{2}\exp(2\pi i\theta)$ and

$$\gamma_\infty(\theta) = \begin{cases} 3/2 - \exp(4\pi i\theta) & \text{, if} 0 \le \theta \le 1/4; \\ 1/2 + 2\exp(-2\pi i(\theta - 1/4)) & \text{, if} 1/4 \le \theta \le 3/4; \\ -1/2 - \exp(-4\pi i(\theta - 3/4)) & \text{, if} 3/4 \le \theta \le 1, \end{cases}$$

as shown in Figure B.1. Each loop encircles a singularity of the DE. The third singularity, i.e., at ∞, is encircled by $\gamma_\infty = (\gamma_1 \circ \gamma_0)^{-1}$. We know that $\pi_1(\mathbb{P}^1 - \{0,1,\infty\}) \cong \mathbb{Z} * \mathbb{Z}$, the free group on two generators. We now compute σ_0 and σ_1.

so that a fundamental matrix for the DE is $F = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$. We have

$$\sigma_0(y_1)(z) = y_1(\exp(2\pi i)z) = {}_2F_1(a,b;c;\exp(2\pi i)z) = {}_2F_1(a,b;c;z) = y_1(z).$$

$$\sigma_0(y_2)(z) = y_2(\exp(2\pi i)z) = \{\exp(2\pi i)z\}^{1-c}{}_2F_1(a+1-c,b+1-c;2-c;\exp(2\pi i)z) = \exp(2\pi i(1-c))\,y_2(z).$$

Hence,

$$\sigma_0(F) = F\begin{bmatrix} 1 & 0 \\ 0 & \exp(-2\pi i c) \end{bmatrix},$$

i.e.,

$$A^{\sigma_0} = \begin{bmatrix} 1 & 0 \\ 0 & \exp(-2\pi i c) \end{bmatrix}.$$

Next, we have

$$\sigma_1(y_1)(z) = y_1(1-\exp(2\pi i)(1-z)) = {}_2F_1(a,b;c;1-\exp(2\pi i)(1-z)). \tag{B.3}$$

Now, with

$$\lambda_{1,1} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{and} \quad \lambda_{1,2} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \tag{B.4}$$

we have

$$\begin{aligned} y_1(z) &= {}_2F_1(a,b;c;z) \\ &= \lambda_{1,1}\,{}_2F_1(a+1-c,b+1-c;a+b+1-c;1-z) \\ &+ \lambda_{1,2}\,(1-z)^{c-a-b}{}_2F_1(c-a,c-b;c-a-b+1;1-z), \end{aligned} \tag{B.5}$$

so that (B.3) becomes

$$\begin{aligned} \sigma_1(y_1)(z) &= \lambda_{1,1}\,{}_2F_1(a+1-c,b+1-c;a+b+1-c;\exp(2\pi i)(1-z)) \\ &+ \lambda_{1,2}\exp(2\pi i(c-a-b))(1-z)^{c-a-b}{}_2F_1(c-a,c-b;c-a-b+1;\exp(2\pi i)(1-z)) \\ &= \lambda_{1,1}\,{}_2F_1(a+1-c,b+1-c;a+b+1-c;1-z) \\ &+ \lambda_{1,2}\exp(2\pi i(c-a-b))(1-z)^{c-a-b}{}_2F_1(c-a,c-b;c-a-b+1;1-z). \end{aligned}$$

We now compute $\sigma_1(y_2)$. Using (B.5) we have

$$\begin{aligned}{}_2F_1(a+1-c,b+1-c;2-c;z) = \lambda_{2,1}\,{}_2F_1(a+1-c,b+1-c;a+b+1-c;1-z)\\ +\lambda_{2,2}\,(1-z)^{c-a-b}{}_2F_1(1-a,1-b;c-a-b+1;1-z),\end{aligned}$$

where

$$\lambda_{2,1} = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \quad \text{and} \quad \lambda_{2,2} = \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}. \tag{B.6}$$

Hence,

$$\begin{aligned}y_2(z) = z^{1-c}\lambda_{2,1}\,{}_2F_1(a+1-c,b+1-c;a+b+1-c;1-z)\\ +z^{1-c}\lambda_{2,2}\,(1-z)^{c-a-b}{}_2F_1(1-a,1-b;c-a-b+1;1-z).\end{aligned} \tag{B.7}$$

Euler's transformation is

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z),$$

so that

$${}_2F_1(a+1-c,b+1-c;a+b+1-c;1-z) = z^{c-1}{}_2F_1(b,a;c-a-b+1;1-z)$$

and

$${}_2F_1(1-a,1-b;c-a-b+1;1-z) = z^{c-1}{}_2F_1(c-b,c-a;c-a-b+1;1-z),$$

and the RHS of (B.7) simplifies to

$$y_2(z) = \lambda_{2,1}\,{}_2F_1(b,a;c-a-b+1;1-z)+\lambda_{2,2}\,(1-z)^{c-a-b}{}_2F_1(c-b,c-a;c-a-b+1;1-z). \tag{B.8}$$

Writing (B.5) and (B.8) in matrix form gives

$$\begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} \\ \lambda_{2,1} & \lambda_{2,2} \end{bmatrix} \begin{pmatrix} {}_2F_1(a,b;c-a-b+1;1-z) \\ (1-z)^{c-a-b}{}_2F_1(c-b,c-a;c-a-b+1;1-z) \end{pmatrix}.$$

Linear combinations of (B.5) and (B.8) give the relations

$$\lambda_{2,2}y_1(z) - \lambda_{1,2}y_2(z) = (\lambda_{2,2}\lambda_{1,1} - \lambda_{1,2}\lambda_{2,1}){}_2F_1(a,b;c-a-b+1;1-z)$$

and

$$\lambda_{2,1}y_1(z)-\lambda_{1,1}y_2(z)=(\lambda_{2,1}\lambda_{1,2}-\lambda_{1,1}\lambda_{2,2})(1-z)^{c-a-b}{}_2F_1(c-b,c-a;c-a-b+1;1-z),$$

which can be written in matrix form as

$$\frac{1}{\lambda_{1,1}\lambda_{2,2}-\lambda_{1,2}\lambda_{2,1}}\begin{bmatrix}\lambda_{2,2} & -\lambda_{1,2}\\ \lambda_{2,1} & -\lambda_{1,1}\end{bmatrix}\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}=\begin{pmatrix}{}_2F_1(a,b;c-a-b+1;1-z)\\ (1-z)^{c-a-b}{}_2F_1(c-b,c-a;c-a-b+1;1-z)\end{pmatrix}.$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{\lambda_{1,1}\lambda_{2,2}-\lambda_{1,2}\lambda_{2,1}}\begin{bmatrix}\lambda_{2,2} & -\lambda_{1,2}\\ \lambda_{2,1} & -\lambda_{1,1}\end{bmatrix}\sigma_1\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}\\
&=\begin{pmatrix}{}_2F_1(a,b;c-a-b+1;\exp(2\pi i)(1-z))\\ \exp(2\pi i(c-a-b))(1-z)^{c-a-b}{}_2F_1(c-b,c-a;c-a-b+1;\exp(2\pi i)(1-z))\end{pmatrix}\\
&=\begin{bmatrix}1 & 0\\ 0 & \exp(2\pi i(c-a-b))\end{bmatrix}\begin{pmatrix}{}_2F_1(a,b;c-a-b+1;1-z)\\ (1-z)^{c-a-b}{}_2F_1(c-b,c-a;c-a-b+1;1-z)\end{pmatrix}\\
&=\frac{1}{\lambda_{1,1}\lambda_{2,2}-\lambda_{1,2}\lambda_{2,1}}\begin{bmatrix}1 & 0\\ 0 & \exp(2\pi i(c-a-b))\end{bmatrix}\begin{bmatrix}\lambda_{2,2} & -\lambda_{1,2}\\ \lambda_{2,1} & -\lambda_{1,1}\end{bmatrix}\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}.
\end{aligned}$$

Hence, with $\epsilon=\exp(2\pi i(c-a-b))$,

$$\begin{aligned}
&\sigma_1\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}\\
&=\begin{bmatrix}\lambda_{2,2} & -\lambda_{1,2}\\ \lambda_{2,1} & -\lambda_{1,1}\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\ 0 & \epsilon\end{bmatrix}\begin{bmatrix}\lambda_{2,2} & -\lambda_{1,2}\\ \lambda_{2,1} & -\lambda_{1,1}\end{bmatrix}\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}\\
&=\frac{1}{-\lambda_{1,1}\lambda_{2,2}+\lambda_{1,2}\lambda_{2,1}}\begin{bmatrix}-\lambda_{1,1} & \lambda_{1,2}\\ -\lambda_{2,1} & \lambda_{2,2}\end{bmatrix}\begin{bmatrix}\lambda_{2,2} & -\lambda_{1,2}\\ \epsilon\lambda_{2,1} & -\epsilon\lambda_{1,1}\end{bmatrix}\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}\\
&=\frac{1}{-\lambda_{1,1}\lambda_{2,2}+\lambda_{1,2}\lambda_{2,1}}\begin{bmatrix}-\lambda_{1,1}\lambda_{2,2}+\epsilon\lambda_{1,2}\lambda_{2,1} & \lambda_{1,1}\lambda_{1,2}-\epsilon\lambda_{1,2}\lambda_{1,1}\\ -\lambda_{2,1}\lambda_{2,2}+\epsilon\lambda_{2,2}\lambda_{2,1} & \lambda_{2,1}\lambda_{1,2}-\epsilon\lambda_{2,2}\lambda_{1,1}\end{bmatrix}\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}\\
&=\begin{bmatrix}B_{1,1} & B_{1,2}\\ B_{2,1} & B_{2,2},\end{bmatrix}\begin{pmatrix}y_1(z)\\ y_2(z)\end{pmatrix}
\end{aligned}$$

where $B_{i,j}$, for $i,j\in\{1,2\}$, are implicitly defined above. To aid giving simplified expressions for $B_{i,j}$, we supply and prove the following elementary results.

Lemma B.1. *We have*

$$\epsilon - 1 = \frac{2\pi i \exp(\pi i(c-a-b))}{\Gamma(c-a-b)\Gamma(1+a+b-c)}. \tag{B.9}$$

Proof. Using the identity

$$\Gamma(x)\Gamma(1-x) = \pi \operatorname{cosec}(\pi x), \tag{B.10}$$

we have,

$$\begin{aligned}
\epsilon - 1 &= \exp(2\pi i(c-a-b)) - 1 \\
&= 2i \exp(\pi i(c-a-b)) \frac{\exp(\pi i(c-a-b)) - \exp(-\pi i(c-a-b))}{2i} \\
&= 2i \exp(\pi i(c-a-b)) \sin \pi(c-a-b) \\
&= \frac{2\pi i \exp(\pi i(c-a-b))}{\pi \operatorname{cosec} \pi(c-a-b)} \\
&= \frac{2\pi i \exp(\pi i(c-a-b))}{\Gamma(c-a-b)\Gamma(1+a+b-c)}.
\end{aligned}$$

□

Lemma B.2. *We have*

$$\lambda_{1,1}\lambda_{2,2} - \lambda_{1,2}\lambda_{2,1} = \frac{1-c}{a+b-c}. \tag{B.11}$$

Proof. From (B.4) and (B.6) we have

$$\begin{aligned}
&\lambda_{1,1}\lambda_{2,2} - \lambda_{1,2}\lambda_{2,1} \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \times \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} - \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \times \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \\
&= \Gamma(c)\Gamma(c-a-b)\Gamma(2-c)\Gamma(a+b-c) \\
&\times \left\{ \frac{1}{\Gamma(c-a)\Gamma(c-b)\Gamma(a+1-c)\Gamma(b+1-c)} - \frac{1}{\Gamma(a)\Gamma(b)\Gamma(1-a)\Gamma(1-b)} \right\}.
\end{aligned} \tag{B.12}$$

Making use of the identities (B.10) and

$$\sin \pi(c-a-b) \sin \pi c = \sin \pi(c-a) \sin \pi(c-b) - \sin \pi a \sin \pi b,$$

we have

$$\begin{aligned}
&\frac{1}{\Gamma(c-a-b)\Gamma(1-c+a+b)\Gamma(c)\Gamma(1-c)} \\
&= \frac{1}{\Gamma(c-a)\Gamma(c-b)\Gamma(a+1-c)\Gamma(b+1-c)} - \frac{1}{\Gamma(a)\Gamma(b)\Gamma(1-a)\Gamma(1-b)},
\end{aligned}$$

so that (B.12) becomes

$$\begin{aligned}
&\lambda_{1,1}\lambda_{2,2}-\lambda_{1,2}\lambda_{2,1} \\
&= \frac{\Gamma(c)\Gamma(c-a-b)\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(c-a-b)\Gamma(1-c+a+b)\Gamma(c)\Gamma(1-c)} \\
&= \frac{1-c}{a+b-c},
\end{aligned} \tag{B.13}$$

as desired. □

Using (B.4), (B.6), (B.9) and (B.11), we simplify $B_{1,1}$ as

$$\begin{aligned}
B_{1,1} &= \frac{\lambda_{1,1}\lambda_{2,2}-\epsilon\lambda_{1,2}\lambda_{2,1}}{\lambda_{1,1}\lambda_{2,2}-\lambda_{1,2}\lambda_{2,1}} \\
&= 1-(\epsilon-1)\frac{\lambda_{1,2}\lambda_{2,1}}{\lambda_{1,1}\lambda_{2,2}-\lambda_{1,2}\lambda_{2,1}} \\
&= 1-\frac{a+b-c}{1-c}\,\frac{2\pi i\exp(\pi i(c-a-b))}{\Gamma(c-a-b)\Gamma(1+a+b-c)}\,\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \\
&= 1-2\pi i\exp(\pi i(c-a-b))\frac{\pi\operatorname{cosec}(\pi c)}{\pi\operatorname{cosec}(\pi a)\,\pi\operatorname{cosec}(\pi b)} \\
&= 1-2i\exp(\pi i(c-a-b))\frac{\sin(\pi a)\,\sin(\pi b)}{\sin(\pi c)},
\end{aligned}$$

which agrees with the formula for $B_{1,1}$ given on p94 of [Bat53] and on p156 of [PS03]. Similarly,

$$\begin{aligned}
B_{1,2} &= \frac{\lambda_{1,1}\lambda_{1,2}-\epsilon\lambda_{1,2}\lambda_{1,1}}{\lambda_{1,1}\lambda_{2,2}-\lambda_{1,2}\lambda_{2,1}} \\
&= (\epsilon-1)\frac{\lambda_{1,1}\lambda_{1,2}}{\lambda_{1,1}\lambda_{2,2}-\lambda_{1,2}\lambda_{2,1}} \\
&= \frac{a+b-c}{1-c}\,\frac{2\pi i\exp(\pi i(c-a-b))}{\Gamma(c-a-b)\Gamma(1+a+b-c)}\,\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\
&= -2\pi i\exp(\pi i(c-a-b))\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)},
\end{aligned}$$

which agrees with both the formula for $B_{1,2}$ given on p94 of [Bat53] and the formula for $B_{2,1}$ given on p156 of [PS03]. Continuing, we have

$$\begin{aligned}
B_{2,1} &= (1-\epsilon)\frac{\lambda_{2,1}\lambda_{2,2}}{\lambda_{1,1}\lambda_{2,2}-\lambda_{1,2}\lambda_{2,1}} \\
&= -\frac{a+b-c}{1-c}\,\frac{2\pi i\exp(\pi i(c-a-b))}{\Gamma(c-a-b)\Gamma(1+a+b-c)}\,\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}\,\frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} \\
&= -2\pi i\exp(\pi i(c-a-b))\frac{\Gamma(2-c)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a+1-c)\Gamma(b+1-c)},
\end{aligned}$$

which differs in sign from the formula for $B_{2,1}$ given on p94 of [Bat53], yet agrees with the formula for $B_{1,2}$ given on p156 of [PS03]. Finally,

$$\begin{aligned} B_{2,2} &= \frac{-\lambda_{1,2}\lambda_{2,1} + \epsilon\lambda_{1,1}\lambda_{2,2}}{\lambda_{1,1}\lambda_{2,2} - \lambda_{1,2}\lambda_{2,1}} \\ &= 1 + (\epsilon - 1)\frac{\lambda_{1,1}\lambda_{2,2}}{\lambda_{1,1}\lambda_{2,2} - \lambda_{1,2}\lambda_{2,1}} \\ &= 1 + \frac{a+b-c}{1-c}\,\frac{2\pi i \exp(\pi i(c-a-b))}{\Gamma(c-a-b)\Gamma(1+a+b-c)}\,\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,\frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} \\ &= 1 + 2i\exp(\pi i(c-a-b))\frac{\sin\pi(c-a)\,\sin\pi(c-b)}{\sin(\pi c)}, \end{aligned}$$

which agrees with the formula for $B_{2,2}$ given on p94 of [Bat53] and on p156 of [PS03].

Hence,

$$\sigma_1(F) = F\begin{bmatrix} B_{1,1} & B_{2,1} \\ B_{1,2} & B_{2,2} \end{bmatrix},$$

i.e.,

$$A^{\sigma_1} = \begin{bmatrix} B_{1,1} & B_{2,1} \\ B_{1,2} & B_{2,2} \end{bmatrix}.$$

B.2 The Case $c = 1$

When $c = 1$ the two solutions y_1 and y_2 given in (B.1) and (B.2) are identical and their Wronskian is zero. Thus, we obtain an independent second solution by computing the limit

$$\begin{aligned} &\lim_{c\to 1}\frac{z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z) - {}_2F_1(a,b;1;z)}{c-1} \\ &= \frac{d}{dc} z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z)\Big|_{c=1} \end{aligned} \tag{B.14}$$

In [Daa10], if $c \in \mathbb{Z}^+$ and $a \notin \{1, 2, \ldots, c-1\}$ then an independent second solution

$$\begin{aligned} &\log(z)\,{}_2F_1(a,b;c;z) - \sum_{k=1}^{c-1}\frac{(c-1)!(k-1)!}{(c-k-1)!(1-a)_k(1-b)_k}(-z)^{-k} \\ &+ \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k\{\psi(a+k) + \psi(b+k) - \psi(1+k) - \psi(c+k)\} \end{aligned}$$

is supplied, where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. When $c = 1$, this is

$$\log(z)\,{}_2F_1(a,b;1;z) + \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(1)_k k!}z^k\{\psi(a+k) + \psi(b+k) - 2\psi(1+k)\}.$$

For $n \in \mathbb{N}$ we have

$$\frac{d}{da}(a)_n = (a)_n \sum_{k=0}^{n-1} \frac{1}{a+k}$$

and, since

$$\frac{1}{(c)_n} = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{c+k},$$

we have

$$\frac{d}{dc}\frac{1}{(c)_n} = \frac{1}{(c)_n} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(c)_n}{(n-1)!} \frac{(-1)^{k+1}}{(c+k)^2},$$

and thus formulae for the partial derivatives of ${}_2F_1(a,b;c;z)$ are given by

$$\frac{\partial}{\partial a}({}_2F_1(a,b;c;z)) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \sum_{k=0}^{n-1} \frac{1}{a+k}$$

and

$$\frac{\partial}{\partial c}({}_2F_1(a,b;c;z)) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(c)_n}{(n-1)!} \frac{(-1)^{k+1}}{(c+k)^2}.$$

Thus, we compute the RHS of (B.14) as

$$\begin{aligned}
&-\log(z)\, z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z)\Big|_{c=1} \\
&- z^{1-c} \sum_{n=0}^{\infty} \frac{(a+1-c)_n (b+1-c)_n}{(2-c)_n n!} z^n \Bigg\{ \sum_{k=0}^{n-1} \frac{1}{a+1-c+k} \\
&+ \sum_{k=0}^{n-1} \frac{1}{b+1-c+k} + \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(2-c)_n}{(n-1)!} \frac{(-1)^{k+1}}{(2-c+k)^2} \Bigg\}\Bigg|_{c=1}
\end{aligned}$$

$$\begin{aligned}
&= -\log(z)\, {}_2F_1(a,b;1;z) - \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(1)_n n!} z^n \left\{ \sum_{k=0}^{n-1} \frac{1}{a+k} + \sum_{k=0}^{n-1} \frac{1}{b+k} + \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(1)_n}{(n-1)!} \frac{(-1)^{k+1}}{(1+k)^2} \right\} \\
&= -\log(z)\, {}_2F_1(a,b;1;z) - \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(1)_n n!} z^n \left\{ \sum_{k=0}^{n-1} \frac{1}{a+k} + \sum_{k=0}^{n-1} \frac{1}{b+k} + \sum_{k=0}^{n-1} \binom{n}{k+1} \frac{(-1)^{k+1}}{1+k} \right\}.
\end{aligned}$$

We refrain from further calculation and now focus on particular hypergeometric equations.

B.2.1 Monodromy Group of ${}_2F_1(1/2, 1/2; 1; z)$

On Page 95 of [Bat53], the independent second solution $y_2(z) = i\,{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-z)$ is chosen. We see immediately that

$$\sigma_0(y_1)(z) = y_1(\exp(2\pi i) z) = y_1(z)$$

and

$$\sigma_1(y_2)(z) = y_2(1-\exp(2\pi i)(1-z)) = i\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;\exp(2\pi i)(1-z)) = y_2(z).$$

The relation

$$\frac{\pi}{2}{}_2F_1(\frac{1}{2},\frac{1}{2};1;z)+\frac{1}{2}\log(1-z)\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-z) = \sum_{n=0}^{\infty}\frac{(1/2)_n(1/2)_n}{(1)_n n!}\{\psi(n+1)-\psi(n+1/2)\}(1-z)^n \tag{B.15}$$

supplied on Page 95 of [Bat53], gives

$$\frac{\pi}{2}y_1(z)+\frac{1}{2i}\log(1-z)\,y_2(z) = \sum_{n=0}^{\infty}\frac{(1/2)_n(1/2)_n}{(1)_n n!}\{\psi(n+1)-\psi(n+1/2)\}(1-z)^n.$$

Applying σ_1 to both sides gives

$$\begin{aligned}
&\frac{\pi}{2}\sigma_1(y_1)(z)+\frac{1}{2i}\log(\exp(2\pi i)(1-z))\,\sigma_1(y_2)(z)\\
&=\sum_{n=0}^{\infty}\frac{(1/2)_n(1/2)_n}{(1)_n n!}\{\psi(n+1)-\psi(n+1/2)\}\exp(2\pi i n)(1-z)^n\\
&=\sum_{n=0}^{\infty}\frac{(1/2)_n(1/2)_n}{(1)_n n!}\{\psi(n+1)-\psi(n+1/2)\}(1-z)^n\\
&=\frac{\pi}{2}y_1(z)+\frac{1}{2i}\log(1-z)\,y_2(z),
\end{aligned}$$

and using invariance of y_2 under σ_1, we have

$$\frac{\pi}{2}\sigma_1(y_1)(z)+\frac{1}{2i}\big(2\pi i+\log(1-z)\big)\,y_2(z) = \frac{\pi}{2}y_1(z)+\frac{1}{2i}\log(1-z)\,y_2(z),$$

i.e.,

$$\sigma_1(y_1)(z)+2y_2(z) = y_1(z).$$

Replacing z by $1-z$ in (B.15) gives

$$\frac{\pi}{2}{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-z)+\frac{1}{2}\log(z)\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;z) = \sum_{n=0}^{\infty}\frac{(1/2)_n(1/2)_n}{(1)_n n!}\{\psi(n+1)-\psi(n+1/2)\}z^n,$$

i.e.,

$$\frac{\pi}{2i}y_2(z)+\frac{1}{2}\log(z)\,y_1(z) = \sum_{n=0}^{\infty}\frac{(1/2)_n(1/2)_n}{(1)_n n!}\{\psi(n+1)-\psi(n+1/2)\}z^n.$$

Applying σ_0 to both sides gives

$$\begin{aligned}\frac{\pi}{2i}\sigma_0(y_2)(z) + \frac{1}{2}\log(\exp(2\pi i)z)\,\sigma_0(y_1)(z) &= \sum_{n=0}^{\infty} \frac{(1/2)_n(1/2)_n}{(1)_n n!}\{\psi(n+1)-\psi(n+1/2)\}\exp(2\pi i n)z^n \\ &= \sum_{n=0}^{\infty} \frac{(1/2)_n(1/2)_n}{(1)_n n!}\{\psi(n+1)-\psi(n+1/2)\}z^n \\ &= \frac{\pi}{2i}y_2(z) + \frac{1}{2}\log(z)\,y_1(z),\end{aligned}$$

and using invariance of y_1 gives

$$\frac{\pi}{2i}\sigma_0(y_2)(z) + \pi i\, y_1(z) = \frac{\pi}{2i}y_2(z),$$

i.e.,

$$\sigma_0(y_2)(z) - 2\,y_1(z) = y_2(z).$$

Hence $\sigma_0(y_1(z), y_2(z)) = (y_1(z), y_2(z))A^{\sigma_0}$ and $\sigma_1(y_1(z), y_2(z)) = (y_1(z), y_2(z))A^{\sigma_1}$, where

$$A^{\sigma_0} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{\sigma_1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

The following lemma is needed to prove that $\mathrm{Gal}^{\partial}(K/k) \cong \Gamma(2)$.

Lemma B.3. *We have that* $\Gamma(2) \cong \langle A^{\sigma_0}, A^{\sigma_1}\rangle$.

Proof. Let $H = \langle A^{\sigma_0}, A^{\sigma_1}\rangle$. From Definition 2.4 we see that $A^{\sigma_0}, A^{\sigma_1} \in \Gamma(2)$, and so $H \le \Gamma(2)$. Now let

$$\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(2)$$

be arbitrary. Owing to a and c being coprime, we will construct an element

$$(A^{\sigma_0})^{e_0}(A^{\sigma_1})^{e_1}\dots(A^{\sigma_0})^{e_{2k-1}}(A^{\sigma_1})^{e_{2k}} \in H$$

such that

$$(A^{\sigma_0})^{e_0}(A^{\sigma_1})^{e_1}\dots(A^{\sigma_0})^{e_{2k-1}}(A^{\sigma_1})^{e_{2k}}\beta = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix},$$

where p is an even integer. Since $\begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} = (A^{\sigma_0})^{p/2}$, we will have shown that $\beta \in H$, giving $\Gamma(2) \leq H$.

If $|a| > |c|$ then write $a = q.2c + r$, for some $q, r \in \mathbb{Z}$ such that $|r| \leq |c|$. Then

$$\beta' = (A^{\sigma_0})^{-q}\beta = \begin{bmatrix} a-2qc & b-2qd \\ c & d \end{bmatrix} = \begin{bmatrix} r & b-2qd \\ c & d \end{bmatrix} = \begin{bmatrix} a' & b' \\ c & d \end{bmatrix}.$$

If $|r| = |c|$, then $\gcd(a,c) = 1$ implies $\gcd(r,c) = 1$, i.e., $|r| = |c| = 1$, contradicting c being even.

Now we have $|a| < |c|$. Write $c = q.2a + r$, for some $q, r \in \mathbb{Z}$ such that $|r| \leq |a|$. If $|r| = |a|$, then $\gcd(a,c) = 1$ implies $\gcd(r,a) = 1$, i.e., $|r| = |a| = 1$, which contradicts c being even. Hence,

$$(A^{\sigma_1})^{q}\beta' = \begin{bmatrix} a & b \\ c-2aq & d-2bq \end{bmatrix} = \begin{bmatrix} a & b \\ r & d-2bq \end{bmatrix} = \begin{bmatrix} a & b \\ c' & d' \end{bmatrix},$$

where $|r| < |a|$. Continuing in this manner, we arrive at a matrix

$$(A^{\sigma_1})^{q^{(2k+1)}}(A^{\sigma_0})^{-q^{(2k)}}\dots(A^{\sigma_1})^{q'}(A^{\sigma_0})^{-q}\beta = \begin{bmatrix} 1 & b'' \\ 0 & d'' \end{bmatrix}.$$

It must be that $d'' = 1$ since the determinant is one, and that b'' is even, since parity of matrix entries is preserved in our matrix operations. Hence, we have

$$(A^{\sigma_1})^{q^{(2k+1)}}(A^{\sigma_0})^{-q^{(2k)}}\dots(A^{\sigma_1})^{q'}(A^{\sigma_0})^{-q}\beta = \begin{bmatrix} 1 & b'' \\ 0 & 1 \end{bmatrix} = (A^{\sigma_0})^{b''/2}$$

and, thus, $\beta \in H$. □

B.2.2 Monodromy Group of ${}_2F_1(1/12, 5/12; 1; z)$

The monodromy group of the hypergeometric DE satisfied by ${}_2F_1(1/12, 5/12; 1; z)$ can be inferred from Proposition 8.2 on p78 of [Yos97], which states that the Schwarz map of this hypergeometric DE gives an isomorphism $\mathbb{C} \xrightarrow{\sim} \mathrm{PSL}_2(\mathbb{Z})\backslash\mathcal{H}$. Furthermore, as stated on p76 of [Yos97], the monodromy group of this hypergeometric DE is conjugate in $\mathrm{PGL}_2(\mathbb{C})$ to $\mathrm{PSL}_2(\mathbb{Z})$). Hence, we have $\mathrm{Gal}^{\partial}(K/k) \cong \mathrm{PSL}(\mathbb{C})$, the Zariski-closure of the monodromy group $\mathrm{PSL}_2(\mathbb{Z})$).

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