Modular Forms Modulo Powers of Primes

BY

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Abstract

We study modular forms modulo p^m with level structure $\Gamma_1(N)$, where $m \geqslant 1$, $p \geqslant 5$ is prime, and N is coprime to p. We exhibit properties of the weight filtrations of these modular forms by approaching their construction from the perspective of Serre p-adic modular forms, and present Chen and Kiming's construction [4] of the theta operator θ. The primary motivation for this thesis is to study and compute theta cycles of these modular forms, supplementing the results of Kim and Lee [13], and to use the tools developed by Chen and Kiming to comment on the weight filtration for the differential operator ∂.

Declaration of Authorship

I, Miles Koumouris, declare that this thesis entitled *Modular Forms Modulo Powers of Primes* and the work presented in it are my own. I confirm that:

- The thesis comprises only my original work towards the degree of Master of Science (MC-SCIMAT) except where indicated in the preface;
- due acknowledgement has been made in the text to all other material used; and
- the thesis is fewer than the maximum word limit in length, exclusive of tables, maps, bibliographies and appendices as approved by the Research Higher Degrees Committee.

Signed:

Date: 16 / 10 / 2023

Preface

The work of this thesis

- is appropriately cited when referencing unoriginal results;
- is original and completed without collaboration when uncited, however with supervisory assistance and corrections mentioned in the section below;
- is unpublished at the date of submission;
- was carried out with the assistance of Masters of Science Scholarships from the School of Mathematics and Statistics, but with no additional financial assistance from grants or Government Research Training Program Scholarships.

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Contents

Introduction

Let $p \geqslant 5$ be prime, and $N, m \geqslant 1$ be integers. In this thesis, we introduce modular forms modulo p^m with level structure $\Gamma_1(N)$ by approaching from both classical modular forms and p-adic modular forms (in the sense of Serre, [15]). Our main objective is to develop the theory necessary to understand Chen and Kiming's construction of the theta operator on modular forms modulo p^m, and its corresponding effect on their weight filtrations.

For most level structures, we can identify modular forms with their corresponding q-expansions at ∞, in particular for Serre p-adic modular forms over $\Gamma_1(N)$. In [15], Serre demonstrates the classical theta operator $\theta = q\frac{d}{dq}$ on q-expansions actually defines a map on p-adic modular forms of level 1, sending a form of weight k to a form of weight $k+2$. In fact, the arguments can be generalised to show this map exists in all levels N for which $p \nmid N$.

We then naturally define a theta operator on modular forms modulo p^m; the definition via p-adic modular forms allows Chen and Kiming to show this operator is well-defined, and the congruence results for classical modular forms allow us to determine the weight filtration in certain cases. In this way, we use both approaches in defining modular forms modulo p^m from the theory developed in Chapter 1.

From the congruence results and Chen and Kiming's theorem (see Theorem 3), we use SageMath-9-7 to compute p-filtration theta cycles for some modular forms modulo p. We also test a result due to Kim and Lee on theta cycles for $m \geqslant 2$.

Notation

- $\mathbb{Q}_p := \{\sum_{i=k}^{\infty} a_i p^i \mid k \in \mathbb{Z},\ 0 \leqslant a_i < p \text{ integers}\}$ is the p-adic numbers;
- $\mathbb{Z}_p := \{\sum_{i=0}^{\infty} a_i p^i \mid 0 \leqslant a_i < p \text{ integers}\} \subset \mathbb{Q}_p$ is the p-adic integers;
- for p a prime number, $\nu_p : \mathbb{Q}_p \longrightarrow \mathbb{Z}$ is the p-adic valuation;
- for $N \geqslant 1$ an integer, $\Gamma_1(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0,\ a \equiv d \equiv 1 \pmod{N}\}$ is the arithmetic group giving *level* N level structure to a modular form;
- $\Gamma_1 := \Gamma_1(N) = \mathrm{SL}_2(\mathbb{Z})$;
- $\Gamma_0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N}\}$;
- $\mathcal{H} := \{z \mid \mathrm{im}(z) > 0\} \subset \mathbb{C}$ is the upper half-plane;
- for f a function on the upper half-plane, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ a real matrix of positive determinant, and k an integer, we define the slash operator

  $$(f|_k\,\gamma)(z) := \det(\gamma)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

  Note that $(f|_k\,\gamma)|_k\,\gamma' = f|_k\,\gamma\gamma'$.

1 Defining modular forms modulo p^m

In this chapter, we first define classical modular forms in the standard analytic way, and give an important class of examples: the Eisenstein series. We also state an equivalent algebraic-geometric definition originally due to Katz [11], as the proofs of certain results we will need (for example Theorem 5.4 in [7]) are easier using a geometric approach.

We conclude with the definitions of Serre p-adic modular forms and modular forms modulo p^m, making sure we prove these are in fact equivalent. The theory (especially for the case $m = 1$) is long established, although still has to be elucidated with some care due to subtleties arising from the level and from the structure of the ring of rational modular forms.

1.1 Classical modular forms

1.1.1 Defining classical modular forms

Definition 1. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a group of finite index (called, for convenience, an *arithmetic group* in this thesis). A *modular form* of *level* Γ and *weight* $k \in \mathbb{Z}_{\geqslant 0}$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ from the upper half-plane $\mathcal{H}$ satisfying

(a) for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ (this is the *automorphy condition*);

(b) for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $(cz+d)^{-k} f(\frac{az+b}{cz+d})$ is bounded as $\mathrm{im}(z) \to \infty$ (this is the *growth condition*).

It is clear from the definition that the set of all such modular forms f is a $\mathbb{C}$-vector space, which we denote by $M_k(\Gamma)$. It is also clear from the definition that multiplying two such modular forms f and f' of weights k and k' respectively results in a modular form of level Γ and weight $k + k'$. Thus

$$M(\Gamma) := \bigoplus_{k>0} M_k(\Gamma)$$

is a graded ring, which we refer to as (*classical*) modular forms with level structure Γ, for the purposes of distinguishing them from the forms soon to be defined. If we omit stating the level structure, it is assumed to be $\mathrm{SL}_2(\mathbb{Z})$ unless otherwise obvious from the context.

1.1.2 Fourier expansions for certain modular forms

Observe that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(p) \cap \Gamma_1(N) \cap \mathrm{SL}_2(\mathbb{Z})$, so for a modular form f in $M(\Gamma_0(p))$, $M(\Gamma_1(N))$ or $M(\mathrm{SL}_2(\mathbb{Z}))$, we have $f(z) = f(z+1)$ by the automorphy condition. This means that f has a Fourier expansion (at ∞) given by $\sum_{n=-\infty}^{\infty} a_n q^n$, with $q := e^{2\pi i z}$ and coefficients $a_n \in \mathbb{C}$ for all n. But by the growth condition, we must have $a_n = 0$ for all $n < 0$. That is, every modular form in $M(\Gamma_0(p))$, $M(\Gamma_1(N))$ or $M(\mathrm{SL}_2(\mathbb{Z}))$ can be identified with a formal power series $\sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[\![q]\!]$, called its *q-expansion* (at ∞).

The same argument shows that the $\mathbb{C}$-vector space of modular forms with level structure given by any arithmetic group Γ containing $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is some submodule of $\mathbb{C}[\![q]\!]$.

Definition 2. For $\Gamma = \Gamma_0(p), \Gamma_1(N), \mathrm{SL}_2(\mathbb{Z})$ and any subring $R \subset \mathbb{C}$, we define

$$M_k(\Gamma, R) := \left\{ f \in M_k(\Gamma) \mid f = \sum_{n=0}^{\infty} a_n q^n \text{ with } a_n \in R \;\forall\, n \right\}$$

to be those modular forms in $M_k(\Gamma)$ whose q-expansions have coefficients in R.

Henceforth, to conform to convention, we write

$$M_k(N, R) := M_k(\Gamma_1(N), R) \text{ and } M(N, R) := M(\Gamma_1(N), R).$$

Now that we can describe certain classical modular forms in terms of their q-expansions, we can state the following definition of some very important modular forms:

Definition 3. The *standard Eisenstein series on* $\mathrm{SL}_2(\mathbb{Z})$ are given by

$$E_k := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

for even $k \in \mathbb{Z}_{>0}$. Here, B_k is the k^{th} Bernoulli number, $\sigma_t(n) := \sum_{d|n} d^t$ is the sum of the t^{th} powers of the positive divisors of n, and $q = e^{2\pi i z}$. We write $P := E_2$, $Q := E_4$ and $R := E_6$.

Proposition 1. *(See §2.1 and Proposition 5 in [2]). For even $k \geqslant 4$, we have $E_k \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.*

Proposition 2. *(See Proposition 6 in [2]). Whilst P satisfies the growth condition, it* *does not satisfy the automorphy condition; instead, for any* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$*, we have*

$$P\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 P(z) - \frac{6ic}{\pi}(cz+d).$$

Although P is not a modular form, it is a *p-adic modular form of weight* 2 (this is a result of Proposition 12, or of the identity (9)). In fact, P will be important in the construction of the theta operator on Serre p-adic modular forms, and eventually on modular forms modulo p^m. It is apropos in the context of our discussion of Eisenstein series to state another important classical result on modular forms of level Γ_1:

Proposition 3. *(Proposition 4 in [2]). The ring $M(\Gamma_1)$ is freely generated by the modular forms Q and R. Then every modular form in $M_k(\Gamma_1)$ can be expressed as an isobaric (in the sense of weight) polynomial in Q and R.*

1.1.3 Congruences between modular forms

Definition 4. If $f = \sum a_n q^n \in \mathbb{Q}_p[\![q]\!]$ is a formal series in one variable q, we set

$$\nu_p(f) := \inf \nu_p(a_n),$$

where $\nu_p : \mathbb{Q}_p \longrightarrow \mathbb{Z}$ is the p-adic valuation for some prime p. Note that $\nu_p(f) \geqslant 0$ implies $f \in \mathbb{Z}_p[\![q]\!]$. If $\nu_p(f) \geqslant m > 0$, we write $f \equiv 0 \pmod{p^m}$. Let (f_i) be a sequence of elements of $\mathbb{Q}_p[\![q]\!]$. We say that f_i *converges to* f (and write $\lim f_i = f$) if the coefficients of f_i converge uniformly to those of f, that is if $\nu_p(f - f_i) \to +\infty$.

Theorem 1. *(Based on Théorème 1 in §1.3 of [15]). Let $m \geqslant 1$ be an integer, let $p \geqslant 5$ be a prime number, and let $N \geqslant 1$ be coprime to p. Let $f, f' \in M(N, \mathbb{Q})$ be two modular forms with rational coefficients, of respective weights k and k'. Suppose that $f \neq 0$ and $\nu_p(f - f') \geqslant \nu_p(f) + m$. Then*

$$k' \equiv k \pmod{(p-1)p^{m-1}}.$$

We present the proof of this theorem in §1.3 in the case $N \neq 2, 3, 4$, once we have defined modular forms modulo powers of primes.

1.1.4 Equivalent definition due to Katz

There is also an algebraic geometric approach to defining modular forms, and under certain conditions, this definition gives the same result. By defining modular forms as global sections of a sheaf on some moduli space, one can more easily deduce certain results about them by calling upon more established theory. One such result (originally attributed to Serre and Swinnerton-Dyer, but stated by Goren in [7]) is used in §1.3.

Definition 5. A *(Katz) modular form of weight k and level* $\Gamma = \Gamma_1(N)$ is a global section of the line bundle $\underline{\omega}^{\otimes k}$ over the compactified modular curve $\Gamma \setminus \mathcal{H}^* := \Gamma \setminus (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}))$, i.e., an element of $M_k^{\mathrm{Katz}}(\Gamma) := H^0(\Gamma \setminus \mathcal{H}^*, \underline{\omega}^{\otimes k})$. Here, $\underline{\omega}^{\otimes k}$ is in fact the Hodge bundle defined as the quotient $\Gamma \setminus \mathcal{H}^* \times \mathbb{C}$ with the action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ given by $(z, \alpha) \mapsto (\gamma z, (cz + d)^k \alpha)$.

In Katz's original definition in [5], the modular curve $\Gamma \setminus \mathcal{H}$ is used, thus giving a definition with no growth condition. But it is a fact that $M_k(\Gamma) = M_k^{\mathrm{Katz}}(\Gamma)$ (discussed for example in §4 of Chapter 1 in [7]). Another advantage of this approach is that one can apply base change to the ring over which the modular curve is defined to arrive at a more general definition. In [5], Diamond and Im cite Katz's definition of the space $M_k^{\mathrm{Katz}}(N, R)$ of *(Katz) modular forms of weight k on $\Gamma_1(N)$ over the ring $R \subset \mathbb{C}$*. In [12], Katz uses this definition to prove results used in §2.4.2, which is possible as a result of the following equality:

Proposition 4. *(Theorem 12.3.7 in [5] for $N \geqslant 5$, for $N \leqslant 4$ consult [4]). For $p \geqslant 5$, we have*

$$M_k^{\mathrm{Katz}}(N, \mathbb{F}_p) \cong M_k(N, \mathbb{F}_p) = M_k(N, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p$$

if $k \neq 1$ and $p \nmid N$.

1.2 Serre p-adic modular forms

1.2.1 Defining Serre p-adic modular forms

When defining Serre p-adic modular forms, we assume that $p \neq 2$ to avoid complications (see §1.4 in [15]). This is not an issue, since the main focus of our study is on the effect of the theta operator for primes $p \geqslant 5$.

Definition 6. A *(Serre) p-adic modular form of level N* is a formal series

$$f = \sum_{n=0}^{\infty} a_n q^n$$

with coefficients $a_n \in \mathbb{Q}_p$ such that there exists a sequence (f_i) of classical modular forms of level N with rational coefficients (i.e. $f_i \in M(N, \mathbb{Q})$) satisfying $\lim f_i = f$ (in the sense that $\nu_p(f - f_i) \to +\infty$).

Note that in [15], Serre only defines p-adic modular forms for level 1.

Definition 7. Let $m \geqslant 1$ be an integer and $p \neq 2$ be a prime number. Define

$$X_m := \mathbb{Z}/p^{m-1}(p-1)\mathbb{Z}.$$

As $m \to \infty$, the X_m form a *projective system* of groups with transition morphisms given by the natural inclusions; we denote by X the projective limit of this system. We have

$$X = \varprojlim X_m = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}.$$

1.2.2 Weights of p-adic modular forms

Proposition 5. *(Theorem 2 in §1.4 of [15]). Let f be a nonzero p-adic modular form, and let (f_i) be a sequence of modular forms of weights (k_i), with rational coefficients, converging to f. Then the sequence (k_i) has a limit in the group $X = \operatorname{proj\,lim} X_m$, called the "weight". This limit depends on f, but not on the chosen sequence (f_i).*

Proof By hypothesis, we have $\nu_p(f_i - f_j) \to \infty$; on the other hand, $\nu_p(f_i)$ is equal to $\nu_p(f)$ for sufficiently large i. From Theorem 1 we deduce that, for all $m \geqslant 1$, the image of the sequence k_i in X_m stabilises; this means that the k_i have a limit k in X. The fact this limit does not depend on the chosen sequence is immediate from Theorem 1. □

Note that it is often true the weights (k_i) stabilise simultaneously in all X_m with $m > M$ for some M. In this scenario, the limit k can be interpreted naturally as an integer. Henceforth, we denote by $\mathbf{V}(\mathbb{Z}_p, k, N)$ the set of p-adic modular forms of weight k and level N. In fact, this is a $\mathbb{Q}_p$-module.

Lemma 1. *Serre p-adic modular forms have a natural $\mathbb{Q}_p$-ring structure.*

Proof Clearly Serre p-adic modular forms constitute a $\mathbb{Q}_p$-module with scalar multiplication given by the normal $\mathbb{Q}_p$ multiplication, and addition well-defined by coefficient-wise addition in the q-expansions (superadditive with respect to ν_p).

To complete our proof of the ring structure, let f and f' be p-adic modular forms with respective weights k and k'; we claim that ff' is a p-adic modular form with weight equal to the limit of $k_ik'_i$ in X. Let (f_i) and (f'_i) be sequences of modular forms with respective weights (k_i) and (k'_i) converging respectively to f and f'. From §1.1.1, we know that $(f_if'_i)$ is a sequence of modular forms with weights $(k_ik'_i)$. Now by factoring out the highest power of p from all coefficients in the q-expansion, we have

$$\nu_p(f'(f - f_i)) \geqslant \nu_p(f - f_i) \to +\infty,$$
$$\nu_p(f_i(f' - f'_i)) \geqslant \nu_p(f' - f'_i) \to +\infty.$$

Hence,

$$\nu_p(ff' - f_if'_i) \geqslant \nu_p(f'(f - f_i)) + \nu_p(f_i(f' - f'_i)) \to +\infty$$

and ff' is indeed a p-adic modular form with weight equal to the limit of $k_ik'_i$ in X. □

1.3 Modular forms modulo p^m

1.3.1 First definitions and properties

Definition 8. Let p be prime and $m \geqslant 1$ be an integer. Let Γ be an arithmetic group containing $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ so that $M_k(\Gamma,\mathbb{Z})$ consists of modular forms of weight k and level structure Γ all having q-expansions with integer coefficients.

Let $\widetilde{M}_k(\Gamma) := M_k(\Gamma,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z}$ the set of all these q-expansions but with the coefficients reduced modulo p^m, which forms a ring in the natural way. We call these *modular forms with weight k and level structure Γ modulo p^m.*

Set $\widetilde{M}(\Gamma) := \sum_{k\geqslant 0} \widetilde{M}_k(\Gamma,\mathbb{Z})$. These are *modular forms with level structure Γ modulo p^m*. We write $\widetilde{M}_k(N) := \widetilde{M}_k(\Gamma_1(N))$, $\widetilde{M}(N) := \widetilde{M}(\Gamma_1(N))$ and $\widetilde{M} := \widetilde{M}(\mathrm{SL}_2(\mathbb{Z}))$, and refer to *modular forms modulo p^m* if the level structure is $\mathrm{SL}_2(\mathbb{Z})$, or otherwise obvious from the context.

Proposition 6. *Consider those forms in $M_k(\Gamma,\mathbb{Q})$ whose q-expansions have only p-integral coefficients. We can equivalently define $\widetilde{M}_k(\Gamma)$ as this set of q-expansions but with the coefficients reduced modulo p^m.*

Proof We only need to prove we do not get extra series with this alternative definition. This is equivalent to showing that every $f \in M_k(\Gamma,\mathbb{Q})$ with p-integral coefficients in the q-expansion has the additional property that the denominators of these coefficients are bounded; this way, there exists $c \in \mathbb{Z}$ with $\gcd(c,p) = 1$ such that $cf \in M_k(\Gamma,\mathbb{Z})$. The fact that the denominators of the coefficients are bounded is a well-known result (see, for example, Theorem 3.52 in [16]). □

Definition 9. For $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{Z}[\![q]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$ (i.e. q-expansions with rational coefficients whose denominators are bounded), denote by $\widetilde{f}$ the modular form modulo p^m obtained by reducing the coefficients of the q-expansion modulo p^m. We use the same notation to denote the resultant q-expansion if f is a p-adic modular form (this also always turns out to be a modular form modulo p^m, see Proposition 7).

Notice that in the definition of $\widetilde{M}(\Gamma)$, we use a sum rather than a direct sum. This is because of the following lemma:

Lemma 2. *For $p \geqslant 5$, we have the inclusions*

$$\widetilde{M}_k(\Gamma) \subset \widetilde{M}_{k+p^{m-1}(p-1)}(\Gamma) \subset \widetilde{M}_{k+2p^{m-1}(p-1)}(\Gamma) \subset \cdots .$$

For $\alpha \in \mathbb{Z}/p^{m-1}(p-1)\mathbb{Z}$, we write $\widetilde{M}^\alpha(\Gamma)$ for the union of $\widetilde{M}_k(\Gamma)$ with k in α.

To prove this lemma, we need the following famous fact about Bernoulli numbers (proved for example in [10]):

Lemma 3. *(The Clausen-von Staudt Theorem). For any positive integer n, we have*

$$B_{2n} + \sum_{\substack{p \text{ prime}:\\ p-1 | 2n}} \frac{1}{p} \in \mathbb{Z}$$

Remark. Note the obvious corollary $\nu_p(B_{t(p-1)}) = \nu_p(1/p) = -1$ for any positive integer t.

Proof of Lemma 2 These inclusions are induced by multiplication by $\widetilde{E}_{p-1}^{p^{m-1}}$, which is usually called the *Hasse invariant* for modular forms modulo p^m; hence it suffices to show $E_{p-1}^{p^{m-1}} \equiv 1 \pmod{p^m}$. By the Clausen-von Staudt theorem, we have

$$\nu_p\left(\frac{p-1}{B_{p-1}}\right) = \nu_p(p-1) + 1 = 1,$$

so $E_{p-1} \equiv 1 \pmod{p^m}$. We proceed by induction on m; having just showed the base case $m = 1$, suppose $m \geqslant 2$. By the inductive hypothesis, we have $E_{p-1}^{p^{m-2}} \equiv 1 \pmod{p^{m-1}}$, which is the same as writing $E_{p-1}^{p^{m-2}} = 1 + p^{m-1} f$ for some $f \in \mathbb{Z}[\![q]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$. Now $\nu_p\binom{p}{i} = 1$ for all $0 < i < p$ since p is prime, hence

$$\begin{aligned} E_{p-1}^{p^{m-1}} &= (1 + p^{m-1} f)^p \\ &\equiv 1^p + (p^{m-1} f)^p \pmod{p^m} \\ &\equiv 1 \pmod{p^m} \qquad \text{(since } p(m-1) \geqslant m) \end{aligned}$$

as required. $\square$

Note that by the Clausen-von Staudt theorem,

$$\nu_p\left(\frac{(p-1)p^{m-1}}{B_{(p-1)p^{m-1}}}\right) = \nu_p((p-1)p^{m-1}) + 1 = m - 1 + 1 = m,$$

so $E_{(p-1)p^{m-1}} \equiv 1 \pmod{p^m}$. In other words, one can think of these inclusions also being induced by multiplication by $\widetilde{E}_{(p-1)p^{m-1}} = 1$. We will see in the proof of Theorem 1 that for $m = 1$ and $N \neq 2, 3, 4$, this is the only relation on modular forms modulo p in the sense that two modular forms are congruent modulo p if and only if they are equal up to some power of the Hasse invariant. The more general result for $p \nmid N$ and arbitrary $m \geqslant 1$ is given in Lemma 2.1 in [13].

1.3.2 Proving Theorem 1

Definition 10. An element $x \in \mathbb{Q}_p$ is *p-integral* if $x \in \mathbb{Z}_p \subset \mathbb{Q}_p$, or equivalently if $\nu_p(x) \geqslant 0$.

Proof of Theorem 1 This theorem follows directly from Corollary 4.4.2 in [11]. However, we present a proof by strengthening the argument of Théorème 1 in §1.3 of [15].

For $f \in \mathbb{Q}[\![q]\!]$ with p-integral coefficients, let $\overline{f}$ denote the series with coefficients reduced modulo p (i.e., $\overline{f} = \widetilde{f}$ when $m = 1$). Also, let $\overline{M} := \widetilde{M}(N)$ for $m = 1$.

Since the denominators of the coefficients of f must be bounded, we can multiply f by some scalar to make the coefficients p-integral, leaving the weight unaffected. Thus we may assume without the loss of generality that $\nu_p(f) = 0$, in which case the condition is equivalent to

$$f' \equiv f \pmod{p^m}.$$

Since the coefficients of f and f' are p-integral and $m \geqslant 1$, we certainly have $\overline{f} = \overline{f'} \neq 0$ (and also $\widetilde{f} = \widetilde{f'}$). If $p \geqslant 5$, we see that $\overline{f}$ and $\overline{f'}$ belong to the same component $\overline{M}^\alpha := \overline{M}^\alpha(N)$ of the algebra $\overline{M}$. This follows from Theorem 5.4 in [7]; although the proof of this theorem assumes $N > 4$, the conclusion holds for all $N \geqslant 1$ since $N \mid M \implies \Gamma(M) \subset \Gamma(N)$. Note that a proof for for $N = 1$ is also given by Swinnerton-Dyer in Lemma 5 (i) of [18]. In summary, $k' \equiv k \pmod{p-1}$. The theorem is therefore proved for $m = 1$.

Assume now that $m \geqslant 2$. Let $h := k' - k$. Up to replacing f' by

$$f' E_{(p-1)p^n}$$

with large enough n, we may assume that $h \geqslant 4$. The Eisenstein series E_h is then a modular form of weight h; as h is divisible by $p-1$, we have $E_h \equiv 1 \pmod{p}$. Set $r = \nu_p(h) + 1$. We need to show that $r \geqslant m$. Suppose $r < m$. We have $fE_h - f' = f - f' + f(E_h - 1)$.

Since $h \equiv 0 \pmod{(p-1)p^{r-1}}$, we have by the Clausen-von Staudt theorem

$$\nu_p(E_h - 1) \geqslant \nu_p\left(\frac{h}{B_h}\right) = \nu_p(h) + 1 = r - 1 + 1 = r,$$

so $E_h - 1 \equiv 0 \pmod{p^r}$. But $f - f' \equiv \pmod{p^m}$, so we conclude that $fE_h - f' \equiv 0 \pmod{p^r}$ and

$$p^{-r}(fE_h - f') \equiv p^{-r}f(E_h - 1) \pmod{p}.$$

Again using the Clausen-von Staudt Theorem, we have

$$p^{-r}(E_h - 1) = \lambda\phi,$$

where $\phi = \sum_{n=1}^{\infty} \sigma_{h-1}(n)q^n$ and $\nu_p(\lambda) = 0$. The above congruence is therefore equivalent to

$$f\phi \equiv g \pmod{p},$$

where g is the modular form $\lambda^{-1}p^{-r}(fE_h - f')$, of weight k'. As $\overline{f} \neq 0$, this can be written as $\overline{\phi} = \overline{g}/\overline{f}$ and shows that $\overline{\phi}$ belongs to the fraction field of $\overline{M}$; moreover, $\overline{g}$ and $\overline{f}$ have the same weight modulo $p-1$; we conclude that $\overline{\phi}$ belongs to the fraction field of $\overline{M}^0$. We have

$$\overline{\phi} - \overline{\phi}^p = \overline{\psi}, \quad \text{with } \psi = \sum_{(p,n)=1} \sigma_{h-1}(n)q^n,$$

and we easily check that

$$\psi \equiv \theta^{h-1}\left(\sum_{n=1}^{\infty} \sigma_1(n)q^n\right), \quad \text{where } \theta = q\frac{d}{dq} \text{ (see 2.2)}.$$

In order to get a contradiction, observe that we have

$$\overline{\psi} = -\frac{1}{24}\theta^{h-1}(\overline{P}) = -\frac{1}{24}\theta^{p-2}(\overline{E}_{p+1}),$$

whence $\overline{\psi} \in \overline{M}^0$, given the properties of the operator θ (see 2.2). The equation $\overline{\phi} - \overline{\phi}^p = \overline{\psi}$ shows that $\overline{\phi}$ is *integral* over $\overline{M}^0$, hence belongs to $\overline{M}^0$ since $\overline{M}^0$ is integrally closed (this is a consequence of Proposition 3, confer [1] for details); but this contradicts Lemme (416–11) in [14]. □

1.3.3 Relationship to p-adic modular forms

Concluding this chapter is a simple matter of showing that we can effortlessly switch between lifting a modular form modulo p^m to a Serre p-adic modular form and lifting to a classical modular form with integer coefficients:

Proposition 7. *Let $\widetilde{\mathbf{V}}(\mathbb{Z}_p, k, N)$ be the q-expansion in the $\mathbb{Q}_p$-module $\mathbf{V}(\mathbb{Z}_p, k, N)$ with p-integral coefficients, whose coefficients have been reduced modulo p^m. Then $\widetilde{\mathbf{V}}(\mathbb{Z}_p, k, N) = \widetilde{M_k}(N)$.*

Proof First observe that every classical modular form f with level N and rational coefficients can be interpreted as a p-adic modular form; the constant sequence $f, f, f, \ldots$ indeed converges to f. Hence, $\widetilde{\mathbf{V}}(\mathbb{Z}_p, k, N) \supseteq \widetilde{M_k}(N)$.

Now for any $f \in \mathbf{V}(\mathbb{Z}_p, k, N)$ with p-integral coefficients, there exists a sequence (f_i) of modular forms with level N and rational coefficients converging to f. In particular, this means there exists $\ell \in \mathbb{Z}_{>0}$ such that for all $i \geqslant \ell$, we have $\nu_p(f - f_i) \geqslant m$. Hence, $\widetilde{f} = \widetilde{f_\ell}$ so that $\widetilde{\mathbf{V}}(\mathbb{Z}_p, k, N) \subseteq \widetilde{M_k}(N)$ (we can reduce the coefficients modulo p^m since they are p-integral). □

2 Theta operator for modular forms modulo p^m

2.1 Theta operator on p-adic modular forms

Proposition 8. *(Theorem 5 (a) in §2.1 of [15]). Let $f = \sum a_n q^n$ be a p-adic modular form of weight k and level N. The series*

$$\theta f := q\frac{df}{dq} = \sum n a_n q^n$$

is a p-adic modular form of weight $k+2$. Hence, we have a map $\theta : \mathbf{V}(\mathbb{Z}_p, k, N) \to \mathbf{V}(\mathbb{Z}_p, k+2, N)$.

Before proving this, we define the ∂ operator:

Proposition 9. *Let $f \in M_k(\Gamma, \mathbb{Q})$, where Γ is an arithmetic group containing $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then*

$$\partial f := 12\theta f - kPf \in M_{k+2}(\Gamma, \mathbb{Q}).$$

Proof Clearly ∂f has a q-expansion with rational coefficients and satisfies the growth condition, as the q-expansions for θf, P and f are all rational and satisfy the growth condition. To show ∂f has weight $k+2$, consider any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $\gamma z := \frac{az+b}{cz+d}$, and observe that

$$
\begin{aligned}
(\partial f)(\gamma z) &= 12(\theta f)(\gamma z) - kP(\gamma z)f(\gamma z) \\
&= 12\left(q\frac{d}{dq}(f)\right)(\gamma z) - kP(\gamma z)(cz+d)^k f(z) && \text{(since } f \text{ has weight } k\text{)} \\
&= \frac{6}{\pi i}f'(\gamma z) - kP(\gamma z)(cz+d)^k f(z) && \text{(since } \tfrac{d}{dz} = \tfrac{dq}{dz}\cdot\tfrac{d}{dq} = 2\pi i q\tfrac{d}{dq}\text{)} \\
&= \frac{6}{\pi i}f'(\gamma z) - k\left((cz+d)^2P(z) - \frac{6ic}{\pi}(cz+d)\right)(cz+d)^k f(z) && \text{(from 2)} \\
&= \frac{6}{\pi i}\cdot\frac{\frac{d}{dz}\left(f(\gamma z)\right)}{\frac{d}{dz}\left(\gamma z\right)} - k\left((cz+d)P(z) + \frac{6c}{\pi i}\right)(cz+d)^{k+1}f(z) \\
& && \text{(by the chain rule)} \\
&= \frac{6}{\pi i}\cdot\frac{\frac{d}{dz}\left((cz+d)^k f(z)\right)}{\frac{ad-bc}{(cz+d)^2}} - k\left((cz+d)P(z) + \frac{6c}{\pi i}\right)(cz+d)^{k+1}f(z) \\
&= \frac{6}{\pi i}\cdot(cz+d)^2\left(ck(cz+d)^{k-1}f(z) + (cz+d)^k f'(z)\right) \\
&\quad - k\left((cz+d)P(z) + \frac{6c}{\pi i}\right)(cz+d)^{k+1}f(z) && \text{(since } ad-bc=1\text{)} \\
&= \frac{6}{\pi i}(cz+d)^{k+2}f'(z) - k(cz+d)^{k+2}P(z)f(z) \\
&= (cz+d)^{k+2}\cdot 12\left(q\frac{d}{dq}(f)\right)(z) - (cz+d)^{k+2}kP(z)f(z) \\
&= (cz+d)^{k+2}\left(12(\theta f)(z) - kP(z)f(z)\right) \\
&= (cz+d)^{k+2}(\partial f)(z),
\end{aligned}
$$

as claimed. □

Proof of Proposition 8 Let (f_i) be a sequence of modular forms with rational coefficients such that $\lim f_i = f$, and let k_i be the weight of f_i. Since P is a p-adic modular form of weight 2 (as a result of Proposition 12 or the identity (9)), ∂f_i can be interpreted as a p-adic modular form of weight $k+2$ (by Proposition 9) in the usual way, and f_i can be interpreted as a p-adic modular form in the usual way, it follows that

$$\theta f_i = \frac{1}{12}\partial f_i + \frac{k}{12}Pf_i$$

is a p-adic modular form of weight $k_i + 2$. So there exists a sequence $(h_{i,j})_j$ of classical modular forms of level N with rational coefficients satisfying $\lim_{j\to\infty} h_{i,j} = \theta f_i$.

Note that we can switch a limit on modular forms with the theta operator as it is coefficient-wise continuous. Hence, we use a diagonal argument to deduce

$$\lim h_{i,i} = \lim \theta f_i = \theta \lim f_i = \theta f.$$

So θf is indeed a p-adic modular form of weight $\lim(k_i + 2) = k + 2$. □

2.2 The construction for modular forms modulo p^m

Now we present the construction of the theta operator for modular forms modulo p^m with level N. Now that we have defined the theta operator for Serre p-adic modular forms, and have defined modular forms modulo p^m via p-adic modular forms, the construction is easy:

Definition 11. Abusing the same symbol as for p-adic modular forms, we define the map

$$\begin{aligned} \theta : \widetilde{M}(N) &\longrightarrow \widetilde{M}(N) \\ \widetilde{f} &\mapsto \widetilde{\theta f}, \end{aligned}$$

where we interpret $f \in M(N, \mathbb{Z})$ as a p-adic modular form in order to evaluate θf.

Observe that the lift $f \in M(N, \mathbb{Z})$ of $\widetilde{f}$ is not unique (see Lemma 2), so we need to justify this is indeed a well-defined operator. This is simple: if $f = \sum_{n=0}^{\infty} a_n q^n$ and $f' = \sum_{n=0}^{\infty} a'_n q^n$ are level N modular forms with integer coefficients, then

$$\begin{aligned} & f \equiv f' \pmod{p^m} \\ \iff & a_n \equiv a'_n \pmod{p^m} \ \forall \ n \\ \implies & n a_n \equiv n a'_n \pmod{p^m} \ \forall \ n \\ \iff & \theta f \equiv \theta f' \pmod{p^m}. \end{aligned}$$

2.3 Defining filtrations

For a modular form modulo p^m, it only makes sense to talk about weight with respect to its lifts to classical modular forms. Hence, we define filtrations:

Definition 12. For p a prime, $m \in \mathbb{Z}_{>0}$, and $f \in M(N, \mathbb{Z})$, the *p^m-filtration of f* (sometimes referred to as the *(weight) filtration*) is defined to be

$$w_{p^m}(f) := \inf\{k' \mid f \equiv g \pmod{p^m} \text{ for some } g \in M_{k'}(N, \mathbb{Z})\}.$$

Note that the filtration is defined above on $M(N, \mathbb{Z})$, but it is also a map $\widetilde{M}(N) \longrightarrow \mathbb{Z}$ by virtue of the fact that $f, g \in M(N, \mathbb{Z})$ satisfying $f \equiv g \pmod{p^m}$ implies $w_{p^m}(f) = w_{p^m}(g)$. Henceforth, we use $w_{p^m}(f)$ and $w_{p^m}(\widetilde{f})$ interchangeably. We say that a lift f of $\widetilde{f}$ (a modular form modulo p^m) has *exact* p^m-filtration if its weight is equal to its filtration $w_{p^m}(f) = w_{p^m}(\widetilde{f})$.

From this definition, if $m' \geqslant m \geqslant 1$ are integers and $f \in M_k(N, \mathbb{Z})$, then

$$w_{p^m}(f) \leqslant w_{p^{m'}}(f) \leqslant k.$$

Moreover, for $p \geqslant 5$ and $p \nmid N$ we must have $w_{p^m}(f) \equiv k \pmod{(p-1)p^{m-1}}$ by Theorem 1.

One can now ask the question: what effect does the theta operator θ have on the p^m-filtration of modular forms modulo p^m? Again working with $p \geqslant 5$ and $p \nmid N$, observe that for f of exact filtration k, we must have

$$w_{p^m}(\theta\widetilde{f}) \equiv k + 2 \pmod{p^{m-1}(p-1)}. \tag{1}$$

This is because $\theta\widetilde{f}$ lifts to a p-adic modular form of weight $k + 2$, meaning there must exist some modular form g with weight congruent to $k + 2$ modulo $p^{m-1}(p - 1)$ such that $\widetilde{g} = \widetilde{\theta f} = \theta\widetilde{f}$. The congruence (1) then follows from Theorem 1.

2.3.1 Theta operator on p-filtrations

Having defined the theta operator on modular forms modulo p^m as well as filtrations, we now have the tools to examine the effect of θ on filtrations. In the case $m = 1$, this effect is well known for all primes $p \geqslant 5$ and levels satisfying $p \nmid N$. This effect is described in the theorem below:

Theorem 2. *(See [12]). Let $p \geqslant 5$ be prime, $m = 1$, and $N \geqslant 1$ be an integer such that $p \nmid N$. If $f \in M_k(N, \mathbb{Z})$ has exact filtration (i.e. $w_p(f) = k$), then*

$$w_p(\theta \overline{f}) \leqslant w_p(f) + p + 1 \tag{2}$$

with equality if and only if $p \nmid w_p(f) = k$.

While Katz provides a proof for all levels N satisfying $p \nmid N$ in [12] using an algebraic-geometric approach, a few authors prove this result for level 1 (see Lemma 5 (ii) in [18], and Corollaire 3 in [14]). We present a sketch of the proof for level 1 with the aid of the following lemma:

Lemma 4. *For $p \geqslant 5$ prime, set $A := E_{p-1}$ and $B := E_{p+1}$. Then $\overline{B} = \overline{P}$ (here, $\overline{\cdot}$ denotes reduction of the coefficients modulo p as usual), and A and B are relatively prime inside $M(\Gamma_1)$.*

Proof We provide a sketch of the proof based on Serre's Corollaire 3 in [14]: the first part of the lemma follows from (8) with $m = 1$. Extending the definition of ∂ to modular forms modulo p in the same way as for θ, one can check directly that $\partial^2 A \equiv -E_4 A$ and $\partial^2 B \equiv -4E_6 A - E_4 B$. Recalling that $M(\Gamma_1)$ can be finitely generated by E_4 and E_6, we have that A and B are polynomials in E_4 and E_6 satisfying a second order system. Then Igusa shows in [9] that we must have A coprime to B since $\partial A \equiv B$, as claimed. □

Proof of Theorem 2 for $N = 1$. Recall that $\overline{E_{p-1}} = 1$ and $\overline{E_{p+1}} = \overline{E_2}$, so

$$\overline{\theta f} = \overline{\tfrac{1}{12}\left(kBf + A\partial f\right)}.$$

If $p \nmid k$, then $w_p(\theta f) = w_p(kBf + A\partial f)$. But A does not divide $kBf + A\partial f$ since A and B are coprime in $M(\Gamma_1)$, $\theta f \in M(\Gamma_1)$, and f has exact p-filtration. Now recall that two modular forms of level 1 are congruent modulo p if and only if they are equal up to some power of the Hasse invariant (see Lemma 5 (i) in [18]). Hence, $kBf + A\partial f$ has exact p-filtration, meaning $w_p(\theta \overline{f}) = k + p + 1$. In the case $p \mid k$, we have $w_p(\theta \overline{f}) = w_p(A\partial f) = w_p(\partial f) \leqslant k + 2 < k + p + 1$. □

2.4 Theta operator on p^m-filtrations for $m \geqslant 2$

One might expect the generalisation to $m \geqslant 1$ of the previous result to be

$$w_{p^m}(\theta\widetilde{f}) \leqslant w_{p^m}(f) + 2 + p^{m-1}(p-1)$$

with equality in some cases, by virtue of the fact that the lift of the Hasse invariant for modular forms modulo p^m has weight $p^{m-1}(p-1)$, and this inequality would match our result for $m = 1$. However, Chen and Kiming prove this is false:

Theorem 3. *(Theorem 1 (i),(iii) in [4]). Let $m \geqslant 2$ and $f \in M_k(N)$ with $p \geqslant 5$ prime and $p \nmid N$. Then*

$$w_{p^m}(\theta\widetilde{f}) \leqslant k + k(m),$$

where $k(m) := 2 + 2p^{m-1}(p-1)$. We have equality if $f \not\equiv 0 \pmod p$, $p \nmid k$ and $w_p(f) = k$.

Observe that this statement is weaker than the result for $m = 1$ both in the sense that we bound above by the weight and not the filtration, and further, that equality does not guarantee $f \not\equiv 0 \pmod p$, $p \nmid k$ and $w_p(f) = k$. We present the proof of this theorem in two parts: first we prove the inequality, and then the conditional equality.

2.4.1 Finding the upper bound for the p^m-filtration

Proving the inequality in Theorem 3 is the same as proving that θ is a map

$$\theta : \widetilde{M}_k(N) \longrightarrow \widetilde{M}_{k+k(m)}(N). \tag{3}$$

In this subsection, we present the proof of this fact, breaking down all components of the proof successively:

By Proposition 12, we have

$$E_2 \equiv -24 \sum_{j=0}^{m-1} p^j f_j \pmod{p^m} \tag{4}$$

where for all j, $f_j \in M_{k_j}(N, \mathbb{Q})$ satisfies $\nu_p(f_j) = 0$, and

$$k_j := \begin{cases} 2 + p^{m-j-1}(p^{j+1} - 1) & \text{for } j = 0, \ldots, m-2 \\ p^{m-1}(p+1) & \text{for } j = m-1 \end{cases}. \tag{5}$$

The point here is that $k(m) - k_j$ is a (positive integer) multiple t_j of $(p-1)p^{m-j-1}$, which is the weight of the lift of the Hasse invariant for modular forms modulo p^{m-j}; indeed, we compute

$$t_j = \begin{cases} p^j - p^{j-1} - \cdots - p - 1 & \text{for } j = 0, \ldots, m-2 \\ p^{m-1} - 2p^{m-2} - \cdots - 2p - 2 & \text{for } j = m-1 \end{cases}. \tag{6}$$

So we can adjust each summand in (4) by some power of $E_{p-1}^{p^{m-j-1}}$ to make its weight equal to $k(m)$. This gives rise to the congruence

$$\theta f \equiv \frac{1}{12} E_{p-1}^{2p^{m-1}} \partial f - 2kf \sum_{j=0}^{m-1} p^j E_{p-1}^{p^{m-j-1} t_j} f_j \pmod{p^m}. \tag{7}$$

The RHS has weight $k + k(m)$, so we indeed have the map (3) as claimed.

To see how Chen and Kiming arrive at Proposition 12, we proceed in steps:

Definition 13. The V operator is defined on formal q-expansions as

$$\left(\sum a_n q^n\right) \mid V := \sum a_n q^{np}.$$

If the q-expansion input represents a modular form f, then the operator acts by $(f \mid V)(z) = f(pz)$.

Proposition 10. *(Corollary 2 in [4]). For all even $k \geqslant 4$, define $G_k := \frac{-B_k}{2k} E_k$, where B_k is the k^{th} Bernoulli number. We claim that as q-expansions,*

$$G_2 \equiv \sum_{j=0}^{m-1} p^j \cdot (G_{2+p^{m-j-1}(p-1)} \mid V^j) \pmod{p^m}, \tag{8}$$

where $m \geqslant 1$ and $p \geqslant 5$ is prime.

Before proving this result, we need the famous Kummer congruences for Bernoulli numbers (proved in many places, such as in [3]):

Lemma 5. *(Kummer 1851). For $\ell \geqslant 1$, we*

$$(1-p^{u-1})\frac{B_u}{u} \equiv (1-p^{v-1})\frac{B_v}{v} \pmod{p^\ell}$$

whenever u and v are positive even integers not divisible by $p-1$ satisfying $u \equiv v \pmod{\varphi(p^\ell)}$.

Proof of Proposition 10 In [4], Chen and Kiming prove this result using the identity

$$G_k = G_k^* \mid (1-p^{k-1}V)^{-1} = G_k^* + p^{k-1}G_k^* \mid V + \cdots + p^{m(k-1)}G_k^* \mid V^m + \cdots \tag{9}$$

at the end of Serre's proof of Théorème 4 in [15] (here G_k^* denotes the p-adic limit $G_{k_i} \to G_k^*$), together with some congruences proved by Serre in [14]. We present an alternative, direct proof that is mostly self-contained.

First recall that

$$G_k := -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n, \tag{10}$$

where $\sigma_t(n) := \sum_{d|n} d^t$ is the divisor sum. Suppose $n \geqslant 1$ with $n = p^\ell x$ where $\gcd(x,p) = 1$. Then $a_n(G_2)$ (i.e. the coefficient of q^n on the LHS of (8)) is given by $\sigma_1(n)$, and the coefficient of q^n on the RHS of (8) is given by

$$a_n\left(\sum_{j=0}^{m-1} p^j \cdot (G_{2+p^{m-j-1}(p-1)} \mid V^j)\right) = \sum_{j=0}^{\min\{m-1,\ell\}} p^j \sigma_{1+p^{m-j-1}(p-1)}(p^{\ell-j}x). \tag{11}$$

Note that for all $0 \leqslant j \leqslant m-1$, $\gcd(x, p^{m-j}) = 1$ and $\varphi(p^{m-j}) = p^{m-j-1}(p-1)$, where φ is the totient function. So using Euler's theorem and the definition of σ_t, we have

$$\begin{aligned} \sigma_{1+p^{m-j-1}(p-1)}(x) &\equiv \sigma_1(x) \pmod{p^{m-j}} \\ \implies p^j\sigma_{1+p^{m-j-1}(p-1)}(x) &\equiv p^j\sigma_1(x) \pmod{p^m}. \end{aligned}$$

Moreover, from the fact that p is prime, we have $\sigma_t(p^i x) = \sigma_t(x) + p^t\sigma_t(x) + \cdots + p^{it}\sigma_t(x)$. Putting these facts together, we get that the RHS of (11) is congruent to $S_\ell\sigma_1(x)$ modulo p^m, where

$$S_\ell := \sum_{j=0}^{\min\{m-1,\ell\}} p^j \sum_{i=0}^{\ell-j} p^{i(1+\varphi(p^{m-j}))}.$$

Observe that $\varphi(p^{m-j}) > p^{m-j-1} \geqslant m-j-1$ so that for $i \geqslant 1$,

$$j + i(1+\varphi(p^{m-j})) \geqslant j + 1 + \varphi(p^{m-j}) > j + 1 + m - j - 1 = m.$$

So only the $i = 0$ summand of the inner sum in the above expression for S_ℓ contributes modulo p^m. Thus the RHS of (11) further simplifies to

$$S_\ell \sigma_1(x) \equiv \left(\sum_{j=0}^{\min\{m-1,\ell\}} p^j \right) \sigma_1(x) \equiv \left(\sum_{j=0}^{\ell} p^j \right) \sigma_1(x) = \sigma_1(n) \pmod{p^m},$$

which is indeed $a_n(G_2)$. It remains to show

$$a_0(G_2) \equiv a_0 \left(\sum_{j=0}^{m-1} p^j \cdot (G_{2+p^{m-j-1}(p-1)} \mid V^j) \right) \pmod{p^m}$$
$$\iff \frac{-B_2}{4} \equiv \sum_{j=0}^{m-1} p^j \left(\frac{-B_{2+\varphi(p^{m-j})}}{2(2+\varphi(p^{m-j}))} \right) \pmod{p^m}.$$

We achieve this using Kummer's congruences; clearly $p-1 \nmid 2 + \varphi(p^{m-j})$ since $p \geqslant 5$. So for each $j = 0, \ldots, m-1$, we have

$$\begin{aligned} \frac{-B_{2+\varphi(p^{m-j})}}{2+\varphi(p^{m-j})} &\equiv \frac{1-p}{1-p^{1+\varphi(p^{m-j})}} \cdot \frac{-B_2}{4} \pmod{p^{m-j}} \\ &\equiv (1-p) \cdot \frac{-B_2}{4} \pmod{p^{m-j}}, \qquad (\text{since } 1+\varphi(p^{m-j}) \geqslant m-j) \end{aligned}$$

yielding

$$\sum_{j=0}^{m-1} p^j \left(\frac{-B_{2+\varphi(p^{m-j})}}{2(2+\varphi(p^{m-j}))} \right) \equiv \sum_{j=0}^{m-1} p^j (1-p) \cdot \frac{-B_2}{4} \equiv (1-p^m) \cdot \frac{-B_2}{4} \equiv \frac{-B_2}{4} \pmod{p^m},$$

as claimed. □

Lemma 6. *Let* $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$*; then* $f \mid V \in M_k(\Gamma_0(p))$*.*

Proof As explained in §3.1 of [15], to show $f \mid V \in M_k(\Gamma_0(p))$, we must show that $f|_k\,\gamma = f$ for all $\gamma \in \Gamma_0(p)$, and that f is holomorphic at the two cusps 0 and $i\infty$ of $\Gamma_0(p)$. Indeed, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$ we have

$$\begin{aligned}
\left((f \mid V)|_k\,\gamma\right)(z) &= (cz+d)^{-k} f\left(p \cdot \frac{az+b}{cz+d}\right) \\
&\qquad\qquad \text{(since } \det(\gamma) = 1 \text{ and } (f \mid V)(z) = f(pz)) \\
&= \left(\tfrac{c}{p}(pz) + d\right)^{-k} f\left(\frac{a(pz)+pb}{\frac{c}{p}(pz)+d}\right) \\
&= f(pz) \qquad \text{(since } f \in M_k(\mathrm{SL}_2(\mathbb{Z}) \text{ and } \left(\begin{smallmatrix} a & pb \\ c/p & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})) \\
&= (f \mid V)(z),
\end{aligned}$$

so the first condition holds.

As for the second condition, let $f = \sum_{n=0}^{\infty} a_n q^n$, and note from §3.1 of [15] that $f \mid V$ is holomorphic at the cusps of $\Gamma_0(p)$ if and only if the series

$$f \mid V = \sum_{n=0}^{\infty} a_n q^{pn} \quad \text{and} \quad (f \mid V)|_k\,W = \sum_{n=0}^{\infty} b_n q^n$$

converge for all $|q| < 1$, where $W := \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$. The first series indeed converges since $|q^p| < 1$ and $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. As for the second series, observe that

$$\begin{aligned}
(f \mid V)|_k\,W &= \left(p^{-k/2}\, f|_k \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\Big|_k \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right) \\
&= p^{-k/2}\, f|_k \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right) \\
&= p^{-k/2}\, f|_k \left(\begin{smallmatrix} 0 & -p \\ p & 0 \end{smallmatrix}\right) \\
&= p^{-k/2}\, f|_k \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right) \\
&= p^{-k/2}\, f|_k \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right) \qquad \text{(since } \det\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = 1 \text{ and } f \in M_k(\mathrm{SL}_2(\mathbb{Z}))) \\
&= p^{-k/2} \left(\det\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)\right)^{k/2} p^{-k} f \\
&= p^{-k/2} \left(p^2\right)^{k/2} p^{-k} f \\
&= p^{-k/2} f,
\end{aligned}$$

so $b_n = p^{-k/2} a_n$, and this series converges (again because $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$). □

Lemma 7. *Let* Tr *denote the trace from* $\Gamma_0(p)$ *to* $\mathrm{SL}_2(\mathbb{Z})$*, defined by Serre in §3.2 of [15] to be*

$$\mathrm{Tr}(f) := \sum_{j=1}^{p+1} f|_k\, \gamma_j,$$

for $f \in M_k(\Gamma_0(p))$*, where* $\gamma_1, \ldots, \gamma_{p+1}$ *are representatives of the quotient* $\Gamma_0(p)\backslash \mathrm{SL}_2(\mathbb{Z})$*. Then* $\mathrm{Tr}(f)$ *does not depend on the choice of* γ_j*, and* $\mathrm{Tr}(f) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$*.*

Proof The fact that Tr does not depend on the choice of γ_j is an immediate consequence of Lemma 6; for another set of representatives $\gamma_1', \ldots, \gamma_{p+1}'$, we have (without loss of generality) that γ_j' is the same as γ_j up to some element $\alpha_j \in \Gamma_0(p)$. But $f|_k\, \alpha_j\gamma_j = f|_k\, \gamma_j$ by Lemma 6.

To show $\mathrm{Tr}(f) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$, first observe that if $[\gamma_j]$ denotes the coset represented by γ_j, then multiplying the representatives by any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ corresponds to permuting the cosets. In other words, $\{[\gamma_1], \ldots, [\gamma_{p+1}]\} = \{[\gamma_1\gamma], \ldots, [\gamma_{p+1}\gamma]\}$. So

$$\begin{aligned}
\mathrm{Tr}(f)(\gamma z) &= (cz+d)^k \sum_{j=1}^{p+1} \left(f|_k\, \gamma_j\gamma\right)(z) \quad (\text{since } g(\gamma z) = (cz+d)^k (g|_k\, \gamma)(z) \text{ for any } g) \\
&= (cz+d)^k \sum_{j=1}^{p+1} \left(f|_k\, \gamma_j\right)(z) \\
&\qquad\qquad (\text{Tr does not depend on the choice of representatives}) \\
&= (cz+d)^k \mathrm{Tr}(f)(z).
\end{aligned}$$

In order to show $\mathrm{Tr}(f)$ satisfies the growth condition, it now suffices to prove $\mathrm{Tr}(f)(z)$ is bounded as $\mathrm{im}(z) \to \infty$. But this follows immediately from the fact that the growth condition applies to $f \in M_k(\Gamma_0(p))$ (i.e., $(f|_k\, \gamma_j)(z)$ is bounded as $\mathrm{im}(z) \to \infty\ \forall\, j$). □

Remark. In Theorem 10 of §3.2 in [15], Serre uses the trace Tr from $\Gamma_0(p)$ to $\mathrm{SL}_2(\mathbb{Z})$ to prove that a modular form of weight k on $\Gamma_0(p)$ is in fact a p-adic modular form of weight k and level 1.

Proposition 11. *(Lemma 2 in [4]). Let* $f \in M_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q})$ *and suppose that* $\nu_p(f) = 0$*. Let* $t \in \mathbb{Z}_{>0}$ *and suppose that* $s \in \mathbb{Z}_{\geqslant 0}$ *is such that*

$$\inf\{s+1, p^s+1-k\} \geqslant t.$$

Then there is $h \in M_{k+p^s(p-1)}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q})$ with $\nu_p(h) = 0$ such that

$$f \mid V \equiv h \pmod{p^t}.$$

Proof The idea is to take $h := \mathrm{Tr}((f \mid V) \cdot (E_{p-1} - p^{p-1}(E_{p-1} \mid V))^{p^s})$. From Lemma 6, we know that $f \mid V \in M_k(\Gamma_0(p), \mathbb{Q})$ and $E_{p-1} - p^{p-1}(E_{p-1} \mid V) \in M_{p-1}(\Gamma_0(p), \mathbb{Q})$, so

$$(f \mid V) \cdot (E_{p-1} - p^{p-1}(E_{p-1} \mid V))^{p^s} \in M_{k+p^s(p-1)}(\Gamma_0(p)).$$

Then by Lemma 7, we have $h \in M_{k+p^s(p-1)}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q})$. Now clearly $\nu_p(f \mid V) = 0$ since $\nu_p(f) = 0$. Our goal is to use Lemme 9 in [15] to argue

$$\begin{aligned} \nu_p(h - (f \mid V)) &\geqslant \inf(s+1, p^s + 1 + \nu_p((f \mid V)|_k W) - k/2) \\ &= \inf(s+1, p^s + 1 - k) \\ &\geqslant t, \end{aligned}$$

where $W = \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$. Then $f \mid V \equiv h \pmod{p^t}$, and since $\nu_p(f \mid V) = 0$, we must have $\nu_p(h) = 0$, and we will be done.

For the inequality above to work, we just need $\nu_p((f \mid V)|_k W) = -k/2$. But from the proof of Lemma 6, we have $(f \mid V)|_k W = p^{-k/2} f$. Then since $\nu_p(f) = 0$ by assumption, we indeed have $\nu_p((f \mid V)|_k W) = -k/2$. □

Proposition 12. *(Proposition 1 in [4]). Let $m \in \mathbb{N}$. For $m \geqslant 2$, define the positive even integers $k_0, \ldots, k_{m-1}$ as in (5), with $k_0 := p+1$ if $m = 1$. Then $k_0 < \cdots < k_{m-1}$ and there are modular forms $f_0, \ldots, f_{m-1}$, depending only on p and m, of level one and of weights $k_0, \ldots, k_{m-1}$ respectively, that have rational q-expansions, satisfy $\nu_p(f_j) = 0$ for all j, and are such that*

$$G_2 = \frac{-E_2}{24} \equiv \sum_{j=0}^{m-1} p^j f_j \pmod{p^m}$$

as a congruence between q-expansions. Recall from (5) that

$$k_j := \begin{cases} 2 + p^{m-j-1}(p^{j+1} - 1) & \text{for } j = 0, \ldots, m-2 \\ p^{m-1}(p+1) & \text{for } j = m-1 \end{cases}. \tag{12}$$

Proof By Proposition 10, it suffices to prove there exist modular forms $f_0, \dots, f_{m-1}$ of weights $k_0, \dots, k_{m-1}$ with rational q-expansions and $\nu_p(f_j) = 0$ such that $f_j \equiv G_{2+p^{m-j-1}(p-1)} \mid V^j \pmod{p^{m-j}}$ for each $j = 0, \dots, m-1$. In the case $m = 1$, set $f_0 := G_{p+1}$. Then indeed $k_0 = p+1$, $f_0 \equiv G_{p+1} \mid V^0 \pmod{p^m}$, and $\nu_p(f_0) = 0$ (in fact $\nu_p(G_k) = 0$ for all k as the q^p-coefficient in the q-expansion is never divisible by p).

Henceforth, suppose $m \geqslant 2$. Set $f_{m-1} := G_{p+1}^{p^{m-1}}$, and observe that

$$f_{m-1} = G_{p+1}^{p^{m-1}} \equiv G_{p+1} \mid V^{m-1} \pmod{p},$$

and f_{m-1} has weight $k_{m-1} = p^{m-1}(p+1)$ (see §1.1.1). From the above congruence, we also conclude that $\nu_p(f_0) = 0$ (as $\nu_p(G_{p+1}) = 0$). Now suppose $j \leqslant m-2$: to find the required f_j, we prove (by induction) the existence of modular forms $f_{j,0}, \dots, f_{j,j}$, where $f_{j,r}$ has weight $2 + p^{m-j-1}(p^{r+1} - 1)$ and rational q-expansion with $\nu_p(f_{j,r}) = 0$ and

$$f_{j,r} \equiv G_{2+p^{m-j-1}(p-1)} \mid V^r \pmod{p^{m-j}}.$$

We then get the desired form by setting $f_j := f_{j,j}$. Now set $f_{j,0} := G_{2+p^{m-j-1}(p-1)}$, and observe that $\nu_p(f_{j,0}) = 0$ so that $f_{j,0}$ clearly satisfies the required properties. Now suppose $f_{j,r}$ satisfies the required properties for some $0 \leqslant r < m-2$. Set $s = m-j+r$, $t = m-j$ and $k = 2 + p^{m-j-1}(p^{r+1} - 1)$, and observe that

$$s + 1 = p^{m-j+r} + 2 > p^{m-j} > m - j = t$$

and also that

$$p^s + 1 - k = p^{m-j+r} - p^{m-j-1}(p^{r+1} - 1) - 1 = p^{m-j-1} - 1 \geqslant m - j = t,$$

where the last inequality follows from the fact that $p > 2$ and $m - j \geqslant 2$. So by Proposition 11 ($h = f_{j,r}$ in the language of the proposition), there exists a modular form $f_{j,r+1}$ with rational q-expansion and $\nu_p(f_{j,r+1}) = 0$ such that

$$f_{j,r+1} \equiv f_{j,r} \mid V \equiv G_{2+p^{m-j-1}(p-1)} \mid V^{r+1} \pmod{p^{m-j}},$$

where $f_{j,r+1}$ has weight

$$\begin{aligned} k + p^s(p-1) &= 2 + p^{m-j-1}(p^{r+1}-1) + p^{m-j+r}(p-1) \\ &= 2 + p^{m-j-1}(p^{r+1}-1+p^{r+1}(p-1)) \\ &= 2 + p^{m-j-1}(p^{r+2}-1). \end{aligned}$$

This completes the induction. □

Remark. One could naïvely ask why we cannot take $f_j = G_{2+p^{m-j-1}(p^{j+1}-1)}$ for $0 \leqslant j < m-1$. In other words, why do we not (in general) have

$$G_{2+p^{m-j-1}(p-1)} \mid V^j \equiv G_{2+p^{m-j-1}(p^{j+1}-1)} \pmod{p^{m-j}}?$$

Using the formula for the q-expansion of G_k given in (10) and Euler's Theorem, it turns out that

$$\begin{aligned} a_n\left(G_{2+p^{m-j-1}(p-1)} \mid V^j\right) &= \sigma_{1+p^{m-j-1}(p-1)}(n/p^j) \\ &\equiv \sigma_{1+p^{m-j-1}(p^{j+1}-1)}(n) \pmod{p^{m-j}} \\ &= a_n\left(G_{2+p^{m-j-1}(p^{j+1}-1)}\right) \end{aligned}$$

for all n with $p^j \mid n$, lending credence to this hope. However, we do not in general have

$$a_n\left(G_{2+p^{m-j-1}(p^{j+1}-1)}\right) \equiv 0 \pmod{p^{m-j}}$$

whenever $p^j \nmid n$. Take for example $n = 1$, and the q^1-coefficient of $G_{2+p^{m-j-1}(p^{j+1}-1)}$ is just equal to 1.

2.4.2 When the upper bound on the p^m-filtration is achieved

To complete the proof of Theorem 3, we must show that if $f \in M_k(N)$ with $f \not\equiv 0 \pmod{p}$, $p \nmid k$ and $w_p(f) = k$, then $w_{p^m}(\theta \widetilde{f}) = k + k(m)$. Suppose for the sake of contradiction that $w_{p^m}(\theta \widetilde{f}) = k' < k + k(m)$. So there exists some $g \in M_{k'}(N)$ such that $\theta f \equiv g \pmod{p^m}$. Then by Theorem 1, we have $k + k(m) = k' + t \cdot p^{m-1}(p-1)$ for some $t \geqslant 1$. Set

$$h := E_{p-1}^{p^{m-1}(t-1)} g$$

so that

$$\theta f \equiv E_{p-1}^{p^{m-1}} h \pmod{p^m}.$$

Substituting this into (7), we get the congruence

$$2kp^{m-1}E_{p-1}^{t_{m-1}} f_{m-1} f \equiv -E_{p-1}^{p^{m-1}} h + \frac{1}{12} E_{p-1}^{2p^{m-1}} \partial f - 2kf \sum_{j=0}^{m-2} p^j E_{p-1}^{p^{m-j-1} t_j} f_j \pmod{p^m}. \tag{13}$$

Observe that

$$t_{m-1} = p^{m-1} - 2p^{m-2} - \cdots - 2p - 2 < p^{m-1}$$

and also

$$\begin{aligned} t_{m-1} &= p^{m-1} - 2p^{m-2} - \cdots - 2p - 2 \\ &< p^{m-1} - p^{m-2} - \cdots - p^{m-j-1} \\ &= p^{m-j-1}(p^j - p^{j-1} - \cdots - 1) \\ &= p^{m-j-1} t_j \end{aligned}$$

for all $j = 0, 1, \ldots, m-2$. So we can factor out $E_{p-1}^{t_{m-1}+1}$ on the RHS of (7), and also divide both sides by $2k$ (since $p \nmid k$ and p is odd) to deduce

$$p^{m-1} E_{p-1}^{t_{m-1}} f_{m-1} f \equiv E_{p-1}^{t_{m-1}+1} h' \pmod{p^m}$$

for some $h' \in M_{k+k(m)-(p-1)(t_{m-1}+1)}(N)$. Thus

$$E_{p-1}^{t_{m-1}} f_{m-1} f \equiv E_{p-1}^{t_{m-1}+1} h'' \pmod{p}$$

for $h'' := h'/p^{m-1}$. It follows that

$$\begin{aligned} w_p(f_{m-1} f) &< k + k(m) - t_{m-1}(p-1) \\ &= k + 2 + 2p^{m-1}(p-1) - (3p^{m-1} - 2(p^{m-1} + p^{m-2} + \cdots + 1))(p-1) \\ &= k + 2 + 2p^{m-1}(p-1) - 3p^{m-1}(p-1) + 2(p^m - 1) \\ &= k + p^m + p^{m-1} \\ &= k + p^{m-1}(p+1). \end{aligned}$$

Now recall from the proof of Proposition 12 that $f_{m-1} = G_{p+1}^{p^{m-1}}$. As $G_{p+1} = -(B_{p+1}/2(p+1))E_{p+1}$ with $B_{p+1}/2(p+1)$ invertible modulo p, we deduce

$$w_p(E_{p+1}^{p^{m-1}} f) < k + p^{m-1}(p+1). \tag{14}$$

However, as $w_p(f) = k \neq p$ (as $p \nmid k$), this contradicts Lemma 8:

Lemma 8. *(Lemma 3 in [4]). Suppose that $p \neq \kappa \in \mathbb{Z}_{>0}$ and that $0 \neq \phi \in \widetilde{M}_\kappa(N)$ with $w_p(\phi) = \kappa$. Then, for $a \in \mathbb{Z}_{>0}$,*

$$w_p(E_{p+1}^a \phi) = w_p(\phi) + a(p+1).$$

Proof Suppose $w_p(E_{p+1}\phi) < \kappa + p + 1$; then $E_{p+1}\phi \equiv E_{p-1}\psi \pmod p$ for some $\psi \in M_{2+\kappa}(N)$. By Theorem 4, we can read this congruence as an equality of forms in $M_{\kappa+p+1}^{\text{Katz}}(N, \mathbb{F}_p)$. By the remark after Lemma 1 in [12], $\widetilde{E}_{p-1}$ and $\widetilde{E}_{p+1}$ do not share any zeros. Hence, $\widetilde{\phi}$ vanishes at every zero of $\widetilde{E}_{p-1}$ to at least the order that $\widetilde{E}_{p-1}$ vanishes at that zero. Thus $\kappa \geqslant p-1$ and $\widetilde{\phi} = \widetilde{E}_{p-1}\widetilde{\eta}$ for some $\widetilde{\eta} \in M_{\kappa-(p-1)}^{\text{Katz}}(N, \mathbb{F}_p)$. By Theorem 4, we have a corresponding lift $\eta \in M_{\kappa-(p-1)}(N, \mathbb{Z})$. But then $w_p(\phi) \leqslant \kappa - (p-1)$, a contradiction. □

Remark. Note that if Lemma 8 requires only $p \nmid \kappa$, then there is a simple proof for the case $a = 1$: suppose $w_p(E_{p+1}\phi) < \kappa + p + 1$; then $E_{p+1}\phi \equiv E_{p-1}\psi \pmod p$ for some $\psi \in M_{2+\kappa}(N)$. Then by Theorem 2, we have

$$\begin{aligned}
\kappa + p + 1 &= w_p(\theta\widetilde{\phi}) \\
&= w_p(\kappa \widetilde{E}_2 \widetilde{\phi} + \widetilde{E}_{p-1}\widetilde{\partial\phi}) \\
&= w_p(\kappa \widetilde{E}_{p+1} \widetilde{\phi} + \widetilde{E}_{p-1}\widetilde{\partial\phi}) \\
&= w_p(\kappa \widetilde{E}_{p-1} \widetilde{\psi} + \widetilde{E}_{p-1}\widetilde{\partial\phi}) \\
&= w_p(\kappa\psi + \partial\phi) \\
&\leqslant \kappa + 2 \qquad \text{(since } \psi, \partial\phi \in M_{2+\kappa}(N))
\end{aligned}$$

a contradiction.

This argument does not lead to an unrestricted induction though, and only gives the result up to $a \equiv -\kappa \pmod p$, as for such a value of a we have $p \mid w_p(E_{p+1}^a\phi)$. This value of a is necessarily less than p, and so cannot be used to argue a contradiction in

(14) even for $m = 2$.

We conclude this section by commenting on some effects of ∂ on the p^m-filtration:

Proposition 13. *Let $m \geqslant 2$ and $f \in M_k(N)$ with $p \geqslant 5$ prime, $p \nmid N$, and $w_p(f) = k \neq p$. Then ∂f has exact p^m-filtration if $k < \varphi(p^m)$. If $p^m \mid k$ (but there is no bound on k), then $w_{p^m}(\partial f) = w_{p^m}(\theta \widetilde{f})$, and if also $a_{np}(f) \equiv 0 \pmod{p^m}$ for all $n \in \mathbb{Z}_{>0}$, then $w_p(\partial f) \geqslant k - (\varphi(p^m) - 1)(p + 1)$.*

Proof By Theorem 1, we have $w_{p^m}(\partial f) = k + 2 - t \cdot \varphi(p^m)$ for some $t \in \mathbb{Z}_{\geqslant 0}$. So if $k < \varphi(p^m)$, we must have $t = 0$ to keep $w_{p^m}(f) > 1$. If $p^m \mid k$, then $\partial \widetilde{f} = \theta \widetilde{f}$ so that indeed $w_{p^m}(\partial f) = w_{p^m}(\theta \widetilde{f})$.

If also $a_{np}(f) \equiv 0 \pmod{p^m}$ for all $n \in \mathbb{Z}_{>0}$, then $f \equiv \theta^{\varphi(p^m)} f \pmod{p^m}$ as q-expansions by Euler's Theorem. Since applying θ adds at most $p + 1$ to the p-filtration by Theorem 2, we have

$$w_p(\partial f) = w_p(\theta \widetilde{f}) \geqslant k - (\varphi(p^m) - 1)(p + 1),$$

as claimed. □

The observation that $f \equiv \theta^{\varphi(p^m)} f \pmod{p^m}$ in the last proof naturally heralds the theory developed in the following section.

3 Theta cycles

Let $f \in M_k(N, \mathbb{Z})$ so that $\overline{f} \in \overline{M}_k(N) = M_k(N, \mathbb{F}_p)$, and suppose we have $a_{np}(\overline{f}) = 0$ for all $n \geqslant 1$. We can write $f = \sum a_n q^n$ for $a_n \in \mathbb{F}_p$; recall the effect of θ on f is given by $\sum a_n q^n \mapsto n a_n q^n$. Then by Fermat's Little Theorem, we must have $\theta^{p-1} f \equiv f \pmod{p}$, which implies $\theta^{p-1} f = A^\ell f$ for some $\ell \geqslant 0$, where $A = E_{p-1}$ is the lift of the Hasse invariant. Thus if we remove the restriction on the coefficients of $\overline{f}$, we still always have

$$\theta^p \overline{f} = \theta \overline{f}$$

as an equality of q-expansions for some $\ell \geqslant 0$. This means $w_p(\theta^p \overline{f}) = w_p(\theta \overline{f})$, which naturally leads us to the notion of a *theta cycle*.

Definition 14. Let $\overline{f} \in \overline{M}_k(N)$ be a modular form modulo p. The *p-filtration theta cycle of $\overline{f}$ (or of f)* is the tuple

$$\left(w_p(\theta \overline{f}), w_p(\theta^2 \overline{f}), \ldots, w_p(\theta^{p-1} \overline{f})\right) \in \mathbb{Z}^{p-1}.$$

3.1 Classifying p-filtration theta cycles

Here we present some results on the classification of p-filtration theta cycles by first presenting a weak but general classification, and then stating a stronger but less general classification due to Edixhoven (see Proposition 3.3 in [6]).

Proposition 14. *Let $p \geqslant 5$ be prime and $N \geqslant 1$ be coprime to p. Let $f \in M_k(N, \mathbb{Z})$ so that $\overline{f} \in \overline{M}_k(N) = M_k(N, \mathbb{F}_p)$. Let $x := w_p(\theta \overline{f})$, and let $t \in \{1, \ldots, p\}$ be the unique integer for which $x - t \equiv 0 \pmod{p}$. Then the p-filtration theta cycle of $\overline{f}$ is given by either*

$$\begin{cases} (x, \ldots, x + (p-2)(p+1)) & \text{if } t = 2 \\ (x, \ldots, x + (p-t)(p+1), x + (2-t)(p+1), \ldots, x - (p+1)) & \text{if } t > 2 \end{cases}$$

(where all omitted terms are given by successive increases of $p+1$), or

$$\big(x, \ldots, x + (p-t)(p+1), x + (p-t+1)(p+1) - (p-c_1)(p-1), \ldots,$$
$$x + (p-t+c_1)(p+1) - (p-c_1)(p-1), x + (2-t+c_1)(p+1), \ldots, x - (p+1)\big)$$

for some integer $1 \leqslant c_1 \leqslant p-2$*. Also,* $t=1$ *does not occur, and* $t=2$ *forces the first case.*

Proof By Theorem 2, we have $w_p(\theta f) = k+p+1$. Let

$$n := \#\{1 \leqslant i \leqslant p-1 \mid w_p(\theta^{i+1} f) - w_p(\theta^i f) \neq p+1\}$$

be the number of "drops" in the theta cycle, where clearly $n \geqslant 1$. Then let $\theta^{r_1} f, \dots, \theta^{r_n} f$ (where $1 \leqslant r_1 < \cdots < r_n \leqslant p-1$) be the modular forms whose filtrations occur *before* a drop in the theta cycle. So if $f_i := \theta^{r_i} f$ for all $1 \leqslant i \leqslant n$, we have $w_p(\theta f_i) - w_p(f_i) \neq p+1$. Note that by again by Theorem 2, $f_1, \dots, f_n$ are the only modular forms whose filtrations in the theta cycle are divisible by p, and moreover,

$$w_p(\theta f_i) - w_p(f_i) = p+1-b_i(p-1)$$

for some $b_i \geqslant 1$. For $1 \leqslant i \leqslant p-1$, let

$$c_i := r_{i+1} - r_i,$$

where $r_{n+1} := p-1+r_1$. Then $c_i - 1 \geqslant 0$ is the number of increases by $p+1$ between the i^{th} and $(i+1)^{\text{th}}$ drops in the theta cycle. The cycle is of length $p-1$, and indeed we have

$$\sum_{i=1}^{n} c_i = p-1.$$

Also, we fall as far as we rise, so

$$\sum_{i=1}^{n} b_i(p-1) = \sum_{i=1}^{n} c_i(p+1) = (p-1)(p+1) \Longrightarrow \sum_{i=1}^{n} b_i = p+1.$$

After a fall, we cannot fall again until reaching the next filtration divisible by p. So for all i,

$$c_i + b_i \equiv c_i(p+1) - b_i(p-1) \equiv 0 \pmod{p}.$$

But $c_i + b_i > 0$ for all i, and also

$$\sum_{i=1}^{n} c_i + b_i = \sum_{i=1}^{n} c_i + \sum_{i=1}^{n} b_i = p-1+p+1 = 2p,$$

so this forces two cases: either $c_1 + b_1 = 2p$ (with $n = 1$), or $c_1 + b_1 = c_2 + b_2 = p$ (with $n = 2$).

Case 1: one fall. Then $c_1 + b_1 = 2p \implies c_1 = p - 1, b_1 = p + 1$. This corresponds to the first case in the proposition. The following diagram depicts the resultant theta cycle:

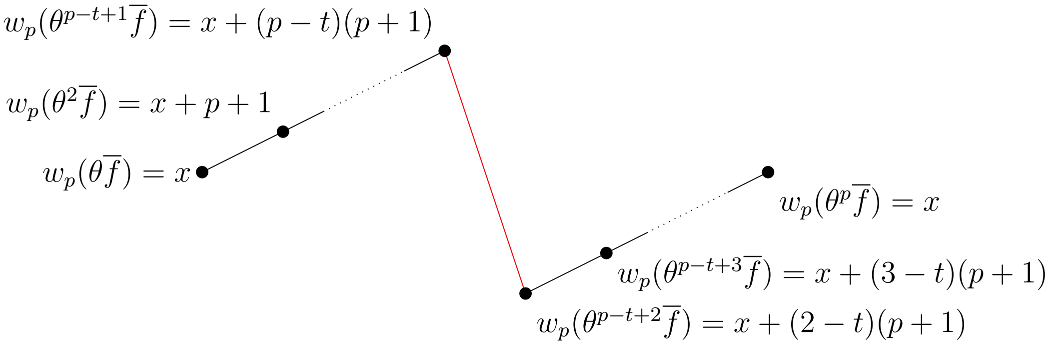

Figure 1: A graphical depiction of the p-filtration theta cycle in the case of one fall with $t > 2$. Black lines denote a rise of $p + 1$ and red lines denote a fall.

Case 2: two falls. Then $c_1 + b_1 = c_2 + b_2 = p$, which implies $b_1 = p - c_1$, $c_2 = p - 1 - c_1$ and $b_2 = c_1 + 1$. This corresponds to the second case in the proposition. The resultant theta cycle is depicted below:

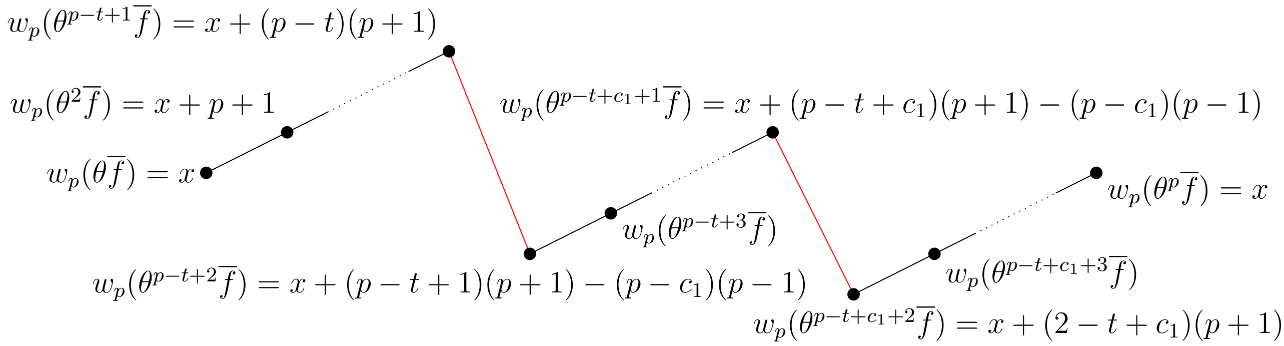

Figure 2: A graphical depiction of the p-filtration theta cycle in the case of two falls with $t - 1 > c_1 > 1$. Black lines denote a rise of $p + 1$ and red lines denote a fall.

Clearly $t = 2$ cannot occur in this case, as otherwise there would be no falls after the first $p - 2$ filtrations in the theta cycle. Now suppose for the sake of contradiction that $t = 1$ does occur in either case; observe that $x + \ell(p + 1) \not\equiv 0 \pmod p$ for all

$0 \leqslant \ell \leqslant p-2$. So by Theorem 2,

$$w_p(\theta^p \overline{f}) = x + (p-1)(p+1) \neq x = w_p(\theta \overline{f}),$$

a contradiction. So indeed, $t = 1$ does not occur. □

For small weights and certain eigenvalues, it is possible to restrict the b_i and obtain a more precise classification of the theta cycles in the second case:

Corollary 1. *(Proposition 3.3 in [6]). Let $p \geqslant 5$ be prime, and let f be a cuspidal eigenform of type (N, k, ε) where $p \nmid N$ and $1 \leqslant k \leqslant p+1$. That is, $a_0(f) = 0$, and f is a modular form of weight k and level N that is an eigenvector for all Hecke operators T_ℓ (see, for example, in §2.1 of [15]), with corresponding eigenvalues λ_ℓ. Suppose that f also has exact p-filtration. Then the possible theta cycles of $\overline{f}$ are given as follows: if $\lambda_p = 0$, then the cycles are*

$$\begin{cases} (p+2, \dots, p+2+(p-2)(p+1)) & \text{if } k = 1 \\ (2+p+1, \dots, 2+(p-2)(p+1), 2) & \text{if } k = 2 \\ (k+p+1, \dots, k+(p-k)(p+1), k_1, \dots, k_1+(k-3)(p+1), k) & \text{if } 3 \leqslant k \leqslant p-1 \\ (3, \dots, 3+(p-3)(p+1), p) & \text{if } k = p \\ \text{does not occur} & \text{if } k = p+1 \end{cases}$$

where $k_1 := p+3-k$, and if $\lambda_p \neq 0$, then the cycles are

$$\begin{cases} (p+2, \dots, p+2+(p-2)(p+1)) & \text{if } k = 1 \\ (k+p+1, \dots, k+(p-k)(p+1), k'+p+1, \dots, k'+(k-1)(p+1)) & \text{if } 2 \leqslant k \leqslant p-1 \\ (p+2, \dots, p+2+(p-2)(p+1)) & \text{if } k = p \\ (2p+2, \dots, 2p+2+(p-2)(p+1)) & \text{if } k = p+1, \end{cases}$$

where $k' := p+1-k$.

3.2 p^m-filtration theta cycles

Let p be prime, $m \geqslant 1$ be an integer, and $f \in M_k(N, \mathbb{Z})$ with $\widetilde{f}$ being the reduction to a modular form modulo p^m. Recalling the effect of the theta operator on q-expansions, observe that $a_{tp}(\theta^m \widetilde{f}) = 0$ for any integer $t \geqslant 1$. This naturally implies

$$a_{tp}\left(\theta^{\varphi(p^m)+m}\widetilde{f}\right) = a_{tp}\left(\theta^m \widetilde{f}\right)$$

for any $t \geqslant 1$, where φ denotes Euler's totient function. But for any integer $n \geqslant 1$ coprime to p, we have

$$a_n\left(\theta^{\varphi(p^m)+m}\widetilde{f}\right) = a_n\left(\theta^m \widetilde{f}\right)$$

by Euler's theorem. Hence,

$$\theta^{\varphi(p^m)+m}\widetilde{f} = \theta^m \widetilde{f}.$$

Note that we do not necessarily have $\theta^{\varphi(p^m)+i}\widetilde{f} = \theta^i \widetilde{f}$ for $i < m$, since we may have

$$a_{tp}(f) \not\equiv 0 \pmod{p^{m-i}}$$

for some $t \geqslant 1$. Hence, we arrive at the following definition:

Definition 15. Let p be prime, $m \geqslant 1$ be an integer, and $f \in M_k(N, \mathbb{Z})$ with $\widetilde{f}$ being the reduction to a modular form modulo p^m. Then the *p^m-filtration theta cycle of $\widetilde{f}$ (or of f)* is the $\varphi(p^m)$-tuple of integers

$$(w_{p^m}(\theta^m \widetilde{f}), w_{p^m}(\theta^{m+1}\widetilde{f}), \ldots, w_{p^m}(\theta^{\varphi(p^m)+m-1}\widetilde{f})).$$

Note that with $m = 1$, this definition is consistent with Definition 14.

3.2.1 Known results

The fact that Chen and Kiming's Theorem 3 is a much weaker analogue of Theorem 2 for the case $m = 1$ vastly inhibits our ability to classify the p^m-filtration theta cycles for $m \geqslant 2$. Nonetheless, Kim and Lee build on the results of Theorem 3 in [13] to obtain the following results:

Theorem 4. *(Theorem 1.7 in [13]). Let $p \geqslant 5$ be prime, $N \in \mathbb{Z}_{>0}$ be coprime to p, and $m \geqslant 2$ be an integer. Let $f \in M_k(N, \mathbb{Z})$ have reduction $\widetilde{f}$ to a modular form modulo p^m as usual, and suppose $\nu_p(f) = 0$. Let n_t be an integer of the form $n_t = tp^{m-1}$ or*

$n_t = tp^{m-1} - k + 1$ with $t \in \mathbb{Z}$. For such n_t with $0 \leqslant n_t \leqslant p^m - p^{m-1} + m - 1$, there exists an integer $b_f(m, n_t) \leqslant 1$ such that

$$w_{p^m}(\theta^{n_1}\widetilde{f}) = k + 2n_t + b_f(m, n_t)p^{m-1}(p-1).$$

Moreover, if $m \geqslant 3$ and $n_t \geqslant p^2$, we have $w_{p^m}(\theta^{n_t}\widetilde{f}) \leqslant k + 2n_t$.

In the case $m = 2$, Kim and Lee compute $b_f(m, n_t)$ more precisely for certain types of cusp forms $\overline{f}$ (those that can be written as linear combinations of modular forms modulo p that are supersingular if they have sufficient p-filtration; confer Theorem 1.8 in [13] for details). It is clear in any case that classifying p^m-filtration theta cycles is a much harder task for $m \geqslant 2$ than it is for $m = 1$. We supplement the results of Theorem 4 in §5 with the calculation of some p^m-filtration theta cycles explained in the following subsection.

3.2.2 SageMath-9-7 computations for p^m-filtration theta cycles

In order to compute p^m-filtration theta cycles, we must be able to use a computer to determine whether the q-expansion of an arbitrary modular form g is equivalent modulo p^m to the q-expansion of some modular form of a given weight. This is, at first glance, not an easy task. Indeed, one might expect this to be an impossible task since it is not clear with which modular form we are trying to compare g; it may be extremely difficult to determine this modular form as $M_k(N, \mathbb{Q})$ may not have an easily computable basis. Also, it is not obvious that showing the congruence for finitely many coefficients of the q-expansions is enough.

Luckily, the Sturm bound is a helpful rejoinder to this last issue (see Corollary 9.20 in [17]), and moreover, there are large classes of modular forms that have easily computable finite bases. For example, modular forms of level 1 and fixed weight have finite Victor-Miller bases (this is a direct corollary of Proposition 3 - alternatively confer Lemma 2.20 in [17]), and cusp forms of any fixed level and fixed weight also have finite bases (see [8]) that are easy for Sage to fetch. We now explain the recipe for computing the p^m-filtration theta cycles for modular forms in these classes by walking through the calculation of the p^m-filtration theta cycle of $\widetilde{f}$ for $p = 7$ and $m = 1, 2$, where f is a modular form of level 1 and weight 12.

In particular, we set

f = `CuspForms(Gamma1(1), 12).q_integral_basis(EXP_LEN)[0]` to be the first cusp form (i.e. $a_0(f) = 0$) in the basis of q-expansions provided by Sage for cusp forms of weight 12 and level 1. Here, `EXP_LEN` is the number of terms of the q-expansion that Sage generates. This gives

$f =$ `q - 24*q^2 + 252*q^3 - 1472*q^4 + 4830*q^5 - 6048*q^6 - 16744*q^7 + 84480*q^8 - 113643*q^9 + O(q^10)`.

Since the level is 1, we know f can be expressed as an isobaric weight-12 polynomial in Q and R, which gives a more precise description of this cusp form. For our calculations in Appendix 5, we provide this description whenever the modular form is of level 1, as it is more precise and versatile in the sense that it does not depend on the arbitrary ordering of bases in Sage. For our specific example, the only ways to obtain 12 by adding 4s and 6s are $4 + 4 + 4$ and $6 + 6$, so we must have

$$f = aQ^3 + bR^2$$

for constants $a, b \in \mathbb{C}$. Equating `q` and `q^2`-coefficients and using the expansion above, we obtain

$$\begin{aligned} a + b &= 0, \\ 3 \times 240a - 2 \times 504b &= 1 \end{aligned}$$

so that $a = 1/1728$ and $b = -1/1728$.

The `.q_integral_basis()` method is extremely useful, and allows us not only to generate modular forms but also to check whether a given modular form lies in a particular weight class. Indeed, to obtain the 7^1-filtration theta cycle of $\overline{f}$, for each $1 \leqslant t \leqslant p - 1$ we see whether $\theta^t \overline{f}$ lies in $\overline{M}_k(N)$ by checking if it lies in the span of the basis of q-expansions (with coefficients reduced modulo 7) for the cusp forms of level 1 and weight k generated by this method, and iterating k to find the smallest k that works. It is enough to check if it lies in the basis for cusp forms, as if it is a linear combination of non cusp forms of weight k modulo 7, we can subtract an appropriate multiple of E_k from each of these non-cusp forms in the linear combination to obtain a linear combination of cusp forms.

To determine if $\theta^t \overline{f}$ lies in the span of the basis of q-expansions with coefficients reduced modulo 7 for the cusp forms of level 1 and weight k, i.e.

```
basis = [g.change_ring(Zmod(7 ** 1)) for g in CuspForms(Gamma1(1),
```

```
w).q_integral_basis(EXP_LEN)],
```

we need only check `w` $\equiv w_7(\theta^{t-1}\widetilde{f})+2 \pmod 6$ by (1) (this saves significant computation time). Sage solves a linear system to match the first `EXP_LEN` coefficients of the q-expansions, which, provided we set `EXP_LEN` greater than the relevant Sturm bound (by making use of the method `.sturm_bound()`), will give a positive indication $\theta^t\overline{f}$ lies in the span if the system has a solution.

This is because, although the Sturm bound applies to modular forms with complex coefficients (and not coefficients in $\mathbb{F}_7$ like $\theta^t\overline{f}$), we know from the end of §1.3.1 that equality of coefficients in the q-expansions of modular forms modulo 7 corresponds to equality of coefficients in the q-expansions of modular forms by multiplying by some power of the lift of the Hasse invariant E_6.

In general, we know the q-expansion $\theta^t f$ is equivalent modulo p^m to some modular form of weight at most $w_{p^m}(\theta^{t-1}f)+k(m)$ by Theorem 3, so we should be safe by setting `EXP_LEN` greater than the Sturm bound for forms of weight $w_{p^m}(\theta^{t-1}f)+k(m)$.

For our example f, we obtain the 7^1-filtration theta cycle

```
[20, 28, 12, 20, 28, 12].
```

It is easy to check this is consistent with the results of Theorem 2. We can use the same recipe to get the 7^2-filtration theta cycle for f:

```
[100, 60, 104, 106, 108, 68, 112, 114, 74, 76, 78, 80, 82, 84, 86, 88,
90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 68, 112, 114, 74, 76, 120,
80, 40, 84, 86, 88, 90, 92, 94, 96, 98].
```

Since $f \not\equiv 0 \pmod 7$, $7 \nmid 12$ and $w_7(f) = 12$ (as the corresponding isobaric polynomial is not divisible by E_6), we should expect

$$w_{7^2}(\theta\widetilde{f}) = 12 + 2 + 2 \times 7 \times 6 = 98,$$

by Theorem 3, which matches $w_{7^2}(\theta^{\varphi(7^2)+1}\widetilde{f})$ from the calculation above. This is indeed to be expected, as it turns out $\overline{f}$ is supersingular. We also note that

$$\begin{aligned}
w_{7^2}(\theta^3\widetilde{f}) &= 60 = 12 + 2\cdot 3 + 42\\
w_{7^2}(\theta^7\widetilde{f}) &= 68 = 12 + 2\cdot 7 + 42\\
w_{7^2}(\theta^{10}\widetilde{f}) &= 74 = 12 + 2\cdot 10 + 42\\
w_{7^2}(\theta^{14}\widetilde{f}) &= 82 = 12 + 2\cdot 14 + 42\\
w_{7^2}(\theta^{17}\widetilde{f}) &= 88 = 12 + 2\cdot 17 + 42\\
w_{7^2}(\theta^{21}\widetilde{f}) &= 96 = 12 + 2\cdot 21 + 42\\
w_{7^2}(\theta^{24}\widetilde{f}) &= 102 = 12 + 2\cdot 24 + 42\\
w_{7^2}(\theta^{28}\widetilde{f}) &= 68 = 12 + 2\cdot 28 + 0\cdot 42\\
w_{7^2}(\theta^{31}\widetilde{f}) &= 74 = 12 + 2\cdot 31 + 0\cdot 42\\
w_{7^2}(\theta^{35}\widetilde{f}) &= 40 = 12 + 2\cdot 35 + (-1)\cdot 42\\
w_{7^2}(\theta^{42}\widetilde{f}) &= 96 = 12 + 2\cdot 42 + 0\cdot 42,
\end{aligned}$$

which matches what Kim and Lee predict in Theorem 4.

The most computationally expensive part of this recipe seems to be the fetching of the q-integral bases, which takes a long time for higher level N as more precision is needed in the basis elements due to the increasing Sturm bound. In Appendix 5, we thus compute the p^m-filtration theta cycles for some modular forms of level $N = 1, 2$. We also restrict to $m \leqslant 3$ and $p = 5, 7, 11$ to prevent the cycles from getting impractically long. We also consider only weight k coprime to p, and level N coprime to p.

4 Conclusion

In this thesis, we presented the theory of modular forms modulo powers of primes in order to study their p^m-filtration theta cycles. By defining classical modular forms both in an analytic way (see §1.1.1) and algebraic-geometric way (see Definition 5), and proving the equivalence of these two definitions under certain circumstances (see Theorem 4), Chen and Kiming prove the result (see Lemma 8) that the p-filtration of a modular form of exact filtration $\kappa \neq p$ multiplied by E_{p+1} is simply $\kappa + p + 1$. This allows them to prove the case of equality in their main result (see Theorem 3) on the effect of θ on the p^m-filtration.

An important takeaway is that it is much harder to calculate lower bounds for the p^m-filtration of a modular form of arbitrary weight, which makes Lemma 8 an important result. Indeed, it seems a more tractable approach to use algebraic-geometric properties of modular forms to prove lower bounds (as in the proof of Lemma 8). Nonetheless, we can deduce, for example, modest restrictions on the effect of ∂ on the p^m-filtration for low weights (see Proposition 13).

Whilst classifying p-filtration theta cycles is an old problem with many results (see, for example, Proposition 3.3 in [6]), classifying p^m-filtration theta cycles for $m \geqslant 2$ is a much newer and seemingly more difficult problem. Kim and Lee's results in [13] from 2023 provide some upper bounds on the filtrations for general m (see Theorem 4), and some exact results for certain classes of modular forms with $m = 2$ (see Theorem 1.8 in [13]). We concluded our study by outlining a general approach to use Sage to compute these theta cycles, with the goal of supplementing the contemporary interest in these objects.

After computing some p^m-filtration theta cycles for $m \geqslant 2$ (see §5), strict computational limitations arose as the level N increases and the power m increases. Since these limitations ostensibly emerge from the internal issue of Sage fetching bases, it is not clear how to overcome these ceilings. One idea is to create a hash table of bases for the relevant classes of modular forms of all even weights up to some large number. While computationally expensive, this avoids Sage repeatedly fetching the same bases when computing these theta cycles.

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5 Appendix

The Sage files relevant to this appendix can be found here.

5.1 Level 1 modular forms

5.1.1 Cusp forms

$$\begin{aligned} f &= \texttt{CuspForms(Gamma1(1), 12).q\_integral\_basis()[0]} \\ &= \texttt{q - 24*q\^{}2 + 252*q\^{}3 - 1472*q\^{}4 + 4830*q\^{}5 - 6048*q\^{}6} \\ &\quad \texttt{- 16744*q\^{}7 + 84480*q\^{}8 - 113643*q\^{}9 + O(q\^{}10)} \\ &= \frac{1}{1728}\left(Q^3 - R^2\right) \end{aligned}$$

The cycles for this f are presented on the next page (as the table does not fit).

	$m = 1$	$m = 2$	$m = 3$
$p = 5$	(18, 24, 30, 12)	(56, 58, 40, 42, 64, 66, 68, 50, 52, 54, 76, 58, 40, 42, 64, 46, 48, 70, 72, 54)	(218, 120, 122, 224, 226, 228, 130, 132, 234, 236, 238, 140, 42, 244, 246, 248, 150, 152, 254, 256, 258, 60, 62, 264, 266, 268, 70, 72, 174, 276, 178, 80, 82, 184, 186, 188, 90, 92, 194, 196, 198, 100, 102, 204, 206, 208, 110, 112, 214, 216, 218, 120, 122, 224, 226, 228, 130, 132, 234, 236, 238, 140, 42, 244, 246, 248, 150, 152, 254, 256, 258, 60, 62, 264, 166, 168, 70, 72, 174, 176, 178, 80, 82, 184, 186, 188, 90, 92, 194, 196, 198, 100, 102, 204, 206, 208, 110, 112, 214, 216)
$p = 7$	(20, 28, 12, 20, 28, 12)	(100, 60, 104, 106, 108, 68, 112, 114, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 68, 112, 114, 74, 76, 120, 80, 40, 84, 86, 88, 90, 92, 94, 96, 98)	(312, 608, 610, 612, 320, 616, 618, 326, 622, 624, 626, 334, 630, 632, 340, 636, 638, 640, 348, 644, 646, 354, 650, 652, 654, 362, 658, 660, 368, 664, 666, 668, 82, 672, 674, 88, 678, 680, 682, 96, 686, 688, 102, 692, 694, 696, 110, 700, 702, 116, 412, 708, 416, 124, 420, 422, 130, 426, 428, 430, 138, 434, 436, 144, 440, 442, 444, 152, 448, 450, 158, 454, 456, 458, 166, 462, 464, 172, 468, 470, 472, 180, 476, 478, 186, 482, 484, 486, 194, 490, 492, 200, 496, 498, 500, 208, 504, 506, 214, 510, 512, 514, 222, 518, 520, 228, 524, 526, 528, 236, 532, 534, 242, 538, 540, 542, 250, 546, 548, 256, 552, 554, 556, 264, 560, 562, 270, 566, 568, 570, 278, 574, 576, 284, 580, 582, 584, 292, 588, 590, 298, 594, 596, 598, 306, 602, 604, 312, 608, 610, 612, 320, 616, 618, 326, 622, 624, 626, 334, 630, 632, 340, 636, 638, 640, 348, 644, 646, 354, 650, 652, 654, 68, 658, 660, 368, 664, 666, 668, 376, 672, 674, 382, 678, 680, 682, 390, 686, 688, 396, 692, 694, 696, 404, 700, 702, 410, 412, 708, 416, 124, 420, 422, 130, 426, 428, 430, 138, 434, 436, 144, 440, 442, 444, 152, 448, 450, 158, 454, 456, 458, 166, 462, 464, 172, 468, 470, 472, 180, 476, 478, 186, 482, 484, 486, 194, 490, 492, 200, 496, 498, 500, 208, 504, 506, 214, 510, 512, 514, 222, 518, 520, 228, 524, 526, 528, 236, 532, 534, 242, 538, 540, 542, 250, 546, 548, 256, 552, 554, 556, 264, 560, 562, 270, 566, 568, 570, 278, 574, 576, 284, 580, 582, 584, 292, 588, 590, 298, 594, 596, 598, 306, 602, 604)

Table 1: Table of p^m-filtration theta cycles for f

$$
\begin{aligned}
f &= \texttt{CuspForms(Gamma1(1), 16).q\_integral\_basis()[0]} \\
&= \texttt{q + 216*q^2 - 3348*q^3 + 13888*q^4 + 52110*q^5 + O(q^6)} \\
&= \frac{1}{1728}\left(Q^4 - R^2 Q\right)
\end{aligned}
$$

	$m = 1$	$m = 2$	$m = 3$
$p = 5$	(18, 24, 30, 12)	(60, 62, 64, 46, 48, 30, 52, 54, 36, 58, 60, 42, 64, 66, 68, 50, 52, 54, 76, 58)	(222, 224, 126, 228, 230, 232, 234, 136, 238, 240, 242, 244, 146, 248, 250, 252, 254, 156, 258, 260, 262, 264, 66, 168, 270, 272, 174, 76, 178, 180, 182, 184, 86, 188, 190, 192, 194, 96, 198, 200, 202, 204, 106, 208, 210, 212, 214, 116, 218, 220, 222, 224, 126, 228, 230, 232, 234, 136, 238, 240, 242, 244, 46, 248, 250, 252, 254, 56, 258, 260, 162, 264, 66, 268, 170, 172, 174, 176, 178, 180, 182, 184, 86, 188, 190, 192, 194, 96, 198, 200, 202, 204, 106, 208, 210, 212, 214, 116, 218, 220)
$p = 7$	(24, 32, 40, 48, 56, 16)	(104, 106, 108, 110, 70, 72, 116, 118, 120, 122, 124, 84, 86, 88, 132, 134, 136, 96, 98, 100, 102, 104, 148, 108, 110, 112, 114, 116, 118, 78, 80, 124, 84, 86, 130, 48, 92, 52, 96, 140, 142, 102)	(610, 612, 614, 322, 324, 620, 622, 624, 626, 628, 336, 338, 634, 636, 638, 640, 642, 350, 352, 648, 650, 652, 654, 656, 364, 366, 662, 664, 666, 668, 670, 378, 86, 676, 678, 680, 682, 684, 392, 394, 690, 692, 694, 696, 698, 112, 114, 704, 706, 708, 710, 712, 126, 128, 424, 720, 722, 724, 432, 140, 142, 438, 440, 736, 444, 446, 154, 156, 452, 454, 456, 458, 460, 168, 170, 466, 468, 470, 472, 474, 182, 184, 480, 482, 484, 486, 488, 196, 198, 494, 496, 498, 500, 502, 210, 212, 508, 510, 512, 514, 516, 224, 226, 522, 524, 526, 528, 530, 238, 240, 536, 538, 540, 542, 544, 252, 254, 550, 552, 554, 556, 558, 266, 268, 564, 566, 568, 570, 572, 280, 282, 578, 580, 582, 584, 586, 294, 296, 592, 594, 596, 598, 600, 308, 310, 606, 608, 610, 612, 614, 322, 324, 620, 622, 624, 626, 628, 336, 338, 634, 636, 638, 640, 642, 350, 352, 648, 650, 652, 654, 656, 70, 72, 662, 664, 666, 668, 670, 378, 380, 676, 678, 680, 682, 684, 392, 394, 690, 692, 694, 696, 698, 406, 408, 704, 706, 414, 416, 712, 126, 128, 718, 426, 428, 430, 432, 140, 142, 438, 440, 442, 444, 446, 154, 156, 452, 454, 456, 458, 460, 168, 170, 466, 468, 470, 472, 474, 182, 184, 480, 482, 484, 486, 488, 196, 198, 494, 496, 498, 500, 502, 210, 212, 508, 510, 512, 514, 516, 224, 226, 522, 524, 526, 528, 530, 238, 240, 536, 538, 540, 542, 544, 252, 254, 550, 552, 554, 556, 558, 266, 268, 564, 566, 568, 570, 572, 280, 282, 578, 580, 582, 584, 586, 294, 296, 592, 594, 596, 598, 600, 308, 310, 606, 608)

Table 2: Table of p^m-filtration theta cycles for f

5.1.2 Non-cusp forms

In this subsection, we just compute theta cycles for Q and R.

$f =$ `ModularForms(Gamma1(1), 4).q_integral_basis()[0]`

$=$ `1 + 240*q + 2160*q^2 + 6720*q^3 + 17520*q^4 + 30240*q^5 + O(q^6)`

$= Q$

	$m = 1$	$m = 2$	$m = 3$
$p = 7$	(12, 20, 28, 12, 20, 28)	(92, 94, 54, 98, 100, 60, 104, 106, 108, 68, 112, 114, 74, 76, 120, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 68, 112, 114, 74, 76, 120, 80, 82, 84, 86, 46, 90)	(598, 306, 602, 604, 312, 608, 610, 612, 320, 616, 618, 326, 622, 624, 626, 334, 630, 632, 340, 636, 638, 640, 348, 644, 646, 354, 650, 652, 654, 362, 658, 660, 368, 664, 666, 668, 376, 672, 674, 88, 678, 680, 682, 96, 686, 688, 102, 692, 694, 696, 110, 700, 702, 116, 412, 708, 416, 124, 420, 422, 130, 426, 428, 430, 138, 434, 436, 144, 440, 442, 444, 152, 448, 450, 158, 454, 456, 458, 166, 462, 464, 172, 468, 470, 472, 180, 476, 478, 186, 482, 484, 486, 194, 490, 492, 200, 496, 498, 500, 208, 504, 506, 214, 510, 512, 514, 222, 518, 520, 228, 524, 526, 528, 236, 532, 534, 242, 538, 540, 542, 250, 546, 548, 256, 552, 554, 556, 264, 560, 562, 270, 566, 568, 570, 278, 574, 576, 284, 580, 582, 584, 292, 588, 590, 298, 594, 596, 598, 306, 602, 604, 312, 608, 610, 612, 320, 616, 618, 326, 622, 624, 626, 334, 630, 632, 340, 636, 638, 640, 348, 644, 646, 354, 650, 652, 654, 362, 658, 660, 368, 664, 666, 668, 376, 672, 674, 382, 678, 680, 682, 390, 686, 688, 396, 692, 694, 696, 404, 700, 702, 410, 412, 708, 416, 124, 420, 422, 130, 426, 428, 430, 138, 434, 436, 144, 440, 442, 444, 152, 448, 450, 158, 454, 456, 458, 166, 462, 464, 172, 468, 470, 472, 180, 476, 478, 186, 482, 484, 486, 194, 490, 492, 200, 496, 498, 500, 208, 504, 506, 214, 510, 512, 514, 222, 518, 520, 228, 524, 526, 528, 236, 532, 534, 242, 538, 540, 542, 250, 546, 548, 256, 552, 554, 556, 264, 560, 562, 270, 566, 568, 570, 278, 574, 576, 284, 580, 582, 584, 292, 588, 590, 298, 594, 596)
$p = 11$	(16, 28, 40, 52, 64, 76, 88, 20, 32, 44)	(228, 230, 232, 234, 236, 238, 130, 242, 244, 136, 248, 250, 252, 254, 256, 258, 260, 152, 264, 266, 158, 160, 272, 274, 276, 278, 280, 172, 174, 176, 68, 180, 182, 184, 296, 298, 300, 192, 194, 196, 198, 200, 202, 204, 206, 208, 320, 212, 214, 216, 218, 220, 222, 224, 226, 228, 230, 232, 234, 236, 238, 240, 242, 244, 246, 248, 250, 252, 254, 256, 148, 260, 262, 264, 266, 268, 270, 162, 164, 276, 168, 170, 172, 284, 176, 178, 290, 182, 184, 186, 188, 190, 192, 194, 196, 308, 310, 202, 204, 206, 208, 210, 212, 214, 216, 218, 220, 222, 114, 226)	Too long

Table 3: Table of p^m-filtration theta cycles for f

$f =$ `ModularForms(Gamma1(1), 6).q_integral_basis()[0]`

$=$ `1 - 504*q - 16632*q^2 - 122976*q^3 - 532728*q^4 + O(q^5)`

$= R$

	$m = 1$	$m = 2$	$m = 3$
$p = 5$	(12, 18, 24, 30)	(50, 52, 54, 36, 58, 60, 62, 64, 46, 28, 70, 72, 54, 56, 58, 60, 42, 64, 46, 68)	(212, 214, 116, 218, 220, 222, 224, 126, 228, 230, 232, 234, 136, 238, 240, 242, 244, 146, 248, 250, 252, 254, 156, 258, 260, 262, 264, 66, 168, 270, 272, 174, 76, 178, 180, 182, 184, 86, 188, 190, 192, 194, 96, 198, 200, 202, 204, 106, 208, 210, 212, 214, 116, 218, 220, 222, 224, 126, 228, 230, 232, 234, 136, 238, 240, 242, 244, 146, 248, 250, 252, 254, 156, 258, 260, 162, 264, 66, 268, 170, 172, 174, 76, 178, 180, 182, 184, 86, 188, 190, 192, 194, 96, 198, 200, 202, 204, 106, 208, 210)
$p = 11$	(18, 30, 42, 54, 66, 18, 30, 42, 54, 66)	(230, 232, 234, 236, 128, 240, 242, 244, 246, 138, 250, 252, 254, 256, 258, 150, 262, 264, 266, 268, 160, 162, 274, 276, 278, 170, 172, 174, 286, 288, 180, 182, 184, 186, 298, 190, 192, 194, 196, 198, 90, 202, 204, 206, 208, 210, 212, 214, 216, 218, 220, 222, 224, 226, 228, 230, 232, 234, 236, 238, 240, 242, 244, 246, 248, 250, 252, 254, 256, 258, 150, 262, 154, 156, 268, 160, 162, 274, 276, 278, 170, 172, 174, 286, 288, 180, 182, 184, 186, 298, 190, 192, 194, 196, 198, 200, 202, 204, 206, 208, 210, 212, 214, 216, 218, 220, 222, 224, 116, 228)	Too long

Table 4: Table of p^m-filtration theta cycles for f

5.2 Level 2 modular forms

Modular forms of level $N > 1$ and fixed weight have bases that are computationally expensive for Sage to fetch. So we compute theta cycles just for cusp forms.

5.2.1 Cusp forms

$$f = \texttt{CuspForms(Gamma1(2), 8).q\_integral\_basis()[0]}$$
$$= \texttt{q - 8*q^2 + 12*q^3 + 64*q^4 - 210*q^5 - 96*q^6}$$
$$\texttt{+ 1016*q^7 - 512*q^8 - 2043*q^9 + O(q^10)}$$

	$m = 1$	$m = 2$
$p = 5$	(14, 20, 10, 8)	(52, 34, 56, 38, 60, 62, 44, 46, 48, 50, 32, 54, 56, 58, 40, 42, 44, 66, 48, 50)
$p = 7$	(16, 24, 32, 40, 48, 56)	(96, 98, 100, 102, 104, 64, 108, 110, 70, 72, 116, 118, 78, 80, 124, 126, 128, 130, 90, 92, 94, 96, 140, 142, 102, 104, 106, 108, 110, 112, 30, 116, 118, 120, 122, 124, 84, 86, 88, 132, 92, 136)

Table 5: Table of p^m-filtration theta cycles for f

$$f = \texttt{CuspForms(Gamma1(2), 12).q\_integral\_basis()[0]}$$
$$= \texttt{q + 252*q^3 - 2048*q^4 + 4830*q^5 - 16744*q^7 + O(q^8)}$$

	$m = 1$	$m = 2$
$p = 5$	(18, 24, 30, 12)	(56, 58, 40, 42, 64, 66, 68, 50, 52, 54, 76, 58, 40, 42, 64, 46, 48, 70, 72, 54)
$p = 7$	(20, 28, 12, 20, 28, 12)	(100, 60, 104, 106, 108, 68, 112, 114, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 68, 112, 114, 74, 76, 120, 80, 40, 84, 86, 88, 90, 92, 94, 96, 98)

Table 6: Table of p^m-filtration theta cycles for f

Note that these are the same cycles as in Table 1.