A geometric approach to Maass' operator on Siegel modular forms

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A thesis submitted for the degree of
Master of Science

at

The University of Melbourne
School of Mathematics and Statistics

May 2018

Abstract

There is a geometric construction, due to Katz, that can be used to define differential operators on modular forms of various types. Harris has applied this construction to define an operator ϑ on spaces of C^∞ Siegel modular forms. In this thesis, we give a detailed exposition of Harris' work in [13]. We endeavour to present the theory within its proper algebraic-geometric context, while at the same time making full use of explicit analytic constructions. In addition, by iterating twists of ϑ, we define an operator Θ on scalar-valued forms. We verify that this Θ is Maass' operator when the dimension $g = 2$.

Contents

Chapter 1

Introduction

1.1 Maass' operator in dimension one

To orient ourselves, we will begin with an example. The ideas we touch on here will be developed much more fully in the subsequent chapters.

Let $\mathfrak{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$ denote the upper half of the complex plane. The modular group $SL_2(\mathbb{Z}) = \{ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \mid \det\gamma = 1 \}$ acts on $\mathfrak{H}$ by linear fractional transformations:

$$\gamma \cdot z = \frac{az+b}{cz+d}.$$

Suppose we have a function $f : \mathfrak{H} \to \mathbb{C}$ that, for some $k \in \mathbb{Z}$, transforms under this action according to the rule

$$f(\gamma \cdot z) = (cz+d)^k f(z) \tag{1.1}$$

for all $z \in \mathfrak{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. If f is holomorphic, we will say that it is a *holomorphic modular form*; if f is merely smooth, we will call it a C^∞ *modular form*. [1] The integer k is called the *weight* of f. The C^∞ modular forms of weight k form a complex vector space $\mathcal{M}_k^\infty$, which contains the holomorphic modular forms $\mathcal{M}_k$ as a finite-dimensional subspace.

We are going to define a differential operator on $\mathcal{M}^\infty := \oplus_{k\in\mathbb{Z}} \mathcal{M}_k^\infty$. Suppose we differentiate (1.1) with respect to z:

$$f'(\gamma \cdot z) = (cz+d)^{k+2} f'(z) + kc(cz+d)^{k+1} f(z). \tag{1.2}$$

Note that f' is almost a modular form of weight $k+2$, but not quite – there is an unwanted extra factor of $kc(cz+d)^{k+1}f(z)$. However, observe that

$$\frac{1}{\operatorname{Im}(\gamma\cdot z)} = \frac{|cz+d|^2}{y} = \frac{(cz+d)^2}{y} - 2ci(cz+d),$$

where $y = \operatorname{Im} z$. Substituting the resulting expression for $c(cz+d)$ into (1.2), we get

$$\begin{aligned} f'(\gamma\cdot z) &= (cz+d)^{k+2} f'(z) + \frac{k}{2i}\left(\frac{(cz+d)^2}{y} - \frac{1}{\operatorname{Im}(\gamma\cdot z)}\right)(cz+d)^k f(z) \\ &= (cz+d)^{k+2}\left(f'(z) + \frac{k}{2iy} f(z)\right) - \frac{k}{2i\operatorname{Im}(\gamma\cdot z)} f(\gamma\cdot z). \end{aligned}$$

[1]Strictly speaking, we should also impose a condition of boundedness at infinity.

This shows $f'(z) + \frac{k}{2iy} f(z)$ satisfies (1.1), with weight $k+2$.

Definition 1.1.1. The differential operator

$$\delta : \mathcal{M}_k^\infty \to \mathcal{M}_{k+2}^\infty, \qquad \delta(f)(z) := f'(z) + \frac{k}{2iy} f(z)$$

is known as the *Maass operator*.

Remark. Note that $\delta(f)$ will not be holomorphic in general, since $y = \operatorname{Im} z$ is not holomorphic in z.

The operator δ has a geometric interpretation. To explain this interpretation, we must first understand the connection between the function-theoretic definition of modular forms that we have given, and certain aspects of the geometry of complex elliptic curves.

We begin with the upper half-plane $\mathfrak{H}$. For every $\tau \in \mathfrak{H}$, there is an elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$, where Λ_τ is the lattice $\mathbb{Z}\tau \oplus \mathbb{Z}$. As we vary τ, the E_τ vary holomorphically, forming a family of elliptic curves over $\mathfrak{H}$ which we will denote by E_{univ}. (The family is 'universal' because every complex elliptic curve is isomorphic to E_τ for some τ.) If $\tau, \tau' \in \mathfrak{H}$, then the fibres E_τ and $E_{\tau'}$ are isomorphic if and only if τ and τ' are related by the action of $SL_2(\mathbb{Z})$. Hence, the set of orbits $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ is in bijection with the set of isomorphism classes of elliptic curves. Moreover, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ is such that $\tau' = \gamma\tau$, then the corresponding isomorphism on the fibres is given by

$$\varphi_\gamma : E_\tau \xrightarrow{\sim} E_{\tau'}, \qquad z \mapsto (c\tau + d)^{-1} z.$$

Unfortunately, there does not exist a universal family over $SL_2(\mathbb{Z})\backslash\mathfrak{H}$. If we want to construct universal families other than $E_{\text{univ}}/\mathfrak{H}$, we are forced to replace the full modular group $SL_2(\mathbb{Z})$ with a finite index subgroup whose action on $\mathfrak{H}$ is free. We will choose to focus on certain subgroups $\Gamma(N)$, for $N \geq 3$, called *principal congruence subgroups*. The quotient $Y(N) := \Gamma(N)\backslash\mathfrak{H}$ is known as a *modular curve*. It is a moduli space for elliptic curves equipped with a certain kind of N-torsion data, and there exists a universal family $E_{\text{univ},N}$ over $Y(N)$. This family is a quotient of E_{univ} by $\Gamma(N)$: the fibre of $E_{\text{univ},N}$ over a point $\Gamma(N)\tau \in Y(N)$ is the elliptic curve obtained from the equivalence class $\{ E_{\gamma\tau} \mid \gamma \in \Gamma(N) \}$ via the identifications φ_γ above.

The modularity condition (1.1) can be explained in terms of the geometry of the universal family $E_{\text{univ},N}/Y(N)$. Recall that the first de Rham cohomology of a complex torus $X = \mathbb{C}/\Lambda$ has a Hodge decomposition

$$H^1_{\text{dR}}(X) = H^{1,0}(X) \oplus H^{0,1}(X) = \mathbb{C}\, du \oplus \mathbb{C}\, d\bar{u},$$

where u is the standard coordinate function on $\mathbb{C}$. Assigning to each $\tau \in \mathfrak{H}$ the vector space $H^1_{\text{dR}}(E_\tau)$ defines a vector bundle over $\mathfrak{H}$, which turns out to be trivial:

$$\mathcal{H}^1_{\text{dR}}(E_{\text{univ}}/\mathfrak{H}) = \mathfrak{H} \times \mathbb{C}\, du \oplus \mathbb{C}\, d\bar{u}. \tag{1.3}$$

We will write ω and $\overline{\omega}$ for the subbundles $\mathfrak{H} \times \mathbb{C}\, du$ and $\mathfrak{H} \times \mathbb{C}\, d\bar{u}$. Similarly, we have a vector bundle $\mathcal{H}^1_{\text{dR}}(E_{\text{univ},N}/Y(N))$ over $Y(N)$, which is a quotient of $\mathcal{H}^1_{\text{dR}}(E_{\text{univ}}/\mathfrak{H})$

by the action $\gamma \mapsto (\varphi_\gamma^{-1})^*$ of $\Gamma(N)$. If we choose to represent the fibre above $\Gamma(N)\tau \in Y(N)$ by E_τ, and the holomorphic part of its de Rham cohomology by $du \in H^1_{dR}(E_\tau)$, then this action identifies du with the pullback

$$(\varphi_\gamma^{-1})^*(du) = (c\tau + d)\, du \in H^1_{dR}(E_{\gamma\tau}). \tag{1.4}$$

Note that the decomposition (1.3) is preserved, so we can write the quotient $\mathcal{H}^1_{dR}(E_{\mathrm{univ},N}/Y(N))$ as a sum of holomorphic and anti-holomorphic pieces:

$$\mathcal{H}^1_{dR}(E_{\mathrm{univ},N}/Y(N)) = \omega_N \oplus \overline{\omega}_N.$$

Modular forms are global sections of the kth tensor power $\omega_N^{\otimes k}$. A global section of $\omega_N^{\otimes k}$ lifts to a global section of $\omega^{\otimes k}$, which is the same thing as a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$. Because of (1.4), f necessarily satisfies (1.1) for all $z \in \mathfrak{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)$. Conversely, if f is a modular form of weight k, then f descends to a global section of $\omega_N^{\otimes k}$ for all $N \geq 3$. Clearly, this correspondence preserves holomorphicity/smoothness.

We can finally give the promised geometric interpretation of Maass' operator. For any holomorphic family X/S, there is a canonical way to differentiate the sections of $\mathcal{H}^1_{dR}(X/S)$, known as the *Gauss-Manin connection*. If we identify $\mathcal{H}^1_{dR}(X/S)$ with its sheaf of holomorphic sections, then the Gauss-Manin connection is a $\mathbb{C}$-linear map of sheaves

$$\nabla : \mathcal{H}^1_{dR}(X/S) \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{H}^1_{dR}(X/S),$$

that satisfies a version of the Leibniz rule. We also have a C^∞ version of this connection:

$$\nabla_\infty : \mathcal{H}^1_\infty(X/S) \to \mathcal{A}^{1,0}_S \otimes_{\mathcal{C}^\infty_S} \mathcal{H}^1_\infty(X/S),$$

where $\mathcal{H}^1_\infty(X/S) := \mathcal{C}^\infty_S \otimes_{\mathcal{O}_S} \mathcal{H}^1_{dR}(X/S)$, and $\mathcal{A}^{1,0}_S = \mathcal{C}^\infty_S \otimes_{\mathcal{O}_S} \Omega^1_S$ is the sheaf of smooth differential forms on S of type $(1,0)$.

The Gauss-Manin connection is characterised by its kernel, the local system of *horizontal sections*. In the case of our family $E_{\mathrm{univ}}/\mathfrak{H}$, the global horizontal sections for ∇_∞ are given by the $\mathbb{C}$-span of

$$\alpha := d\tilde{x} - \frac{\operatorname{Re} z}{\operatorname{Im} z} d\tilde{y}, \quad \beta := \frac{1}{\operatorname{Im} z} d\tilde{y}$$

where $u = \tilde{x} + i\tilde{y}$, and $z \in \mathfrak{H}$. We have the identities

$$du = \alpha + z\beta, \quad d\bar{u} = \alpha + \bar{z}\beta, \tag{1.5}$$

and a formula for ∇_∞ is given by $\nabla_\infty(du) = dz \otimes \beta$.

Writing $\omega_{N,\infty} := \mathcal{C}^\infty_{Y(N)} \otimes_{\mathcal{O}_{Y(N)}} \omega_N$ and $\mathcal{H}^1_\infty := \mathcal{H}^1_\infty(E_{\mathrm{univ},N}/Y(N))$, we define a map ϑ by the diagram

$$\begin{array}{ccc}
\omega_{N,\infty}^{\otimes k} & \longrightarrow & (\mathcal{H}^1_\infty)^{\otimes k} \\
\Big\downarrow{\scriptstyle \vartheta} & & \Big\downarrow{\scriptstyle (\nabla_\infty)^{\otimes k}} \\
 & & \mathcal{A}^{1,0}_{Y(N)} \otimes (\mathcal{H}^1_\infty)^{\otimes k} \\
 & & \Big\downarrow{\scriptstyle \mathrm{KS}^{-1} \otimes \mathrm{id}} \\
\omega_{N,\infty}^{\otimes(k+2)} & \longleftarrow & \omega_{N,\infty}^{\otimes 2} \otimes (\mathcal{H}^1_\infty)^{\otimes k}.
\end{array}$$

Here the horizontal arrows are induced by the canonical inclusion and projection, $\nabla_\infty^{\otimes k}$ is induced from ∇_∞ via the product rule, and KS is the Kodaira-Spencer isomorphism $du^{\otimes 2} \mapsto dz$. To compute ϑ on a section of $\omega_N^{\otimes k}$, we lift it to a section of $\omega^{\otimes k}$, apply the Gauss-Manin connection and KS^{-1} for $E_{univ}/\mathfrak{H}$, and project the result onto $\omega^{\otimes(k+2)}$. Note that the projection of β onto ω is $\frac{1}{2iy}\,du$ (from (1.5)), so that $\vartheta(du) = du^{\otimes 2} \otimes \frac{1}{2iy}\,du$. Hence, if $f : \mathfrak{H} \to \mathbb{C}$ is a smooth function, we have

$$\begin{aligned}\vartheta(f\,du^{\otimes k}) &= f'(z)du^{\otimes(k+2)} + kf\,\vartheta(du) \otimes du^{\otimes(k-1)} \\ &= \left(f'(z) + \frac{k}{2iy}f\right) du^{\otimes(k+2)},\end{aligned}$$

which shows $\vartheta = \delta$.

1.2 Theta operators for Siegel modular forms

We consider a g-dimensional generalisation of the picture sketched above. In the new picture, the symplectic group $Sp_{2g}(\mathbb{Z})$ replaces the special linear group $SL_2(\mathbb{Z})$, the Siegel upper half space $\mathfrak{S}_g$ replaces the upper half plane $\mathfrak{H}$, and abelian varieties replace elliptic curves. Modular forms in this setting are known as *Siegel modular forms*. As with elliptic modular forms, they have both an analytic and a geometric definition. The weight of a Siegel modular form f is a holomorphic representation $\kappa : GL_g(\mathbb{C}) \to GL(V_\kappa)$. If $\kappa = \det^{\otimes k}$ for some $k \in \mathbb{Z}$, then we say that f is *scalar-valued* and of *weight* k; otherwise, we call f *vector-valued*.

In [21], Maass defined a differential operator

$$\delta_g : \mathcal{M}_{g,k}^\infty \to \mathcal{M}_{g,k+2}^\infty$$

on the space $\mathcal{M}_{g,k}^\infty$ of C^∞ Siegel modular forms of degree g and weight k (our notation here is not Maass', but is adapted from [13]). Maass defined this operator via an explicit (but complicated) formula. In [13], Harris reinterpreted this operator in the language of algebraic geometry. In fact, he defined a map

$$\vartheta : \{\, C^\infty \text{ forms of weight std} \,\} \to \left\{\, C^\infty \text{ forms of weight } \mathrm{Sym}^2(\mathrm{std}) \otimes \mathrm{std} \,\right\}, \quad (1.6)$$

where std is the standard representation of $GL_g(\mathbb{C})$ on $\mathbb{C}^g$. He then showed that both δ_g and ϑ arise from certain canonical differential operators on the universal enveloping algebra of the complexification of $Sp_{2g}(\mathbb{R})$, and thereby established a connection between them.

The geometric point of view adopted in [13] (and featured in the previous section) is due to Katz. Katz's techniques are applicable to many different kinds of modular forms. For example, in [17], Katz defined a theta operator on mod p modular forms. This operator has recently been generalised by Ghitza and Flander to mod p Siegel modular forms of arbitrary degree [9]. Working independently, Yamauchi has considered essentially the same generalisation for forms of degree 2 [29]. Suppose κ is a representation of $GL_g(\mathbb{C})$ arising from a Schur functor. The mod p operator is of the form

$$\vartheta_{FG} : \{ \text{ mod p forms of weight } \kappa \} \to \\ \Big\{ \text{ mod p forms of weight } \mathrm{Sym}^2(\mathrm{std}) \otimes \det^{\otimes(p-1)}(\mathrm{std}) \otimes \kappa \Big\}. \quad (1.7)$$

Compare the weights of the forms in the targets of (1.6) and (1.7). The factor of $\mathrm{Sym}^2(\mathrm{std})$ appears in both (and for the same reason), whereas the appearance of the extra factor of $\det^{\otimes(p-1)}(\mathrm{std})$ in (1.7) is a phenomenon peculiar to characteristic p (due to multiplication by the Hasse invariant).

It does not seem to be especially desirable to produce modular forms whose weight contains $\mathrm{Sym}^2(\mathrm{std})$ as an irreducible factor. In particular, it would be nice to be able to produce scalar-valued forms from scalar-valued forms. However, because of the intricacies of representation theory in characteristic p, an analysis of ϑ_{FG} in the style of [13] would not seem to be feasible. In this direction, Yamauchi noticed that if he applied his operator twice, the weight of the resulting form had a factor of $(\mathrm{Sym}^2 \mathrm{std})^{\otimes 2}$, which is reducible if $g = 2$. By projecting onto the irreducible factors of $(\mathrm{Sym}^2 \mathrm{std})^{\otimes 2}$, he obtained modular forms of various new weights.

'Theta operators' have many applications in number theory. For example, Harris was motivated by proving rationality results for special values of L-functions attached to Siegel modular forms. The characteristic p versions of these operators are closely related to mod p Galois representations; the connection has been studied by Ghitza and McAndrew in [12].

1.3 What this thesis is about

We have two goals in this thesis. Our first goal is to give a detailed account of the background necessary to understand Sections 4.0–4.4 of Harris' paper [13]. Our second goal is define to an operator $\Theta : \mathcal{M}^{\infty}_{g,k} \to \mathcal{M}^{\infty}_{g,k+2}$, using Yamauchi's method of iteration-then-projection. We will then verify, in the case $g = 2$, that Θ is the Maass operator δ_2.

Although we have taken a more elementary route to its definition, the operator Θ is already implicit in Harris' work. It must be acknowledged that in some ways, Harris' point of view is more conceptual and powerful than ours. For example, one can deduce from the results of [13] that $\Theta = \delta_g$ for general g. However, the low-tech approach we adopt to the representation theory in Section 5.1 is well-suited to the characteristic p setting, and it should be possible to define a mod p analogue of the Maass operator Θ using our methods. It would be interesting to compare such an operator to the one defined by Böcherer and Nagaoka in [3], using completely different techniques.

1.4 How this thesis is organised

Our main object of study will be a universal family of abelian varieties A_{univ} over the Siegel upper half space $\mathfrak{S}_g$. This is analogous to the family of elliptic curves

$E_{univ}/\mathfrak{H}$ introduced in Section 1.1. We will be especially interested in its quotients $A_{univ,N}/\mathfrak{S}_{g,N}$ by certain finite index subgroups $\Gamma_g(N)$ of the modular group $Sp_{2g}(\mathbb{Z})$.

In Chapter 2, we define $A_{univ}/\mathfrak{S}_g$ as a family of complex tori, and discuss its cohomology from a topological point of view. This will be important later when we define and calculate the Gauss-Manin connection. In Chapter 3, we define the de Rham cohomology sheaf $\mathcal{H}^1_{dR}(A_{univ}/\mathfrak{S}_g)$, and show that it satisfies a relative version of the Hodge decomposition. In Chapter 4, we consider $A_{univ}/\mathfrak{S}_g$ as a family of abelian varieties. We define the families $A_{univ,N}/\mathfrak{S}_{g,N}$, and give the geometric definition of Siegel modular forms. In Chapter 5, we define the C^∞ theta operator Θ, and calculate it in the case $g = 2$.

There is an index of notation following Chapter 5.

Chapter 2

Cohomology of a family of complex tori

A complex abelian variety is, in particular, a complex torus. In this chapter, we define $\pi : A_{\text{univ}} \to \mathfrak{S}_g$ as a family of complex tori, and discuss its singular cohomology. The fibre of this family lying above a point $\tau \in \mathfrak{S}_g$ is the complex torus $A_\tau = \mathbb{C}^g/\Lambda_\tau$, where Λ_τ is the lattice spanned by the standard basis vectors $e_1, \ldots, e_g$ of $\mathbb{C}^g$ and the columns $\tau_1, \ldots, \tau_g$ of the matrix τ. We will see that 1st homology group $H_1(A_\tau; \mathbb{Z})$ of A_τ can be identified with Λ_τ itself. It follows that

$$H^1(A_\tau; \mathbb{C}) = \mathbb{C}\, e_1^* \oplus \ldots \oplus \mathbb{C}\, e_g^* \oplus \mathbb{C}\, \tau_1^* \oplus \ldots \oplus \mathbb{C}\, \tau_g^*, \tag{2.1}$$

since $H^1(A_\tau; \mathbb{C}) = \text{Hom}(\Lambda_\tau \otimes_{\mathbb{Z}} \mathbb{C}; \mathbb{C})$.

The main technical work in this chapter is to show how the vector spaces $H^1(A_\tau; \mathbb{C})$ fit together to form a locally free $\mathcal{O}_{\mathfrak{S}_g}$-module $\mathcal{H}^1(A_{\text{univ}}/\mathfrak{S}_g)$. In fact, $\mathcal{H}^1(A_{\text{univ}}/\mathfrak{S}_g)$ is free: we will show that it has a basis of global sections $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ that restricts to (2.1) on the fibres. These are the 'horizontal sections' that define the Gauss-Manin connection.

In Section 2.1, we show that the singular cohomology group $H_1(X; \mathbb{Z})$ of a complex torus $X = V/\Lambda$ may be identified with Λ. In Section 2.2, we construct the family $A_{\text{univ}}/\mathfrak{S}_g$. In Section 2.3, we define a notion of singular cohomology for a family $f : X \to S$ of complex manifolds, and apply this to $A_{\text{univ}}/\mathfrak{S}_g$.

2.1 $H_1(V/\Lambda; \mathbb{Z}) = \Lambda$

Let V be a complex vector space of dimension g. A discrete subgroup Λ of V of rank $2g$ is called a *lattice*. A *complex torus* is a quotient $X = V/\Lambda$ where Λ is a lattice in V. It is a g-dimensional complex Lie group. We will denote by $q : V \to X$ the canonical projection.

Proposition 2.1.1. *There is a canonical isomorphism* $\Lambda \cong \pi_1(X, 0)$*, given by mapping a lattice point* λ *to the homotopy class of the loop* $s \mapsto s\lambda$, $s \in [0, 1]$.

Proof. Since V is simply connected, we can regard the canonical projection q as the universal covering map. We denote by

$$\text{Aut}_q(V) := \left\{ \text{homeomorphisms } \varphi : V \xrightarrow{\sim} V \text{ such that } q \circ \varphi = q \right\}$$

the automorphism group of the covering q. Let $\pi_1(X, x)$ denote the fundamental group of X at a point x, and recall that it acts on the fibre $q^{-1}(x)$ in the following way: given $\gamma \in \pi_1(X, x)$ and $v \in q^{-1}(x)$, we set

$$v \cdot \gamma := \widetilde{\gamma}(1),$$

where $\widetilde{\gamma} : [0, 1] \to V$ is the unique lift of γ such that $\widetilde{\gamma}(0) = v$. This action is known as the *monodromy action*.

First, we claim that $\mathrm{Aut}_q(V)$ is precisely the group of translations t_λ, where $\lambda \in \Lambda$ is a lattice point. Clearly, every t_λ belongs to $\mathrm{Aut}_q(V)$, so suppose $\varphi \in \mathrm{Aut}_q(V)$. Set $\lambda := \varphi(0)$; since $q \circ \varphi = q$, we have $\lambda \in \Lambda$. Note that both t_λ and φ send 0 to λ. Since covering homomorphisms are determined by their value on a single point [20, Proposition 11.36], we must have $\varphi = t_\lambda$.

Now, because V is simply connected, $\pi_1(X) := \pi_1(X, 0)$ is isomorphic to $\mathrm{Aut}_q(V)$ via the map $\gamma \mapsto \varphi_\gamma$, where φ_γ is the unique automorphism of q satisfying

$$\varphi_\gamma(v) = v \cdot \gamma$$

for all $v \in \Lambda = q^{-1}(0)$ [20, Corollary 12.9]. In particular, given $\lambda \in \Lambda$, let $\widetilde{\gamma} : [0, 1] \to V$ be the path $\gamma(s) = s\lambda$, and let $\gamma = q \circ \widetilde{\gamma}$. Then by definition of the monodromy action, we have

$$\varphi_\gamma(0) = \widetilde{\gamma}(1) = \lambda,$$

since $\widetilde{\gamma}$ is a lift of γ beginning at $0 \in V$. Hence, $\varphi_\gamma = t_\lambda$.

We have shown that there is an isomorphism

$$\pi_1(X) \xrightarrow{\sim} \mathrm{Aut}_q(V) = \{\, t_\lambda \mid \lambda \in \Lambda \,\},$$

which assigns to the homotopy class of the loop $s \mapsto s\lambda$ the q-automorphism t_λ. It remains to note that $\{\, t_\lambda \mid \lambda \in \Lambda \,\} \cong \Lambda$, since the action of Λ on V by translations is faithful. $\square$

Corollary 2.1.2. *There is canonical isomorphism* $\Lambda \cong H_1(X; \mathbb{Z})$. *In particular,* $H_1(X; \mathbb{Z})$ *is a free abelian group of rank* $2g$.

Proof. Note that since $\pi_1(X) \cong \Lambda$ is abelian, it is canonically isomorphic to its abelianization $H_1(X; \mathbb{Z})$. $\square$

In fact, the assignment $V/\Lambda \mapsto \Lambda$ is functorial. Suppose we are given a continuous map $f : V/\Lambda \to V'/\Lambda'$ such that $f(0) = 0$. Then the fact that V is simply connected and $V' \to V'/\Lambda'$ is a covering map implies that f has a unique lift

$$\begin{array}{ccc} V & \overset{F}{\dashrightarrow} & V' \\ \downarrow & & \downarrow \\ V/\Lambda & \xrightarrow{f} & V'/\Lambda' \end{array}$$

such that $F(0) = 0$ [20, Corollary 11.19]. Given $\lambda \in \Lambda$, the map

$$G_\lambda : V \to V', \quad v \mapsto F(v + \lambda) - F(v)$$

is continuous, and its image lies in Λ'. Since Λ' is discrete, G_λ must be constant. It follows that $F(v + \lambda) = F(v) + F(\lambda)$ for all $v \in V$, and in particular, that $F|_\Lambda$ is a group homomorphism. Hence, the assignment

$$\left(V \xrightarrow{f} V'\right) \longmapsto \left(\Lambda \xrightarrow{F|_\Lambda} \Lambda'\right) \tag{2.2}$$

defines a functor, in the following sense:

Proposition 2.1.3. *The assignment* (2.2) *defines a functor from the category whose objects are* g*-dimensional complex tori and whose morphisms are continuous maps preserving the base point* 0*, to the category of abelian groups. The isomorphism of Corollary 2.1.2 is a natural equivalence between this functor and the homology functor* $H_1(-;\mathbb{Z})$*.*

2.2 Definition of $A_{\text{univ}}/\mathfrak{S}_g$

Suppose X and S are complex manifolds. We say that a holomorphic map $f : X \to S$ is a *family of complex manifolds* or a *holomorphic family* if it is proper, and everywhere of maximal rank. By the preimage theorem, all the fibres $X_s := f^{-1}(s)$ are a compact complex manifolds.

Consider a $g \times g$ complex matrix τ with positive definite imaginary part. By definition, such a matrix belongs to the *Siegel upper half-space*

$$\mathfrak{S}_g := \left\{ \tau \in \mathrm{Mat}_{g\times g}(\mathbb{C}) \;\middle|\; {}^t\tau = \tau,\ \mathrm{Im}\,\tau > 0 \right\},$$

a complex manifold of dimension $g(g+1)/2$. Let $e_1, \dots, e_g$ be the columns of the $g \times g$ identity matrix 1_g, and let $\tau_1, \dots, \tau_g$ be the columns of τ. We consider these $2g$ vectors as elements of $\mathbb{R}^{2g}$ via the $\mathbb{R}$-linear isomorphism $z \mapsto (\mathrm{Re}\, z, \mathrm{Im}\, z)$, and write them as a matrix

$$\begin{pmatrix} 1_g & \mathrm{Re}\,\tau \\ 0 & \mathrm{Im}\,\tau \end{pmatrix}.$$

Since $\mathrm{Im}\,\tau$ is positive definite, we have

$$\det \begin{pmatrix} 1_g & \mathrm{Re}\,\tau \\ 0 & \mathrm{Im}\,\tau \end{pmatrix} = \det 1_g \cdot \det \mathrm{Im}\,\tau > 0,$$

so we see that the vectors $e_1, \dots, e_g, \tau_1, \dots, \tau_g$ are linearly independent over $\mathbb{R}$. Hence,

$$\Lambda_\tau := \mathrm{Span}_{\mathbb{Z}}(e_1, \dots, e_g, \tau_1, \dots, \tau_g)$$

is a lattice in $\mathbb{C}^g$, and $A_\tau := \mathbb{C}^g/\Lambda_\tau$ is a complex torus.

We now define $A_{\text{univ}}/\mathfrak{S}_g$, following [5]. Let $\mathbb{Z}^{2g}$ act on the product manifold $\mathbb{C}^g \times \mathfrak{S}_g$ in the following way:

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \cdot (z, \tau) := (z + \tau n_1 + n_2, \tau).$$

Since this action is proper and free, the quotient space $A_{\text{univ}} := \mathbb{Z}^{2g} \backslash \mathbb{C}^g \times \mathfrak{S}_g$ is a complex manifold [18, Theorem 2.2]. The projection $\pi : A_{\text{univ}} \to \mathfrak{S}_g$ onto the second factor is then a family of complex manifolds, whose fibre above τ is A_τ.

This family is trivial as a family of smooth manifolds. To see this, consider the point $i1_g \in \mathfrak{S}_g$. If τ is any point in $\mathfrak{S}_g$, then we can use the identification $\mathbb{C}^g \cong \mathbb{R}^{2g}$ given by $z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$ to define an $\mathbb{R}$-linear isomorphism

$$\Phi_\tau := \begin{pmatrix} 1_g & \operatorname{Re}\tau \\ 0 & \operatorname{Im}\tau \end{pmatrix}^{-1} : \mathbb{C}^g \to \mathbb{C}^g. \tag{2.3}$$

Since Φ_τ maps Λ_τ onto Λ_{i1_g} isomorphically, it lifts to a diffeomorphism $A_\tau \cong A_{i1_g}$, which we also denote by Φ_τ. We define

$$\Phi : A_{\mathrm{univ}} \to A_{i1_g} \times \mathfrak{S}_g, \quad (z, \tau) \mapsto (\Phi_\tau(z), \tau).$$

Note that Φ is a diffeomorphism, and that it makes the diagram

$$\begin{array}{ccc} A_{\mathrm{univ}} & \xrightarrow{\Phi} & A_{i1_g} \times \mathfrak{S}_g \\ \downarrow \pi & \swarrow \mathrm{proj}_2 & \\ \mathfrak{S}_g & & \end{array} \tag{2.4}$$

commute. We say that Φ is a *smooth trivialisation* of the family $A_{\mathrm{univ}}/\mathfrak{S}_g$. In fact, since the base $\mathfrak{S}_g$ is contractible, the existence of a smooth trivialisation for $A_{\mathrm{univ}}/\mathfrak{S}_g$ is a particular case of Ehresmann's lemma:

Theorem 2.2.1. *Let $f : X \to S$ be a proper submersion between smooth manifolds. If S is contractible, then for any point $0 \in S$, there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & X_0 \times S \\ \downarrow f & \swarrow \mathrm{proj}_2 & \\ S & & \end{array},$$

where Φ is a diffeomorphism.

Proof. See [7, Lemma 10.2]. □

2.3 Cohomology varying in a family

In this section, we define a relative notion of cohomology for a holomorphic family of complex manifolds, following the approach taken in [27] and [1]. Our aim is to define a holomorphic vector bundle on the base S (or what is the same, a locally free sheaf of $\mathcal{O}_S$-modules), in such a way that the fibre at a point s can be identified with the singular cohomology space $H^k(X_s; \mathbb{C})$.

Remark. When we say 'relative', we will always mean it in the sense of algebraic geometry (which has to do with Grothendieck's emphasis on studying properties of morphisms rather than objects), and not in the sense of algebraic topology (which is about the singular (co)homology of a space relative to a subspace).

We begin by recalling the notions of constant sheaf and local system. Fix a finite-dimensional complex vector space V. If U is an open subset of S (or X), a map $g : U \to V$ is said to be *locally constant* if it is continuous, when V is considered to have the discrete topology. The *constant sheaf* $\mathcal{V}$ of stalk V has sections

$$\mathcal{V}(U) := \{ \text{locally constant maps } U \to V \};$$

it is the sheaf on S associated to the presheaf $U \mapsto V$. A sheaf $\mathcal{V}$ on S is *locally constant* if there is an open covering $\mathfrak{U}$ of S such that for each $U \in \mathfrak{U}$, the restriction $\mathcal{V}|_U$ is isomorphic to a constant sheaf. We also say that $\mathcal{V}$ is a *local system*.

Now let $\underline{\mathbb{C}}$ denote the constant sheaf of stalk $\mathbb{C}$ on X, and consider the higher pushforward $R^k f_* \underline{\mathbb{C}}$. This is the sheaf on S associated to the presheaf

$$U \mapsto H^k(f^{-1}(U), \underline{\mathbb{C}}|_U).$$

Lemma 2.3.1. *The stalk of* $R^k f_* \underline{\mathbb{C}}$ *at a point* $s \in S$ *is isomorphic to* $H^k(X_s; \mathbb{C})$*.*

Proof. Since the contractible neighbourhoods of s form a local basis for s, in computing the stalk

$$\left(R^k f_* \underline{\mathbb{C}}\right)_s = \varinjlim_{U \ni s} H^k(f^{-1}(U), \underline{\mathbb{C}}|_U)$$

we may assume U is contractible. In this case, we have $f^{-1}(U) \cong X_s \times U$ as differentiable manifolds by Ehresmann's lemma, so

$$H^k(f^{-1}(U), \underline{\mathbb{C}}|_U) \cong H^k(f^{-1}(U); \mathbb{C}) \cong H^k(X_s \times U; \mathbb{C}).$$

But since U is contractible, $X_s \times U$ is homotopy equivalent to U, so we have

$$H^k(X_s \times U; \mathbb{C}) \cong H^k(X_s; \mathbb{C}),$$

by the homotopy invariance of singular cohomology. □

Proposition 2.3.2. *The sheaf* $R^k f_* \underline{\mathbb{C}}$ *is a local system of vector spaces on* S*.*

Proof. Let $U \subseteq S$ be a contractible open set, and choose some base point $0 \in U$. By Ehresmann's lemma, the family is trivial over U, so that we have a commuting diagram

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\Phi} & X_0 \times U \\ \downarrow f & \swarrow \mathrm{proj}_2 & \\ U & & \end{array}.$$

Since Φ is a homeomorphism, we have

$$\begin{aligned} H^k(f^{-1}(V); \mathbb{C}) &= H^k(\Phi^{-1}(\mathrm{proj}_2^{-1}(V)); \mathbb{C}) \\ &\cong H^k(\mathrm{proj}_2^{-1}(V); \mathbb{C}), \end{aligned}$$

so the sheaves $R^k f_* \underline{\mathbb{C}}$ and $R^k \mathrm{proj}_{2*} \underline{\mathbb{C}}$ are isomorphic (here $\underline{\mathbb{C}}$ denotes the constant sheaves of stalk $\mathbb{C}$ on $f^{-1}(U)$ and $X_0 \times U$ respectively).

We claim that $R^k \mathrm{proj}_{2*}\underline{\mathbb{C}}$ is isomorphic to a constant sheaf. Its stalk at every point $s \in U$ is canonically isomorphic to $H^k(X_0;\mathbb{C})$, since U is contractible. However, we still have to show that its sections over an open set $V \subseteq U$ can be identified with the locally constant functions $V \to H^k(X_0;\mathbb{C})$.

By definition, elements of $R^k \mathrm{proj}_{2*}\underline{\mathbb{C}}(V)$ are collections of germs

$$(g_s)_{s\in V} \in \prod_{s\in V} H^k(X_0;\mathbb{C})$$

that are *compatible*, in the sense that for every point $s \in V$, there exists an open neighbourhood $V' \subseteq V$ of s and a section $\sigma \in H^k(\mathrm{proj}_2^{-1}(V'), \underline{\mathbb{C}}|_{V'})$ such that $\sigma_{s'} = g_{s'}$ for all $s' \in V'$. Note that we may assume V' is contractible, since S is locally contractible. In this case, we have

$$\begin{aligned} H^k(\mathrm{proj}_2^{-1}(V'), \underline{\mathbb{C}}|_{V'}) &= H^k(X_0 \times V', \underline{\mathbb{C}}|_{V'}) \\ &\cong H^k(X_0;\mathbb{C}). \end{aligned}$$

It follows that the section $(g_s)_{s\in V}$ must be constant on V', and hence on the connected components of V. □

The next two lemmas imply that $R^k f_*\underline{\mathbb{C}}$ is a locally free $\underline{\mathbb{C}}$-module, where $\underline{\mathbb{C}}$ is the constant sheaf *on the base* S.

Lemma 2.3.3. *Let S be a complex manifold. Then every sheaf $\mathcal{V}$ of finite-dimensional complex vector spaces on S has the structure of a $\underline{\mathbb{C}}$-module.*

Proof. Let $f \in \underline{\mathbb{C}}(U)$ and $\sigma \in \mathcal{V}(U)$ be sections over an open subset U of S. Since f is locally constant, it is constant on the connected components U_i of U, which are open since S is locally connected. Let $c_i \in \mathbb{C}$ be the value of f on U_i, and note that $c_i\sigma|_{U_i} \in \mathcal{V}(U_i)$. We define $f \cdot \sigma \in \mathcal{V}(U)$ to be the section obtained by gluing together the $c_i\sigma|_{U_i}$. □

Lemma 2.3.4. *Let V be an n-dimensional complex vector space, and let $\mathcal{V}$ be the constant sheaf on S of stalk V. Then $\mathcal{V} \cong \underline{\mathbb{C}}^{\oplus n}$ as $\underline{\mathbb{C}}$-modules.*

Proof. If $v_1, \dots, v_n$ is a basis of V, then the constant functions $x \mapsto v_i$ on S are a basis of global sections for $\mathcal{V}$. □

Corollary 2.3.5. *Every local system $\mathcal{V}$ of complex vector spaces is a locally free sheaf of $\underline{\mathbb{C}}$-modules.*

Hence, we can make $R^k f_*\underline{\mathbb{C}}$ into a locally free $\mathcal{O}_S$-module by extending scalars.

Lemma 2.3.6. *If $\mathcal{V}$ is a locally free $\underline{\mathbb{C}}$-module on S, then $\mathcal{O}_S \otimes_{\underline{\mathbb{C}}} \mathcal{V}$ is a locally free $\mathcal{O}_S$-module.*

Proof. Given $f \in \mathcal{O}_S(U)$ and $g \otimes v \in \mathcal{O}_S(U) \otimes_{\underline{\mathbb{C}}(U)} \mathcal{V}(U)$, set

$$f \cdot (g \otimes v) := fg \otimes v.$$

This makes $U \mapsto \mathcal{O}_S(U) \otimes_{\underline{\mathbb{C}}(U)} \mathcal{V}(U)$ into a presheaf of $\mathcal{O}_S$-modules. Hence, its sheafification $\mathcal{O}_S \otimes_{\underline{\mathbb{C}}} \mathcal{V}$ is a sheaf of $\mathcal{O}_S$-modules.

If $U \subseteq S$ is an open set on which $\mathcal{V}$ is constant, then by Lemma 2.3.4, $\mathcal{V}|_U \cong \underline{\mathbb{C}}^{\oplus n}$ for some $n \geq 1$, so we have

$$(\mathcal{O}_S \otimes_{\underline{\mathbb{C}}} \mathcal{V})|_U \cong \mathcal{O}_U \otimes_{\underline{\mathbb{C}}} \mathcal{V}|_U \cong \mathcal{O}_U \otimes_{\underline{\mathbb{C}}} \underline{\mathbb{C}}^{\oplus n} \cong \mathcal{O}_U^{\oplus n},$$

which shows that $\mathcal{O}_S \otimes_{\underline{\mathbb{C}}} \mathcal{V}$ is locally free. □

Combining these facts, we make the following definition:

Definition 2.3.7. Let $f : X \to S$ be a family of complex manifolds. The k*th relative cohomology sheaf* of f is the locally free $\mathcal{O}_S$-module

$$\mathcal{H}^k(X/S) := \mathcal{O}_S \otimes_{\underline{\mathbb{C}}} R^k f_* \underline{\mathbb{C}}.$$

Remark. Note that the stalk of $\mathcal{H}^k(X/S)$ at $s \in S$ is

$$\mathcal{H}^k(X/S)_s = \mathcal{O}_{S,s} \otimes_{\underline{\mathbb{C}}} \left(R^k f_* \underline{\mathbb{C}}\right)_s = \mathcal{O}_{S,s} \otimes_{\underline{\mathbb{C}}} H^k(X_s; \mathbb{C}),$$

while the fibre $\mathcal{O}_{S,s}/\mathfrak{m}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{H}^k(X/S)_s$ may be identified with $H^k(X_s;\mathbb{C})$, since $\mathcal{O}_{S,s}/\mathfrak{m}_s \cong \mathbb{C}$. (Here $\mathfrak{m}_s$ is the maximal ideal of the local ring $\mathcal{O}_{S,s}$.)

Since our family of complex tori $A_{\text{univ}}/\mathfrak{S}_g$ is trivial in the smooth sense, the sheaf $R^k(\text{proj}_2)_* \underline{\mathbb{C}}$ is actually constant. Consequently, $\mathcal{H}^k(A_{\text{univ}}/\mathfrak{S}_g)$ is free. For $k = 1$, we have the following explicit basis.

Proposition 2.3.8. *There exists a basis of global sections*

$$\alpha_1, \dots \alpha_g, \beta_1, \dots, \beta_g$$

of $\mathcal{H}^1(A_{\text{univ}}/\mathfrak{S}_g)$ *such that for all* $\tau \in \mathfrak{S}_g$*, we have* $(\alpha_j)_\tau = e_j^*$ *and* $(\beta_j)_\tau = \tau_j^*$ *in the fibre* $(\Lambda_\tau \otimes_{\mathbb{Z}} \mathbb{C})^*$.

Proof. First, we define sections for the family $\text{proj}_2 : A_{i1_g} \times \mathfrak{S}_g \to \mathfrak{S}_g$. Write $f_j := ie_j$, and note that the $\mathbb{Z}$-basis $e_1, \dots, e_g, f_1, \dots, f_g$ for the lattice Λ_{i1_g} also forms a basis for the $2g$-dimensional complex vector space $\Lambda_{i1_g} \otimes_{\mathbb{Z}} \mathbb{C}$ (of course, the map $\Lambda_{i1_g} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{C}^g$ is not injective). Hence, the constant functions $\tau \mapsto e_j^*$, $\tau \mapsto f_j^*$ define global sections of the sheaf $R^1\text{proj}_{2*}\underline{\mathbb{C}}$, which is constant of stalk $H^1(A_{i1_g};\mathbb{C}) = (\Lambda_{i1_g} \otimes_{\mathbb{Z}} \mathbb{C})^*$.

Next, recall that we have an isomorphism $R^1\text{proj}_{2*}\underline{\mathbb{C}} \cong R^1\pi_*\underline{\mathbb{C}}$, which is given on the level of the underlying presheaves by

$$\Phi^* : H^1(\text{proj}_2^{-1}(V);\mathbb{C}) \to H^1(\Phi^{-1}(\text{proj}_2^{-1}(V));\mathbb{C}),$$

where Φ is the diffeomorphism in Diagram (2.4). So we may define global sections of $R^1\pi_*\underline{\mathbb{C}}$ by setting

$$\alpha_j := \Phi^*(\tau \mapsto e_j^*), \quad \beta_j := \Phi^*(\tau \mapsto ie_j^*).$$

These sections form a $\underline{\mathbb{C}}$-basis for $R^1\pi_*\underline{\mathbb{C}}$, so by extending scalars, we get an $\mathcal{O}_{\mathfrak{S}_g}$-basis for $\mathcal{H}^1(A_{\text{univ}}/\mathfrak{S}_g)$, which we denote by the same symbols.

It remains to check that the α_j and β_j restrict to the right thing on the fibres. Let $\tau \in \mathfrak{S}_g$. For every contractible neighbourhood V of τ, we have a commuting diagram

$$\begin{array}{ccc} H^1(A_{i1_g};\mathbb{C}) & \xrightarrow{\Phi_\tau^*} & H^1(A_\tau;\mathbb{C}) \\ \uparrow & & \uparrow \\ H^1(A_{i1_g} \times V;\mathbb{C}) & \xrightarrow{\Phi^*} & H^1(\pi^{-1}(V);\mathbb{C}), \end{array}$$

where Φ_τ is the $\mathbb{R}$-linear automorphism of $\mathbb{C}^g$ which fixes the e_j's and sends τ_j to ie_j (see Equation (2.3)), and the vertical maps may be interpreted as passing to the stalks of the presheaves at τ. So on the stalks, the isomorphism $R^1\mathrm{proj}_{2*}\underline{\mathbb{C}} \cong R^1\pi_*\underline{\mathbb{C}}$ is given by Φ_τ^*. But under the natural equivalence of Proposition 2.1.3, Φ_τ^* corresponds to the map

$$(\Lambda_{i1_g} \otimes_\mathbb{Z} \mathbb{C})^* \to (\Lambda_\tau \otimes_\mathbb{Z} \mathbb{C})^*, \quad \psi \mapsto \psi \circ \Phi_\tau.$$

Because $e_j^* \circ \Phi_\tau = e_j^*$ and $f_j^* \circ \Phi_\tau = \tau_j^*$, we have

$$(\alpha_j)_\tau = e_j^* \circ \Phi_\tau = e_j^*, \quad (\beta_j)_\tau = f_j^* \circ \Phi_\tau = \tau_j^*,$$

as required. □

Chapter 3

de Rham cohomology

This chapter is about the de Rham cohomology of the family $A_{\mathrm{univ}}/\mathfrak{S}_g$. The fibre A_τ has a Hodge decomposition

$$H^1_{\mathrm{dR}}(A_\tau) = \mathbb{C}\, du_1 \oplus \cdots \oplus \mathbb{C}\, du_g \oplus \mathbb{C}\, d\bar{u}_1 \oplus \cdots \oplus \mathbb{C}\, d\bar{u}_g, \qquad (3.1)$$

where $u_1, \ldots, u_g$ are the standard coordinates on $\mathbb{C}^g$. Our main goal is to show that this decomposition holds on the cohomology of the family, which we interpret here as the de Rham cohomology sheaf $\mathcal{H}^1_{\mathrm{dR}}(A_{\mathrm{univ}}/\mathfrak{S}_g)$. We will show that this sheaf is free on certain sections which globalise the differential forms in (3.1).

In Section 3.1, we review differential forms on a complex manifold X. In Section 3.2, we define the relative de Rham cohomology sheaf $\mathcal{H}^k_{\mathrm{dR}}(X/S)$. We then show that $\mathcal{H}^k_{\mathrm{dR}}(X/S)$ is computed by the smooth relative de Rham complex, and use this fact to establish its basic properties. In Section 3.3, we prove the relative Hodge decomposition for $\mathcal{H}^1_{\mathrm{dR}}(X/S)$, assuming that the usual Hodge decomposition holds on the fibres, and we use the Hodge-to-de Rham spectral sequence to identify the holomorphic part of the decomposition $\mathcal{H}^{1,0}$ with $f_*\Omega^1_{X/S}$. Finally, in Section 3.4, we give a short proof of the Hodge decomposition for a complex torus, and deduce the Hodge decomposition for $\mathcal{H}^1_{\mathrm{dR}}(A_{\mathrm{univ}}/\mathfrak{S}_g)$.

3.1 Differential forms

In this section, we review the concepts of differential forms and de Rham cohomology on a complex manifold X, following Chapter 2 of [27] and Sections 2.1 and 2.2 of [4].

Let X be a complex manifold, and let $T_{X,\mathbb{R}}$ denote the tangent bundle on X, considered as a real manifold. Recall that the complexified tangent bundle $T_{X,\mathbb{C}} := T_{X,\mathbb{R}} \otimes \mathbb{C}$ can be written as a direct sum

$$T_{X,\mathbb{C}} = T_X \oplus T'_X,$$

where T_X is the holomorphic tangent bundle, and $T'_X = \overline{T_X}$ is its complex conjugate [27, Proposition 2.13]. Hence, the complexified cotangent bundle, which is the dual bundle of $T_{X,\mathbb{C}}$, decomposes as

$$T^\vee_{X,\mathbb{C}} = T^\vee_X \oplus (T'_X)^\vee.$$

These bundles are trivial over any holomorphic chart $(U, (z_1, \dots, z_n))$: local frames for the tangent subbundles T_X and T'_X are given by the vector fields $\partial/\partial z_i$ and $\partial/\partial \bar{z}_i$, while their duals $T_X^\vee$ and $(T'_X)^\vee$ are trivialised by the dual frames dz_i and $d\bar{z}_i$.

If we take exterior powers of the cotangent bundle, we see that it decomposes as

$$\bigwedge^k T_{X,\mathbb{C}}^\vee = \bigoplus_{p+q=k} \bigwedge^p T_X^\vee \otimes_{\mathbb{C}} \bigwedge^q (T'_X)^\vee. \tag{3.2}$$

We denote the sheaf of smooth sections of $T_{X,\mathbb{C}}^\vee$ by $\mathcal{A}_X^k$ – it is the *sheaf of smooth* k*-forms* on X. The decomposition (3.2) induces a decomposition of $\mathcal{A}_X^k$:

$$\mathcal{A}_X^k = \bigoplus_{p+q=k} \mathcal{A}^{p,q}, \tag{3.3}$$

where $\mathcal{A}^{p,q}$ is the sheaf of smooth sections of $\bigwedge^p T_X^\vee \otimes_{\mathbb{C}} \bigwedge^q (T'_X)^\vee$. We say that a section ω of $\mathcal{A}^{p,q}$ has *type* (p, q); in local coordinates $z_1, \dots, z_n$, we can write ω as a linear combination of basic (p, q)-forms

$$dz_I \wedge d\bar{z}_J := dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

where the coefficients are smooth complex-valued functions. We refer to (3.3) as the *type decomposition*.

We now recall the definition of de Rham cohomology. The *de Rham complex* is the complex of sheaves

$$0 \longrightarrow \mathcal{C}_X^\infty \xrightarrow{d} \mathcal{A}_X^1 \xrightarrow{d} \mathcal{A}_X^2 \longrightarrow \cdots,$$

where $\mathcal{C}_X^\infty$ is the sheaf of smooth complex-valued functions on X, and the exterior derivative d is given locally by

$$d(f\, dz_I \wedge d\bar{z}_J) := df \wedge dz_I \wedge d\bar{z}_J.$$

Taking global sections, we get a complex of $\mathbb{C}$-vector spaces

$$0 \longrightarrow C^\infty(X) \xrightarrow{d} A^1(X) \xrightarrow{d} A^2(X) \longrightarrow \cdots$$

whose cohomology in degree k is by definition the k*th de Rham cohomology group* of X, denoted by $H_{dR}^k(X)$.

The type decomposition can sometimes pass to the cohomology of X. Let $H^{p,q}(X)$ be the subspace of $H_{dR}^{p+q}(X; \mathbb{C})$ consisting of classes representable by a closed (p, q)-form.

Definition 3.1.1. We say that the de Rham cohomology of X carries a *Hodge decomposition* if for all $k \geq 0$, we have

$$H_{dR}^k(X; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

Remark. The main theorem of Hodge theory states that the de Rham cohomology of X has a Hodge decomposition if X is compact and admits a Kähler metric; see e.g. [7, Theorem 8.5].

Next, we show how the type decomposition induces a decomposition of the exterior derivative d. Suppose u is a section of $\mathcal{A}^{p,q}$ over some open set $U \subseteq X$. If we write u in local coordinates $z_1, \ldots, z_n$ as $u = \sum u_{I,J}\, dz_I \wedge d\bar{z}_J$, then du is a sum of terms of the form

$$\sum_{1 \le i \le n} \frac{\partial u_{I,J}}{\partial z_i} dz_i \wedge dz_I \wedge d\bar{z}_J + \sum_{1 \le i \le n} \frac{\partial u_{I,J}}{\partial \bar{z}_i} d\bar{z}_i \wedge dz_I \wedge d\bar{z}_J.$$

It follows that $du \in \mathcal{A}^{p+1,q}(U) \oplus \mathcal{A}^{p,q+1}(U)$, since the notion of type is a coordinate-invariant. Letting ∂u and $\bar{\partial} u$ denote the components of du of type $(p+1, q)$ and $(p, q+1)$ defines sheaf maps

$$\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}, \qquad \bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1},$$

such that $d = \partial + \bar{\partial}$. Note that $d^2 = (\partial + \bar{\partial})^2 = 0$ implies the relations

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

In the language of homological algebra, these relations say exactly that the diagram

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathcal{A}_X^{0,2} & \xrightarrow{\partial} & \mathcal{A}_X^{1,2} & \xrightarrow{\partial} & \mathcal{A}_X^{2,2} & \longrightarrow & \cdots \\
\bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \\
\mathcal{A}_X^{0,1} & \xrightarrow{\partial} & \mathcal{A}_X^{1,1} & \xrightarrow{\partial} & \mathcal{A}_X^{2,1} & \longrightarrow & \cdots \\
\bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \\
\mathcal{A}_X^{0,0} & \xrightarrow{\partial} & \mathcal{A}_X^{1,0} & \xrightarrow{\partial} & \mathcal{A}_X^{2,0} & \longrightarrow & \cdots
\end{array} \tag{3.4}$$

is a bicomplex, for which the de Rham complex $\mathcal{A}_X^{\cdot}$ is the associated total complex.

We denote by

$$\Omega_X^p := \ker\left(\bar{\partial} : \mathcal{A}_X^{p,0} \to \mathcal{A}_X^{p,1}\right)$$

the sheaf of *holomorphic* p*-forms on* X, and we define the *holomorphic de Rham complex* $\Omega_X^{\cdot}$ to be the complex

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\partial} \Omega_X^1 \xrightarrow{\partial} \Omega_X^2 \longrightarrow \cdots,$$

where the differential ∂ is the restriction of the differential of $\mathcal{A}_X^{\cdot,0}$ to $\Omega_X^{\cdot}$.

3.2 de Rham cohomology in a family

Now suppose $f : X \to S$ is a family of complex manifolds. Let $X_s = f^{-1}(s)$ be the fibre of f over some point $s \in S$. Since the inclusion $\iota : X_s \to X$ is a smooth embedding, the tangent map $T\iota$ identifies the tangent bundle $T_{X_s,\mathbb{R}}$ of X_s with a subbundle of $T_{X,\mathbb{R}}$. We have $Tf \circ T\iota = 0$ since $f \circ \iota$ is constant, so there is a complex of vector bundles on X_s:

$$0 \longrightarrow T_{X_s,\mathbb{R}} \longrightarrow (T_{X,\mathbb{R}})|_{X_s} \xrightarrow{Tf|_{X_s}} (f^*T_{S,\mathbb{R}})|_{X_s} \longrightarrow 0, \qquad (3.5)$$

which turns out to be exact, as one sees by applying the dimension formula for linear maps on the fibres [19, Proposition 5.38]. We define the *relative* (or *vertical*) *tangent bundle* $T_{X/S,\mathbb{R}}$ by the short exact sequence

$$0 \longrightarrow T_{X/S,\mathbb{R}} \longrightarrow T_{X,\mathbb{R}} \xrightarrow{Tf} f^*T_{S,\mathbb{R}} \longrightarrow 0;$$

by (3.5), we have $(T_{X/S,\mathbb{R}})|_{X_s} = T_{X_s,\mathbb{R}}$. The complexified relative tangent bundle is denoted $T_{X/S,\mathbb{C}} := T_{X/S,\mathbb{R}} \otimes \mathbb{C}$. It follows from the fact that f is holomorphic that $f^*T_{S,\mathbb{C}} = f^*T_S \oplus f^*T'_S$, and that $Tf \otimes \mathbb{C}$ is the direct sum of the maps

$$(Tf)^{1,0} : T_X \to f^*T_S \quad \text{and} \quad (Tf)^{0,1} : T'_X \to f^*T'_S$$

(see [27, Subsection 2.2.1]), so we have a decomposition

$$T_{X/S,\mathbb{C}} = T_{X/S} \oplus T'_{X/S},$$

where $T_{X/S} := T_X/f^*T_S$ and $T'_{X/S} := T'_X/f^*T'_S$.

The *complexified relative cotangent bundle* is the dual bundle

$$T^\vee_{X/S,\mathbb{C}} = T^\vee_{X/S} \oplus (T'_{X/S})^\vee.$$

Sections of its kth exterior power are called *relative* k*-forms*; they form a locally free $\mathcal{C}^\infty_X$-module which we will denote by $\mathcal{A}^k_{X/S}$. As in the previous section, we have a decomposition of $\mathcal{A}^k_{X/S}$ into types:

$$\mathcal{A}^k_{X/S} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_{X/S}.$$

The next lemma will allow us to define a relative version of the de Rham complex.

Lemma 3.2.1. *We have* $\mathcal{A}^k_{X/S} = \mathcal{A}^k_X/f^*\mathcal{A}^k_S$*, where* $f^*\mathcal{A}^k_S := \mathcal{C}^\infty_X \otimes_{f^{-1}\mathcal{C}^\infty_S} f^{-1}\mathcal{A}^k_S$ *is the pullback sheaf.*

Proof. We have a short exact sequence of vector bundles on X:

$$0 \longrightarrow f^*\big(\textstyle\bigwedge^k T^\vee_{S,\mathbb{C}}\big) \longrightarrow \textstyle\bigwedge^k T^\vee_{X,\mathbb{C}} \longrightarrow \textstyle\bigwedge^k T^\vee_{X/S,\mathbb{C}} \longrightarrow 0.$$

Since the category of complex vector bundles on X is equivalent to the category of finite locally free $\mathcal{C}^\infty_X$-modules [28, Proposition 8.45], taking sheaves of sections gives a short exact sequence in the latter category. But the sheaf of sections of a pullback bundle is the pullback of its sheaf of sections (see [28, Problem 1.18]), so this sequence is

$$0 \longrightarrow f^*\mathcal{A}^k_S \longrightarrow \mathcal{A}^k_X \longrightarrow \mathcal{A}^k_{X/S} \longrightarrow 0.$$

□

Note also that the pullback of the differential $f^* d_S$ defines a subcomplex $f^*\mathcal{A}^\bullet_S$ of $\mathcal{A}^\bullet_X$.

Definition 3.2.2. We define the *relative de Rham complex* to be the quotient complex $\mathcal{A}^\bullet_{X/S} = \mathcal{A}^\bullet_X / f^*\mathcal{A}^\bullet_S$.

Note that $(\mathcal{A}^k_{X/S})|_{X_s} = \mathcal{A}^k_{X_s}$. Hence we can recover $\mathcal{A}^\bullet_{X_S}$ from $\mathcal{A}^\bullet_{X/S}$ by restricting to the fibre X_s. As in the previous section, the exterior derivative decomposes as $d_{X/S} = \partial_{X/S} + \bar{\partial}_{X/S}$, and $\mathcal{A}^\bullet_{X/S}$ is the totalisation of the bicomplex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathcal{A}^{0,2}_{X/S} & \xrightarrow{\partial} & \mathcal{A}^{1,2}_{X/S} & \xrightarrow{\partial} & \mathcal{A}^{2,2}_{X/S} & \longrightarrow & \cdots \\
\bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \\
\mathcal{A}^{0,1}_{X/S} & \xrightarrow{\partial} & \mathcal{A}^{1,1}_{X/S} & \xrightarrow{\partial} & \mathcal{A}^{2,1}_{X/S} & \longrightarrow & \cdots \\
\bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \\
\mathcal{A}^{0,0}_{X/S} & \xrightarrow{\partial} & \mathcal{A}^{1,0}_{X/S} & \xrightarrow{\partial} & \mathcal{A}^{2,0}_{X/S} & \longrightarrow & \cdots
\end{array} \tag{3.6}$$

Definition 3.2.3. The *relative holomorphic de Rham complex* is defined by

$$\Omega^\bullet_{X/S} := \ker\left(\bar{\partial} : \mathcal{A}^{\bullet,0}_{X/S} \to \mathcal{A}^{\bullet,1}_{X/S}\right).$$

Note that $\Omega^\bullet_{X/S}$ is a complex of $\mathcal{O}_X$-modules, but that its differential $\partial_{X/S}$ is not $\mathcal{O}_X$-linear, only $f^{-1}\mathcal{O}_S$-linear.

Definition 3.2.4. The k*th relative de Rham cohomology sheaf* is defined by

$$\mathcal{H}^k_{dR}(X/S) := \mathbb{R}^k f_*(\Omega^\bullet_{X/S}),$$

where $\mathbb{R}^k f_*$ is the kth right derived functor of the pushforward functor f_*.

To actually compute $\mathcal{H}^k_{dR}(X/S)$, we will use the complex of smooth relative forms $\mathcal{A}^\bullet_{X/S}$. Recall that a module over a sheaf of rings $\mathcal{R}$ on X is called *fine* if for every open cover of X, there exists a partition of unity with respect to $\mathcal{R}$ that is subordinate to this cover [27, Definition 4.35], and that fine sheaves are acyclic. Since $\mathcal{C}^\infty_X$ has this property, $\mathcal{C}^\infty_X$-modules such as $\mathcal{A}^k_{X/S}$ and $\mathcal{A}^{p,q}_{X/S}$ are acyclic. The following lemma is one consequence.

Lemma 3.2.5. *The presheaf* $U \mapsto \ker d^k_{X/S}(U)/\operatorname{im} d^{k-1}_{X/S}(U)$ *on* X *is already a sheaf.*

Proof. Apply $\Gamma(U, -)$ to the short exact sequence

$$0 \longrightarrow \operatorname{im} d^k_{X/S}\big|_U \longrightarrow \ker d^k_{X/S}\big|_U \longrightarrow \frac{\ker d^k_{X/S}}{\operatorname{im} d^{k-1}_{X/S}}\Big|_U \longrightarrow 0,$$

and note that the resulting sequence is still exact, since the sheaf $\operatorname{im} d^k_{X/S}|_U$ is fine. □

The next result is an extension of the classical Dolbeault resolution to the relative situation.

Lemma 3.2.6. *The inclusion $\Omega^{\bullet}_{X/S} \hookrightarrow \mathcal{A}^{\bullet}_{X/S}$ is a quasi-isomorphism.*

Proof. Because the de Rham complex $\mathcal{A}^{\bullet}_{X/S}$ is the totalisation of the bicomplex $\mathcal{A}^{\bullet,\bullet}_{X/S}$, is suffices to show that the columns of Diagram (3.6) are exact, so that for each $p \geq 0$, the complex $(\mathcal{A}^{p,\bullet}_{X/S}, \bar{\partial})$ is an acyclic resolution of $\Omega^{p}_{X/S}$ via the inclusion $\Omega^{p}_{X/S} \hookrightarrow \mathcal{A}^{p,0}_{X/S}$ [27, Lemma 8.5]. Since we already know that the sheaves $\mathcal{A}^{p,q}_{X/S}$ are acyclic, it suffices to check that $\mathcal{A}^{p,\bullet}_{X/S}$ is exact in positive degrees.

Let $q > 0$. We have to show that the complex of sheaves

$$\mathcal{A}^{p,q-1}_{X/S} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,q}_{X/S} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,q+1}_{X/S} \tag{3.7}$$

is exact. Restricting to the fibre X_s, we obtain a sequence

$$\mathcal{A}^{p,q-1}_{X_s} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,q}_{X_s} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,q+1}_{X_s} \ , \tag{3.8}$$

which we already know to be exact, since it is part of the usual Dolbeault resolution of $\Omega^{p}_{X_s}$ [27, Proposition 2.31]. To deduce the exactness of (3.7) from that of (3.8), we will make use of the following fact: since X is a locally compact space, for any compact subset $K \subseteq X$ and any abelian sheaf $\mathcal{F}$ on X, we have

$$\Gamma(K, \mathcal{F}|_K) := \Gamma\left(K, \iota^{-1}\mathcal{F}\right) \cong \varinjlim_{U \supseteq K} \mathcal{F}(U), \tag{3.9}$$

where $\iota : K \to X$ is the inclusion, and the colimit is taken over open sets U containing K [16, Theorem 2.2]. (In general, the association

$$W \mapsto \varinjlim_{U \supseteq W} \mathcal{F}(U),$$

for W an open subset of K, only defines a presheaf on K, for which $\iota^{-1}\mathcal{F}$ is the associated sheaf.)

We will show exactness of (3.7) on the stalks. Let $x \in X_s \subseteq X$, and let

$$\omega_x \in \ker\left(\bar{\partial}_{X/S}\right)_x \subseteq \left(\mathcal{A}^{p,q}_{X/S}\right)_x$$

be the germ at x of a section ω of $\mathcal{A}^{p,q}_{X/S}$, defined in some open subset of X. Note that if we restrict ω to X_s, we obtain a section $\omega|_{X_s}$ of $\mathcal{A}^{p,q}_{X_s}$ such that

$$(\omega|_{X_s})_x \in \ker\left(\bar{\partial}_{X_s}\right)_x \subseteq \left(\mathcal{A}^{p,q}_{X_s}\right)_x.$$

By exactness of (3.8) at x, there exists an open subset $W \subseteq X_s$ and a section α of $\mathcal{A}^{p,q-1}_{X_s}$ over W, such that $\omega|_W = \bar{\partial}_{X_s}\alpha$. Let K be a compact subset of W such that $x \in K$. Now

$$\alpha|_K \in \Gamma\left(K, \mathcal{A}^{p,q-1}_{X_s}|_K\right) = \Gamma\left(K, \mathcal{A}^{p,q-1}_{X/S}|_K\right),$$

so by (3.9), there exists an open subset U of X containing K, and a section β of $\mathcal{A}^{p,q-1}_{X/S}$ over U, such that $\beta|_K = \alpha|_K$. We have

$$\bar{\partial}_{X/S}\beta|_K = \bar{\partial}_{X_s}(\beta|_K) = \bar{\partial}_{X_s}(\alpha|_K) = \omega|_K.$$

By definition of the colimit, there must be some open set V with $K \subseteq V \subseteq U$, such that $\bar{\partial}_{X/S}\beta|_V = \omega|_V$. Hence, $\omega_x = (\bar{\partial}_{X/S}\beta)_x$, as required. □

Theorem 3.2.7. *The kth relative de Rham cohomology sheaf is the pushforward of the kth cohomology sheaf of the complex $\mathcal{A}^{\bullet}_{X/S}$:*

$$\mathcal{H}^k_{dR}(X/S) = f_*\left(\frac{\ker d^k_{X/S}}{\operatorname{im} d^{k-1}_{X/S}}\right).$$

Proof. Note that the sheaves $\mathcal{A}^k_{X/S}$ are acyclic for the functor f_*, since they are acyclic for the global sections functor, and for any abelian sheaf $\mathcal{F}$ on X, $R^i f_*(\mathcal{F})$ is the sheaf on S associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(U)})$ [14, Proposition 8.1]. Hence, to compute $\mathcal{H}^k_{dR}(X/S)$, we apply f_* to $\mathcal{A}^{\bullet}_{X/S}$ and take cohomology. Since f_* preserves the exactness of the sequence

$$0 \longrightarrow \ker d^k_{X/S} \longrightarrow \mathcal{A}^k_{X/S} \xrightarrow{d^k_{X/S}} \operatorname{im} d^k_{X/S} \longrightarrow 0,$$

we have

$$\mathcal{H}^k_{dR}(X/S) = \frac{\ker\left(f_* d^k_{X/S}\right)}{\operatorname{im}\left(f_* d^{k-1}_{X/S}\right)} = f_*\left(\frac{\ker d^k_{X/S}}{\operatorname{im} d^{k-1}_{X/S}}\right).$$

□

We can use this to verify that $\mathcal{H}^k_{dR}(X/S)$ restricts to the right thing on the fibres.

Corollary 3.2.8. *The stalk of $\mathcal{H}^k_{dR}(X/S)$ at a point $s \in S$ can be identified canonically with $\mathcal{O}_{S,s} \otimes H^k_{dR}(X_s)$.*

Proof. Consider the Cartesian square

$$\begin{array}{ccc} X_s & \xrightarrow{\iota} & X \\ \downarrow & & \downarrow f \\ \{s\} & \longrightarrow & S. \end{array}$$

If $\mathcal{F}$ is an abelian sheaf on X, then the pullback of $f_*\mathcal{F}$ along the inclusion $\{s\} \to S$ is the stalk $\mathcal{F}_s$, by definition. On the other hand, if we push $\iota^{-1}\mathcal{F}$ forward onto $\{s\}$, we get $\Gamma(X_s, \iota^{-1}\mathcal{F})$. We claim $\mathcal{F}_s = \Gamma(X_s, \iota^{-1}\mathcal{F})$. Since f is closed, we have

$$\mathcal{F}_s = \varinjlim_{V \ni s} \mathcal{F}(f^{-1}(V)) = \varinjlim_{U \supseteq X_s} \mathcal{F}(U)$$

(see the proof of Theorem 6.2 in [16]). But X_s is compact, so this is equal to $\Gamma(X_s, \iota^{-1}\mathcal{F})$ by (3.9).

If we apply this to the abelian sheaf $\ker d^k_{X/S} / \operatorname{im} d^{k-1}_{X/S}$, we see that $\mathcal{H}^k_{dR}(X/S)_s = H^k_{dR}(X_s)$ as abelian groups. Hence, we have $\mathcal{H}^k_{dR}(X/S)_s = \mathcal{O}_{S,s} \otimes H^k_{dR}(X_s)$ as $\mathcal{O}_{S,s}$-modules. □

Finally, the following comparison theorem shows $\mathcal{H}^k_{dR}(X/S)$ is isomorphic to the cohomology sheaf $\mathcal{H}^k(X/S)$ defined in the previous chapter.

Theorem 3.2.9. *We have an isomorphism*

$$\mathcal{H}^k_{dR}(X/S) \cong \mathcal{H}^k(X/S) := \mathcal{O}_S \otimes_{\underline{\mathbb{C}}} R^k f_* \mathbb{C},$$

which restricts the ordinary de Rham isomorphism on the fibres.

Proof. This follows from the fact that $\Omega^\bullet_{X/S}$ is a resolution for $f^{-1}(\mathcal{O}_S)$. For details, see [1, Section 1.3]. □

3.3 The Hodge decomposition for $\mathcal{H}^1_{dR}(X/S)$

In this section, we prove a relative version of the Hodge decomposition for $\mathcal{H}^1_{dR}(X/S)$, assuming that the Hodge decomposition holds on the fibres. For example, this is the case if X/S is a family of compact Kähler manifolds.

The *Hodge filtration* is the filtration on the de Rham complex given by

$$F^p\mathcal{A}^k_{X/S} := \bigoplus_{r \geq p} \mathcal{A}^{r,k-r}.$$

It induces a filtration on cohomology:

$$F^p\mathcal{H}^k_{dR}(X/S) := \mathbb{R}^k f_* \left(F^p\mathcal{A}^\bullet_{X/S} \hookrightarrow \mathcal{A}^\bullet_{X/S} \right).$$

From Theorem 3.2.7 and Lemma 3.2.5, we see that sections of $F^p\mathcal{H}^k_{dR}(X/S)$ are precisely those cohomology classes which are representable by elements of $\bigoplus_{r\geq p} \mathcal{A}^{r,k-r}$.

Proposition 3.3.1. *Suppose $f : X \to S$ is a holomorphic family such that the Hodge decomposition $H^1_{dR}(X_s) = H^{1,0}(X_s) \oplus H^{0,1}(X_s)$ exists for all $s \in S$. Then $\mathcal{H}^1_{dR}(X/S)$ has a decomposition*

$$\mathcal{H}^1_{dR}(X/S) = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1},$$

where $\mathcal{H}^{1,0}$ and $\mathcal{H}^{0,1}$ are locally free $\mathcal{C}^\infty_S$-submodules whose fibres at s can be identified with $H^{1,0}(X_s)$ and $H^{0,1}(X_s)$ respectively.

Proof. Define $\mathcal{H}^{1,0} := F^1\mathcal{H}^1_{dR}(X/S)$ and $\mathcal{H}^{0,1} := \overline{\mathcal{H}^{1,0}}$. These are the subsheaves of $\mathcal{H}^1_{dR}(X/S)$ consisting of cohomology classes representable by forms of type $(1,0)$ and type $(0,1)$ respectively. Since restricting a form to smaller open subsets preserves its type, we have $\mathcal{H}^{1,0}_s = \mathcal{C}^\infty_{S,s} \otimes H^{1,0}(X_s)$ and $\mathcal{H}^{0,1}_s = \mathcal{C}^\infty_{S,s} \otimes H^{0,1}(X_s)$ for all $s \in S$.

To see that we have a direct sum, suppose $U \subseteq S$ is open. It follows immediately from the type decomposition that $\mathcal{H}^1_{dR}(X/S)(U) = \mathcal{H}^{1,0}(U) + \mathcal{H}^{0,1}(U)$. Let $\omega \in \mathcal{H}^{1,0}(U) \cap \mathcal{H}^{0,1}(U)$. We then have $\omega_s \in H^{1,0}(X_s) \cap H^{0,1}(X_s) = \{0\}$ for all $s \in U$, since the Hodge decomposition holds for the fibre X_s. Hence, $\omega = 0$, and $\mathcal{H}^1_{dR}(X/S)(U) = \mathcal{H}^{1,0}(U) \oplus \mathcal{H}^{0,1}(U)$. □

We will describe this decomposition in another way, using the Hodge-to-de Rham spectral sequence. We begin by recalling the spectral sequence associated to a bicomplex $(E^{\bullet,\bullet}, d_{\rightarrow}, d_{\uparrow})$ of abelian sheaves, following [26, Part I, Section 1.7]. A similar definition applies to a bicomplex of abelian groups (or in any abelian category). For each $r \geq 0$, we have a collection or *page* of abelian sheaves $E_r = \{E_r^{p,q}\}$, where $p, q \in \mathbb{Z}$, together with a sheaf map $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfying $d_r^2 = 0$. On page E_0, we have $E_0^{p,q} = E^{p,q}$ and $d_0 = d_{\uparrow}$. The first page E_1 is then calculated from E_0 by taking cohomology: we have

$$E_1^{p,q} = \frac{\ker\left(d_{\uparrow} : E^{p,q} \to E^{p,q+1}\right)}{\operatorname{im}\left(d_{\uparrow} : E^{p,q-1} \to E^{p,q}\right)},$$

and d_1 is induced from $d_{\rightarrow}$ (this makes sense, because all the squares anticommute). This process can be continued. Eventually, it can happen that for r large enough, the differential $d_r = 0$. We then have

$$E_r = E_{r+1} = E_{r+2} = \cdots,$$

and we say the spectral sequence *degenerates at* E_r. The point is that the limiting objects $E_\infty^{p,q} := E_r^{p,q}$ partially compute the cohomology of the total complex $E^{\bullet} := \operatorname{Tot}(E^{\bullet,\bullet})$ associated to our bicomplex $E^{\bullet,\bullet}$. That is, if we equip $E^{\bullet}$ with the grading $F^pE^k := \oplus_{i\geq p} E^{i,p-i}$, and consider the induced grading $F^pH^k(E^{\bullet})$ on cohomology, then the graded pieces of the cohomology are given by

$$E_\infty^{p,q} = \frac{F^pH^{p+q}(E^{\bullet})}{F^{p+1}H^{p+q}(E^{\bullet})}.$$

Definition 3.3.2. If X is a complex manifold, then its *Hodge-to-de Rham spectral sequence* is the spectral sequence associated to the bicomplex of abelian groups $\Gamma(X, \mathcal{A}_X^{\bullet,\bullet})$ obtained by applying the global sections functor to (3.4).

Note that if X is compact and has a Hodge decomposition, then its Hodge-to-de Rham spectral sequence degenerates on the first page; see the argument preceding the statement of Theorem 9.10 in [7].

Definition 3.3.3. If $f : X \to S$ is a family of complex manifolds, then the *Hodge-to-de Rham spectral sequence* for this family is the spectral sequence associated to the bicomplex $f_*\mathcal{A}_{X/S}^{\bullet,\bullet}$ (i.e. the pushforward of (3.6) onto S).

Theorem 3.3.4. *Let* $f : X \to S$ *be a holomorphic family, such that the Hodge decomposition holds on the fibres. Then the Hodge-to-de Rham spectral sequence for* f *degenerates at the first page* E_1*.*

Proof. Our assumption implies that for all $s \in S$, the Hodge-to-de Rham spectral sequence for X_s degenerates on the first page. Now if we pass to the fibres at s in the Hodge-to-de Rham spectral sequence for f, we get the Hodge-to-de Rham spectral sequence for X_s (c.f. [1, Theorem 1.2, (iv)]), so the maps ∂ in E_1 satisfy $\partial_s = 0$ for all $s \in S$. Since being 0 is a stalk-local property for a morphism, this implies $\partial = 0$, as required. □

Remark. For a proof in the algebraic category, see [6, Theorem 5.5].

Note that the qth row of the first page E_1 looks like

$$R^q f_*(\mathcal{O}_X) \xrightarrow{\partial} R^q f_*(\Omega^1_{X/S}) \xrightarrow{\partial} R^q f_*(\Omega^2_{X/S}) \longrightarrow \cdots,$$

since $\mathcal{A}^{p,\bullet}_{X/S}$ is an acyclic resolution of $\Omega^p_{X/S}$. So if the Hodge-to-de Rham spectral sequence for f degenerates at E_1, it makes the identification

$$R^q f_*(\Omega^p_{X/S}) = \frac{F^p \mathcal{H}^{p+q}_{dR}(X/S)}{F^{p+1}\mathcal{H}^{p+q}_{dR}(X/S)}.$$

Taking $p + q = 1$, we get a short exact sequence

$$0 \longrightarrow f_*\Omega^1_{X/S} \longrightarrow \mathcal{H}^1_{dR}(X/S) \longrightarrow R^1 f_* \mathcal{O}_S \longrightarrow 0, \tag{3.10}$$

which we refer to as the *Hodge exact sequence*.

Theorem 3.3.5. *Suppose* $f : X \to S$ *is a family of complex manifolds, such that for all* $s \in S$*, the Hodge decomposition holds for* $H^1_{dR}(X_s)$*. Then the Hodge exact sequence* (3.10) *can be identified with the split exact sequence*

$$0 \longrightarrow \mathcal{H}^{1,0} \longrightarrow \mathcal{H}^1_{dR}(X/S) \longrightarrow \mathcal{H}^{0,1} \longrightarrow 0,$$

of Proposition 3.3.1. In particular, we have identifications $f_*\Omega^1_{X/S} = \mathcal{H}^{1,0}$ *and* $R^1 f_*\mathcal{O}_S = \mathcal{H}^{0,1}$.

3.4 Complex tori revisited

In this section, we will give a short proof of the Hodge decomposition for a complex torus, following [2, Proposition 1.3.5]. We then deduce some consequences for $\mathcal{H}^1_{dR}(A_{univ}/\mathfrak{S}_g)$.

We begin with differential forms. Suppose $X = V/\Lambda$ is a complex torus, with projection $q : V \to X$. Since q is a surjective submersion, the pullback map $q^* : A^k(X) \to A^k(V)$ is injective, and we can use it to identify $A^k(X)$ with a subspace of $A^k(V)$. Note that if $\omega \in A^k(X)$, then for all $\lambda \in \Lambda$, ω is invariant under pullback by the translation $t_\lambda : v \mapsto v + \lambda$ since it is well-defined on X. In fact, this property characterises $A^k(X)$ as a subspace of $A^k(V)$ [25, Proposition 21.8]:

$$A^k(X) = \{\, \omega \in A^k(V) \mid t_\lambda^*\omega = \omega \text{ for all } \lambda \in \Lambda \,\}.$$

If $\omega \in A^k(X)$ satisfies $t_a^*\omega = \omega$ for all $a \in V$, then we say that ω is *translation invariant*. The translation invariant k-forms on X (or V) form a complex vector space, which we denote by $IF^k(X)$. Note that if $u_1, \ldots, u_g$ are complex coordinates on V, then the 1-forms du_i and $d\bar{u}_i$ belong to $IF^1(X)$. In fact, they form a basis, so we have a decomposition

$$IF^1(X) = \mathbb{C}\, du_1 \oplus \ldots \oplus \mathbb{C}\, du_g \oplus \mathbb{C}\, d\bar{u}_1 \oplus \ldots \oplus \mathbb{C}\, d\bar{u}_g. \tag{3.11}$$

Now consider the de Rham isomorphism $H^1_{dR}(X) \xrightarrow{\sim} H^1(X;\mathbb{C})$. If we identify the singular cohomology group $H^1(X;\mathbb{C})$ with the dual space $H_1(X;\mathbb{C})^*$, the isomorphism is given explicitly by

$$\omega \mapsto \left([c] \mapsto \int_c \omega\right),$$

where $c : [0,1] \to X$ is a smooth representative of the homology class $[c]$ (see Equation (18.14) of [19]). Using Corollary 2.1.2, we can view this as a map

$$\iota : H^1_{dR}(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{C}), \qquad \iota(\omega) : \lambda \mapsto \int_{\gamma_\lambda} \omega,$$

where $\gamma_\lambda : [0,1] \to X$ is the loop $\gamma(s) = s\lambda$.

Proposition 3.4.1. *The cohomology of a complex torus $X = V/\Lambda$ carries a Hodge decomposition.*

Proof. Choose a basis $\lambda_1, \dots, \lambda_{2g}$ of Λ, and let $x_1, \dots, x_{2g}$ be the corresponding real coordinate functions on V. Then $dx_1, \dots, dx_{2g}$ are translation invariant 1-forms on X, and they form a basis for $\mathrm{IF}^1(X)$ over $\mathbb{C}$. On the other hand, since we have $\iota(dx_i)(\lambda_j) = \int_{\lambda_j} dx_i = \delta_{ij}$ by definition, the cohomology classes $\iota(dx_1), \dots, \iota(dx_{2g})$ form a basis of $H^1(X;\mathbb{C})$, dual to the basis $\lambda_1, \dots, \lambda_{2g}$ of $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. We conclude that the canonical map $\mathrm{IF}^1(X) \to H^1_{dR}(X)$ must be an isomorphism.

If we now choose complex coordinates $u_1, \dots, u_g$ on V, then the decomposition (3.11) induces the Hodge decomposition of $H^1_{dR}(X)$ on passing to cohomology.

The case $k > 1$ follows from the fact that the cup product induces an isomorphism $H^k(X;\mathbb{C}) \cong \bigwedge^k H^1(X;\mathbb{C})$ (see [2, Corollary 1.3.4]). □

Remark. Note that even though we called it the 'Hodge decomposition', what we have proved in Proposition 3.4.1 is something weaker than what is implied by the full strength of Hodge theory, which involves the concepts of harmonic forms and Dolbeault cohomology.

We use this to compute the de Rham cohomology sheaf $\mathcal{H}^1_{dR}(A_{univ}/\mathfrak{S}_g)$.

Proposition 3.4.2. *The relative Hodge decomposition holds for $\mathcal{H}^1_{dR}(A_{univ}/\mathfrak{S}_g)$. In fact, there exists a basis of global sections*

$$du_1, \dots, du_g, d\bar{u}_1, \dots, d\bar{u}_g \tag{3.12}$$

for $\mathcal{H}^1_{dR}(A_{univ}/\mathfrak{S}_g)$, such that $du_1, \dots, du_g$ is a basis for $\mathcal{H}^{1,0}$ and $d\bar{u}_1, \dots, d\bar{u}_g$ is a basis for $\mathcal{H}^{0,1}$. On the fibres, (3.12) *restricts to the canonical basis* (3.1) *of $H^1_{dR}(A_\tau)$.*

Proof. The existence of the decomposition follows from Propositions 3.3.1 and 3.4.1. Note that by Lemma 3.2.5, elements of $\Gamma(\mathfrak{S}_g, \mathcal{H}^1_{dR}(A_{univ}/\mathfrak{S}_g))$ are represented by actual global sections of the relative cotangent bundle $T^\vee_{A_{univ}/\mathfrak{S}_g,\mathbb{C}}$. Hence, it is obvious that the sections du_i and $d\bar{u}_i$ exist, and that they have the stated properties. □

Finally, we compute the image of (3.12) under the relative de Rham isomorphism of Theorem 3.2.9, in terms of the basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ of Proposition 2.3.8. The identities (3.13) are Equations (4.2.2.1) and (4.2.2.2) of [13].

Lemma 3.4.3. *Suppose* $X = \mathbb{C}^g/\Lambda$ *is a complex torus, and we are given a basis* $\lambda_1, \dots, \lambda_{2g}$ *of* Λ*. Then*

$$\iota(du_i) = \sum_{1\le j\le 2g} \lambda_{ji}\lambda_j^*, \qquad \iota(d\bar{u}_i) = \sum_{1\le j\le 2g} \bar{\lambda}_{ji}\lambda_j^*,$$

where λ_{ji} *is the* i*th component of* $\lambda_j = (\lambda_{j1}, \dots, \lambda_{jg})$*.*

Proof. Note that

$$\int_{\lambda_j} du_i = \int_0^1 d(t\lambda_{ji}) = \lambda_{ji}\int_0^1 dt = \lambda_{ji},$$

and similarly that $\int_{\lambda_j} d\bar{u}_i = \bar{\lambda}_{ji}$. □

From now on, we will drop ι from the notation.

Proposition 3.4.4. *The relative de Rham isomorphism makes the identifications*

$$du_i = \alpha_i + \sum_{1\le j\le g} z_{ij}\beta_j, \qquad d\bar{u}_i = \alpha_i + \sum_{1\le j\le g} \bar{z}_{ij}\beta_j, \tag{3.13}$$

where $z_{ij} = z_{ji}$ *are the standard coordinates on* $\mathfrak{S}_g$*.*

Proof. It suffices to check (3.13) on the stalks. Let $\tau \in \mathfrak{S}_g$. Applying Lemma 3.4.3 with respect to the basis $e_1, \dots e_g, \tau_1, \dots \tau_g$ of Λ_τ, we see that

$$\begin{aligned}(du_i)_\tau &= e_i^* + \textstyle\sum_{1\le j\le g} \tau_{ji}\tau_j^* \\ &= e_i^* + \textstyle\sum_{1\le j\le g} \tau_{ij}\tau_j^* \\ &= (\alpha_i)_\tau + \textstyle\sum_{1\le j\le g}(z_{ij})_\tau(\beta_j)_\tau \\ &= (\alpha_i + \textstyle\sum_{1\le j\le g} z_{ij}\beta_j)_\tau.\end{aligned}$$

Hence, $du_i = \alpha_i + \sum_{1\le j\le g} z_{ij}\beta_j$. The second identity is proved similarly. □

Chapter 4

Abelian varieties and modular forms

The previous two chapters have been about the cohomology of families of complex manifolds, especially the family of complex tori $A_{\mathrm{univ}}/\mathfrak{S}_g$. We are now in a position to develop in detail the g-dimensional generalisation of the geometric picture we sketched in Section 1.1. In particular, we can give a geometric interpretation of Siegel modular forms.

Central to the discussion in Section 1.1 was the action of the modular group $SL_2(\mathbb{Z})$ on the upper half plane $\mathfrak{H}$. After defining modular forms as functions that transform in a special way under this action, we mentioned that the quotient set $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ classifies the isomorphism classes of elliptic curves. We then considered the modular curves $Y(N) := \Gamma(N)\backslash\mathfrak{H}$, which are fine moduli spaces for elliptic curves with level N-structure. By examining the action of $SL_2(\mathbb{Z})$ which gives rise to the universal family $E_{\mathrm{univ},N}/Y(N)$ as a quotient of $E_{\mathrm{univ}}/\mathfrak{H}$, we obtained an interpretation of the modularity condition (1.1). As a result, we could identify modular forms with tensor powers of de Rham cohomology classes, and hence differentiate them using the Gauss-Manin connection.

The g-dimensional analogue of an elliptic curve is an abelian variety, and Siegel modular forms are related to abelian varieties in the same way that elliptic modular forms are related to elliptic curves. The symplectic group $Sp_{2g}(\mathbb{Z})$ acts on the Siegel upper half space $\mathfrak{S}_g$ in a way that generalises the action of $SL_2(\mathbb{Z})$ on $\mathfrak{H}$, and the quotient $Sp_{2g}(\mathbb{Z})\backslash\mathfrak{S}_g$ classifies the isomorphism classes of principally-polarized abelian varieties. There are Siegel modular varieties $\mathfrak{S}_{g,N} := \Gamma_g(N)\backslash\mathfrak{S}_g$, which are fine moduli spaces for principally-polarized abelian varieties with level N-structure. Our main concern in this chapter is with the de Rham cohomology of the universal families $A_{\mathrm{univ},N}$ over these moduli spaces. We will show that the universal family $A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}$ arises as a quotient of $A_{\mathrm{univ}}/\mathfrak{S}_g$ by an action of the principal congruence subgroup $\Gamma_g(N)$, and that this action induces an action on the relative cotangent bundle $T^\vee_{A_{\mathrm{univ}}/\mathfrak{S}_g}$ such that $T^\vee_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}} = \Gamma_g(N)\backslash T^\vee_{A_{\mathrm{univ}}/\mathfrak{S}_g}$. It follows that a Siegel modular form of weight k is a global section of the sheaf $\omega_N^{\otimes k}$, where $\omega_N := \bigwedge^g \pi_{N*}\Omega^1_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}}$ is the determinant of the Hodge sheaf $\pi_{N*}\Omega^1_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}}$.

In Section 4.1, we define abelian varieties as complex tori admitting a polarization, and we show that $A_{\mathrm{univ}}/\mathfrak{S}_g$ is a family of abelian varieties. In Section 4.2, we discuss the action of the symplectic group $Sp_{2g}(\mathbb{R})$ on $\mathfrak{S}_g$. We show that two fibres A_τ, $A_{\tau'}$ of the family $A_{\mathrm{univ}}/\mathfrak{S}_g$ are isomorphic as polarized abelian varieties if and only if their base points τ, τ' belong to the same $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit, and we give an explicit formula for these isomorphisms. We also define the principal congruence subgroups $\Gamma_g(N)$, and show that the quotients $\mathfrak{S}_{g,N} := \Gamma_g(N)\backslash\mathfrak{S}_g$ are complex manifolds. In Section 4.3, we construct the family $A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}$, and we show that $A_{\mathrm{univ},N}$ can be interpreted as a quotient $\Gamma_g(N)\backslash A_{\mathrm{univ}}$ for an appropriate action of $\Gamma_g(N)$ on A_{univ}. In Section 4.4, we describe the induced $\Gamma_g(N)$-action on $T^{\vee}_{A_{\mathrm{univ}}/\mathfrak{S}_g}$, prove that $T^{\vee}_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}} = \Gamma_g(N)\backslash T^{\vee}_{A_{\mathrm{univ}}/\mathfrak{S}_g}$, and finish by discussing Siegel modular forms.

4.1 Complex abelian varieties and polarizations

We review some very classical material about complex abelian varieties, following the exposition in Birkenhake and Lange's textbook [2].

Let $X = V/\Lambda$ be a complex torus. Recall that the group $\mathrm{Pic}(X)$ of invertible line bundles on X can be identified with the sheaf cohomology group $H^1(X, \mathcal{O}_X^*)$. Now the sheaf of holomorphic functions $\mathcal{O}_X$ sits in a short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}_X^* \longrightarrow 1,$$

called the *exponential exact sequence*. Part of the corresponding long exact sequence on cohomology is

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z});$$

the map we have labelled c_1 is the connecting homomorphism. By the Künneth formula and Corollary 2.1.2, we have

$$H^2(X, \mathbb{Z}) \cong \bigwedge^2 H^1(X; \mathbb{Z}) \cong \bigwedge^2 \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}),$$

so the map c_1 associates to a holomorphic line bundle $L \in H^1(X, \mathcal{O}_X^*)$ an integer-valued alternating form $E = c_1(L)$ on Λ, called its *first Chern class*. If E is extended to a real alternating form on V, then with some work (see [2, Proposition 2.1.6]), one can show that it has the property $E(iu, iv) = E(u, v)$ for all $u, v \in V$. It follows that E is the imaginary part of a form $H : V \times V \to \mathbb{C}$, defined by

$$H(u, v) := E(iu, v) + iE(u, v)$$

for all $u, v \in V$. In fact, it can be easily checked that H is a *Hermitian form*; recall that this means that H is $\mathbb{C}$-linear in its first slot and satisfies $H(u, v) = \overline{H(v, u)}$ for all $u, v \in V$. Conversely, if we start with a Hermitian form H on V, then its imaginary part $E = \mathrm{Im}\, H$ is a real alternating form on V satisfying $E(iu, iv) = E(u, v)$ for all $u, v \in V$. If the restriction of E to the lattice Λ is integer valued, then (with some more work; see [2, Proposition 2.1.6] again) one can show that there is a line bundle on X for which $E_{\Lambda \times \Lambda}$ is the first Chern class. With this correspondence in mind, we make the following definition:

Definition 4.1.1. Suppose $X = V/\Lambda$ is a complex torus, and H is a Hermitian form on V. We say that H is a *Riemann form* on X if its imaginary part $E = \operatorname{Im} H$ takes integral values on $\Lambda \times \Lambda$.

The set of Riemann forms on X form a group under addition, known as the *Néron-Severi group* $NS(X)$. By the theory we have sketched above, we may identify $NS(X)$ with the image of the map c_1.

Definition 4.1.2. If a Riemann form $H \in NS(X)$ is positive definitive, then it is called a *polarization* on X.

If H is a polarization on X, then its imaginary part $E = E|_{\Lambda\times\Lambda}$ can be put into a certain canonical form. Namely, there exists a basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ of Λ with respect to which E is given by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

with $D = \operatorname{diag}(d_1, \ldots d_g)$ a diagonal matrix whose entries are strictly positive and satisfy $d_i \mid d_{i+1}$ for all $1 \le i \le g-1$ [15, Chapter II, Section 4, Lemma 5]. We say that the basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ is *symplectic*, and we refer to the list $(d_1, \ldots d_g)$ as the *type* of the polarization H. If H has type $(1, \ldots, 1)$, then we say that H is a *principal polarization*, and that X is *principally-polarized*.

If $L \in \operatorname{Pic}(X)$ is a line bundle whose first Chern class is a polarization, then we say that L is *positive definite*. One might ask what this entails for L. In fact it is a theorem of Lefschetz that L is positive definite if and only if for all integers $n \ge 3$, the tensor power $L^{\otimes n}$ is *very ample*; roughly speaking, this means that L has enough global sections to set up an embedding of X into projective space. [1] In this case, it follows from Chow's theorem that X has the structure of a projective algebraic variety.

Theorem 4.1.3. *The following conditions on a complex torus X are equivalent:*

1. *X is the complex manifold associated to an algebraic variety;*

2. *X admits a polarization.*

Proof. See [23, Chapter I, Section 3] or [2, Theorem 4.5.4]. □

Definition 4.1.4. A complex torus that admits a polarization is called a *complex abelian variety*.

If X is an abelian variety and H is a polarization on X, then we refer to the pair (X, H) as a *polarized abelian variety*. A *homomorphism of polarized abelian varieties* $f : (Y, H') \to (X, H)$ is a homomorphism of complex tori $f : Y \to X$ such that $H' = f^*H$.

Proposition 4.1.5. *The family $A_{\text{univ}}/\mathfrak{S}_g$ is a family of principally-polarized abelian varieties.*

[1] A line bundle L is called *ample* if some tensor power of it is very ample. For line bundles on complex tori, ample = positive definite.

Proof. Let $\tau \in \mathfrak{S}_g$, and consider the lattice Λ_τ in $\mathbb{C}^g$. We can define a Hermitian form H_τ on $\mathbb{C}^g$ by setting

$$H_\tau(z, w) := {}^t z y^{-1} \overline{w},$$

where y is the imaginary part of τ. Since $y > 0$, H_τ is positive definite. To compute $E = \operatorname{Im} H_\tau$ with respect to the basis $\tau_1, \dots, \tau_g, e_1, \dots, e_g$ of Λ, note that $H_\tau(e_i, e_j)$ is the i, j-entry of the matrix

$${}^t 1_g y^{-1} \overline{1}_g = y^{-1},$$

while $H_\tau(\tau_i, e_j)$ is the i, j-entry of

$${}^t \tau y^{-1} \overline{1}_g = \tau y^{-1} = x y^{-1} + i 1_g,$$

and $H_\tau(\tau_i, \tau_j)$ is the i, j-entry of

$$\begin{aligned} {}^t \tau y^{-1} \overline{\tau} &= (x + iy) y^{-1} (x - iy) \\ &= x y^{-1} x + x y^{-1}(-iy) + iy y^{-1} x + iy y^{-1}(-iy) \\ &= x y^{-1} x + y. \end{aligned}$$

Taking imaginary parts, we find $E(\tau_i, e_j) = \delta_{ij}$ and $E(e_i, e_j) = E(\tau_i, \tau_j) = 0$, so that the matrix of E is

$$\begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}.$$

Therefore, H_τ is a principal polarization, and the complex torus $A_\tau = \mathbb{C}^g / \Lambda_\tau$ is a principally-polarized abelian variety. Note that the basis $\tau_1, \dots, \tau_g, e_1, \dots, e_g$ is symplectic with respect to H_τ. □

4.2 The action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathfrak{S}_g$

Recall that in Section 2.2, we showed $A_{\text{univ}} / \mathfrak{S}_g$ was smoothly trivial: we had a diffeomorphism $A_{\tau'} \cong A_\tau$ for all $\tau, \tau' \in \mathfrak{S}_g$. Of course, since diffeomorphisms need not preserve complex structures, the polarized abelian varieties $(A_{\tau'}, H_{\tau'})$ and (A_τ, H_τ) will not be isomorphic in general. However, suppose we do have an isomorphism

$$f : (A_{\tau'}, H_{\tau'}) \xrightarrow{\sim} (A_\tau, H_\tau).$$

By the discussion preceding Proposition 2.1.3 and Liouville's theorem from complex analysis, f lifts to a vector space automorphism F of $\mathbb{C}^g$ that restricts to an isomorphism of lattices $\Lambda_{\tau'} \cong \Lambda_\tau$. Let

$${}^t \gamma = \begin{pmatrix} {}^t a & {}^t c \\ {}^t b & {}^t d \end{pmatrix} \in \mathrm{GL}_{2g}(\mathbb{Z})$$

be the matrix representation of this isomorphism with respect to the canonical symplectic bases of $\Lambda_{\tau'}$ and Λ_τ (the reason for taking the transpose will become clear shortly). If we identify F with its standard matrix representation, we have

$$F \begin{pmatrix} \tau' & 1_g \end{pmatrix} = \begin{pmatrix} \tau & 1_g \end{pmatrix} {}^t \gamma,$$

or equivalently,

$$F\tau' = \tau\,{}^t a + {}^t b = {}^t(a\tau + b) \quad \text{and} \quad F = \tau\,{}^t c + {}^t d = {}^t(c\tau + d).$$

Since F is invertible and τ' is symmetric, we get

$$\begin{aligned}\tau' &= {}^t(c\tau + d)^{-1}\,{}^t(a\tau + b)\\ &= (a\tau + b)(c\tau + d)^{-1}.\end{aligned}$$

Taking imaginary parts of $f^*H_\tau = H_{\tau'}$, we find that γ belongs to the *real symplectic group*

$$\mathrm{Sp}_{2g}(\mathbb{R}) := \left\{ \gamma \in \mathrm{GL}_{2g}(\mathbb{R}) \;\middle|\; {}^t\gamma\begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}\gamma = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \right\}.$$

Letting $\mathrm{Sp}_{2g}(\mathbb{R})$ act on $\mathfrak{S}_g$ via

$$\gamma\tau := (a\tau + b)(c\tau + d)^{-1},$$

we have shown one direction of the following lemma (the other direction is proved by reversing the above reasoning).

Lemma 4.2.1. *Every isomorphism between fibres of the family of principally-polarized abelian varieties $A_{\mathrm{univ}}/\mathfrak{S}_g$ is given by a diagram of the form*

$$\begin{array}{ccccc} \Lambda_\tau & \hookrightarrow & \mathbb{R}^{2g} & \xrightarrow{j_\tau} & \mathbb{C}^g \\ \downarrow {\scriptstyle {}^t\gamma^{-1}} & & \downarrow {\scriptstyle {}^t\gamma^{-1}} & & \downarrow {\scriptstyle {}^t(c\tau+d)^{-1}} \\ \Lambda_{\tau'} & \hookrightarrow & \mathbb{R}^{2g} & \xrightarrow{j_{\tau'}} & \mathbb{C}^g \end{array}, \tag{4.1}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$, τ and τ' are related by $\tau' = \gamma\tau$, and j_τ is the change of basis

$$x \mapsto \begin{pmatrix} \tau & 1_g \end{pmatrix} x.$$

Conversely, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ is given, then ${}^t(c\tau + d)^{-1}$ determines an isomorphism of polarized abelian varieties $A_\tau \cong A_{\gamma\tau}$.

We will now work out the basic properties of the action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathfrak{S}_g$. It is not immediately obvious that the definition even makes sense; for example, one has to check that $c\tau + d$ is invertible and that $\gamma\tau \in \mathfrak{S}_g$. These facts follow from a couple of short calculations (see for instance [11, Section 2]).

Lemma 4.2.2. *The action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathfrak{S}_g$ is transitive and proper.*

Proof. This proof follows the outline in [8, Exercise 2.1.3]. Given a point $\tau = x + iy \in \mathfrak{S}_g$, we define a map $s : \mathfrak{S}_g \to \mathrm{Sp}_{2g}(\mathbb{R})$ by setting

$$s(\tau) := \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R}),$$

where $y^{\frac{1}{2}}$ is the matrix obtained from y by taking square roots of its eigenvalues. Since $s(\tau) \cdot i1_g = \tau$, the action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathfrak{S}_g$ is transitive. From the orbit-stabilizer theorem, we get a $\mathrm{Sp}_{2g}(\mathbb{R})$-equivariant bijection $\mathrm{Sp}_{2g}(\mathbb{R})/K \cong \mathfrak{S}_g$, where K is the stabilizer of $i1_g$. One computes that

$$K = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R}) \right\}.$$

Since K is canonically isomorphic to the unitary group $U(g)$ via $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mapsto a + ib$, it is, in particular, a compact subgroup of $\mathrm{Sp}_{2g}(\mathbb{R})$.

To prove that the action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathfrak{S}_g$ is proper, it suffices to show that if $L \subseteq \mathfrak{S}_g$ is a compact subset, then the set of $\gamma \in \mathrm{Sp}_{2g}(\mathbb{R})$ such that $\gamma L \cap L \neq \varnothing$ is compact. But for any two points $e_1, e_2 \in \mathfrak{S}_g$, we have $\gamma e_1 = e_2$ if and only if $\gamma \in s(e_2)Ks(e_1)^{-1}$. It follows that

$$\left\{ \gamma \in \mathrm{Sp}_{2g}(\mathbb{R}) \,\middle|\, \gamma L \cap L \neq \varnothing \right\} = s(L)Ks(L)^{-1},$$

and since s is continuous, this is a compact set. □

Remark. The existence of the map s in this proof implies $\mathrm{Sp}_{2g}(\mathbb{R}) \to \mathfrak{S}_g$ is trivial as a principal K-bundle.

Definition 4.2.3. If N is a positive integer, the group

$$\Gamma_g(N) := \left\{ \gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}) \,\middle|\, \gamma \equiv 1_{2g} \pmod{N} \right\}$$

is called a *principal congruence subgroup* of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Proposition 4.2.4. *If $N \geq 3$, the quotient set $\mathfrak{S}_{g,N} := \Gamma_g(N)\backslash\mathfrak{S}_g$ a complex manifold in a canonical way.*

Proof. Since $\Gamma_g(N)$ is a discrete group, Lemma 4.2.2 implies that its action on $\mathfrak{S}_g$ is properly discontinuous. Moreover, by a lemma of Serre [24], $\Gamma_g(N)$ acts freely on $\mathfrak{S}_g$ for all $N \geq 3$. Now apply [18, Theorem 2.2]. □

4.3 Universal families of abelian varieties

In this section, we construct a universal family over the Siegel modular variety $\mathfrak{S}_{g,N}$, following the outline in [5]. As with our family of complex tori over $\mathfrak{S}_g$, this family is defined as a quotient of $\mathbb{C}^g \times \mathfrak{S}_g$ by the action of an appropriate group.

Lemma 4.3.1. *The symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathbb{C}^g \times \mathfrak{S}_g$ via*

$$\gamma(z, \tau) := \left({}^t(c\tau + d)^{-1}z, \gamma\tau\right)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $(z, \tau) \in \mathbb{C}^g \times \mathfrak{S}_g$.

Proof. This follows from the commutativity of Diagram (4.1), and the fact that ${}^t(\gamma\delta)^{-1} = {}^t\gamma^{-1}\,{}^t\delta^{-1}$ for all $\gamma, \delta \in \mathrm{Sp}_{2g}(\mathbb{Z})$. □

We now have two group actions on $\mathbb{C}^g \times \mathfrak{S}_g$: the one of $\mathrm{Sp}_{2g}(\mathbb{Z})$ which we have just defined, and the one of $\mathbb{Z}^{2g}$ which we used to define the family $A_{\text{univ}}/\mathfrak{S}_g$. Although these actions don't commute, we can put them together to get an action of a semidirect product on $\mathbb{C}^g \times \mathfrak{S}_g$:

Lemma 4.3.2. *Let $\mathrm{Sp}_{2g}(\mathbb{Z}) \ltimes \mathbb{Z}^{2g}$ be the semidirect product of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathbb{Z}^{2g}$ with multiplication*

$$(\gamma, m) * (\delta, n) := (\gamma\delta, {}^t\delta m + n).$$

Then $\mathrm{Sp}_{2g}(\mathbb{Z}) \ltimes \mathbb{Z}^{2g}$ acts on $\mathbb{C}^g \times \mathfrak{S}_g$ by

$$(\gamma, m) \cdot (z, \tau) := \left({}^t(c\tau + d)^{-1}(z + j_\tau m), \gamma\tau\right).$$

Proof. The identity element of $\mathrm{Sp}_{2g}(\mathbb{Z}) \ltimes \mathbb{Z}^{2g}$ is $(1_{2g}, 0)$, and it is clear that it acts trivially. It remains to show that

$$((\gamma, m) * (\delta, n)) \cdot (z, \tau) = (\gamma, m) \cdot ((\delta, n) \cdot (z, \tau))$$

for all $(\gamma, m), (\delta, n) \in \mathrm{Sp}_{2g}(\mathbb{Z}) \ltimes \mathbb{Z}^{2g}$ and $(z, \tau) \in \mathbb{C}^g \times \mathfrak{S}_g$. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\delta = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$, then we find that

$$\begin{aligned}((\gamma, m) * (\delta, n)) \cdot (z, \tau) &= (\gamma\delta, {}^t\delta m + n) \cdot (z, \tau) \\ &= \left({}^t\left[(c\tau + d)(c'\tau + d')\right]^{-1}\left(z + j_\tau\left({}^t\delta m + n\right)\right), \gamma\delta\tau\right)\end{aligned}$$

and

$$\begin{aligned}(\gamma, m) \cdot ((\delta, n) \cdot (z, \tau)) &= (\gamma, m) \cdot \left({}^t(c'\tau + d')^{-1}(z + j_\tau n), \delta\tau\right) \\ &= \left({}^t(c\tau + d)^{-1}\left({}^t(c'\tau + d')^{-1}(z + j_\tau n) + j_{\delta\tau} m\right), \gamma\delta\tau\right).\end{aligned}$$

Examining these two equations carefully, we see that it is enough to show that

$$j_{\delta\tau} m = {}^t(c'\tau + d')^{-1} j_\tau\left({}^t\delta m\right).$$

But this follows immediately from the commutativity of (4.1). □

Proposition 4.3.3. *If $N \geq 3$, there exists a universal family $\pi_N : A_{\text{univ},N} \to \mathfrak{S}_{g,N}$ over the Siegel modular variety $\mathfrak{S}_{g,N}$.*

Proof. The action of the subgroup $\Gamma_g(N) \ltimes \mathbb{Z}^{2g}$ on $\mathbb{C}^g \times \mathfrak{S}_g$ is properly discontinuous and free if $N \geq 3$, so the quotient

$$A_{\text{univ},N} := \Gamma_g(N) \ltimes \mathbb{Z}^{2g} \backslash \left(\mathbb{C}^g \times \mathfrak{S}_g\right)$$

is a complex manifold. Moreover, if we let $\Gamma_g(N) \ltimes \mathbb{Z}^{2g}$ act on $\mathfrak{S}_g$ by $(\gamma, m) \cdot \tau = \gamma\tau$, the projection $\mathbb{C}^g \times \mathfrak{S}_g \to \mathfrak{S}_g$ becomes a $\Gamma_g(N) \ltimes \mathbb{Z}^{2g}$-equivariant map. Passing to the quotient defines the family $\pi_N : A_{\text{univ},N} \to \mathfrak{S}_{g,N}$. □

In fact, the total space $A_{\text{univ},N}$ can be obtained directly as a quotient of A_{univ}:

Lemma 4.3.4. *The group* $\Gamma_g(N)$ *acts on* A_{univ} *in such a way that* $\Gamma_g(N)\backslash A_{\mathrm{univ}} = A_{\mathrm{univ},N}$.

Proof. Note that $\mathbb{Z}^{2g}$ is a normal subgroup of $\Gamma_g(N) \ltimes \mathbb{Z}^{2g}$, and that its action on $\mathbb{C}^g \times \mathfrak{S}_g$ coincides with the action with which we defined A_{univ} in Section 2.2. Hence, we have an induced action of $\Gamma_g(N) \cong \Gamma_g(N) \ltimes \mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ on A_{univ} such that $A_{\mathrm{univ},N} = \Gamma_g(N)\backslash A_{\mathrm{univ}}$. $\square$

Explicitly, this action is given as follows: if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_g(N)$, and φ_γ denotes the corresponding automorphism of A_{univ}, then $\varphi_\gamma|_{A_\tau} = {}^t(c\tau + d)^{-1}$ is the isomorphism $A_\tau \cong A_{\gamma\tau}$ of Diagram (4.1). Note that we have a commuting diagram

$$\begin{array}{ccc} A_{\mathrm{univ}} & \xrightarrow{q} & A_{\mathrm{univ},N} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_N} \\ \mathfrak{S}_g & \longrightarrow & \mathfrak{S}_{g,N}, \end{array}$$

where q denotes the quotient map.

Proposition 4.3.5. *The map* $q : A_{\mathrm{univ}} \to A_{\mathrm{univ},N}$ *is a holomorphic covering map, which is normal, in the sense that the action of* $\Gamma_g(N)$ *permutes the fibres of* q *transitively.*

Proof. This follows from the fact that $\Gamma_g(N)$ acts freely and properly discontinuously on A_{univ}; see the discussion after Theorem 2.2 in [18]. $\square$

4.4 The cotangent bundle of $A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}$

In this section, we show that the action of $\Gamma_g(N)$ on A_{univ} induces an action on the relative holomorphic cotangent bundle $T^\vee_{A_{\mathrm{univ}}/\mathfrak{S}_g}$, and that quotienting out by this action gives rise to $T^\vee_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}}$. We then discuss Siegel modular forms. To lighten the notation, we will write $T^\vee := T^\vee_{A_{\mathrm{univ}}/\mathfrak{S}_g}$ and $T^\vee_N := T^\vee_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}}$ for the cotangent bundles, and $\Omega^1 := \Omega^1_{A_{\mathrm{univ}}/\mathfrak{S}_g}$ and $\Omega^1_N := \Omega^1_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}}$ for their sheaves of holomorphic sections.

We begin by quoting a general result, which will give $\Gamma_g(N)\backslash T^\vee$ the structure of a vector bundle over $A_{\mathrm{univ},N}$.

Lemma 4.4.1. *Suppose* Γ *is a discrete group acting by automorphisms on a complex manifold* X. *We assume the action of* Γ *is free and properly discontinuous, so that* $\Gamma\backslash X$ *is a complex manifold and the quotient map* $q : X \to \Gamma\backslash X$ *is a normal holomorphic covering map. Suppose* E *is a complex vector bundle on* X *with the following properties:*

(i) Γ *acts on* E *on the left by vector bundle isomorphisms: each map* $\xi \mapsto \gamma\xi$ *is a vector bundle isomorphism over the left translation* L_γ.

(ii) E *is covered by* Γ*-equivariant charts. This means that around each point in* X*, there is a* q*-saturated open set* $\mathfrak{U}$ *(i.e.* $\mathfrak{U} = q^{-1}(U)$ *for some open* $U \subseteq \Gamma\backslash X$*), and a vector bundle chart*

$$\psi : \mathfrak{U} \times \mathbb{C}^n \xrightarrow{\sim} E_{\mathfrak{U}}$$

such that $\psi(\gamma u, v) = \gamma\psi(u, v)$ *for all* $\gamma \in \Gamma$, $u \in \mathfrak{U}$, *and* $v \in \mathbb{C}^n$.

Then the set of orbits $\Gamma\backslash E$ *has a unique vector bundle structure over* $\Gamma\backslash X$ *such that the natural projection* $\natural : E \to \Gamma\backslash E$ *is a surjective submersion, and a vector bundle morphism over the projection* $q : X \to \Gamma\backslash X$*. Moreover, the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\natural} & \Gamma\backslash E \\ \downarrow & & \downarrow \\ X & \xrightarrow{q} & \Gamma\backslash X \end{array}$$

is a pullback.

Proof. See [22, Proposition 3.1.1] for a more general statement involving the action of a Lie group G on a principal G-bundle. We have adapted this statement to describe our situation – the special case where G is discrete – more explicitly. □

Let $\Gamma_g(N)$ act on $T^\vee = A_{\mathrm{univ}} \times \mathbb{C}\, du_1 \oplus \cdots \oplus \mathbb{C}\, du_g$ by $\gamma \mapsto (\varphi_\gamma^*)^{-1}$. To see that this makes sense, note that if we label the entries of ${}^t(c\tau + d)^{-1}$ as a_{ij}, we find that

$$\varphi_\gamma^*(du_i) = d(u_i \circ \varphi_\gamma) = d(\textstyle\sum_j a_{ij} u_j) = \sum_j a_{ij} du_j.$$

So $(\varphi_\gamma^*)^{-1}$ is given on $T^\vee_{A_\tau}$ by ${}^t(c\tau + d)$; in particular, it preserves the du_i's.

Lemma 4.4.2. *The action* $\gamma \mapsto (\varphi_\gamma^*)^{-1}$ *of* $\Gamma_g(N)$ *on* $T^\vee_{A_{\mathrm{univ}}/\mathfrak{S}_g}$ *satisfies the two conditions of Lemma 4.4.1.*

Proof. First, note that the diagram

$$\begin{array}{ccc} T^\vee & \xrightarrow{(\varphi_\gamma^*)^{-1}} & T^\vee \\ \downarrow & & \downarrow \\ A_{\mathrm{univ}} & \xrightarrow{\varphi_\gamma} & A_{\mathrm{univ}} \end{array}$$

commutes, so condition (i) is satisfied.

Now let U be an evenly covered open subset of $A_{\mathrm{univ},N}$. Let $\mathfrak{U} := q^{-1}(U)$, and choose a section $s : U \to \mathfrak{U}$. This determines a $\Gamma_g(N)$-equivariant isomorphism

$$\Gamma_g(N) \times U \cong \mathfrak{U}, \qquad (\gamma, x) \mapsto \varphi_\gamma(s(x)). \tag{4.2}$$

Define a map

$$\psi : \Gamma_g(N) \times U \times \mathbb{C}^g \to \mathfrak{U} \times \mathbb{C}\, du_1 \oplus \cdots \oplus \mathbb{C}\, du_g = T^\vee_{\mathfrak{U}}$$

by

$$\psi(\gamma, x, \nu) := \big(\varphi_\gamma(s(x)), (\varphi_\gamma^*)^{-1}\big(\textstyle\sum_i \nu_i du_i\big)\big),$$

where $\sum_i \nu_i du_i \in T^\vee_{s(x)}$. Note that ψ is an isomorphism and is linear on the fibres, so it is a local trivialisation of $T^\vee_{\mathfrak{U}}$. Moreover, the diagram

$$\begin{array}{ccc} \Gamma_g(N) \times U \times \mathbb{C}^g & \xrightarrow{\psi} & \mathfrak{U} \times \mathbb{C}\, du_1 \oplus \cdots \oplus \mathbb{C}\, du_g \\ \downarrow{\scriptstyle L_\gamma \times \mathrm{id} \times \mathrm{id}} & & \downarrow{\scriptstyle \varphi_\gamma \times (\varphi_\gamma^*)^{-1}} \\ \Gamma_g(N) \times U \times \mathbb{C}^g & \xrightarrow{\psi} & \mathfrak{U} \times \mathbb{C}\, du_1 \oplus \cdots \oplus \mathbb{C}\, du_g \end{array}$$

is commutative, so ψ is $\Gamma_g(N)$-equivariant. Hence, if we use (4.2) to identify $\Gamma_g(N)\times U$ in the domain of ψ with $\mathfrak{U}$, we have the required equivariant chart. $\square$

It follows that $\Gamma_g(N)\backslash T^\vee$ is a vector bundle over $A_{\mathrm{univ},N}$ in a canonical way.

Proposition 4.4.3. *There is an isomorphism*

$$\Gamma_g(N)\backslash T^\vee_{A_{\mathrm{univ}}/\mathfrak{S}_g} \cong T^\vee_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}} \tag{4.3}$$

of vector bundles over $A_{\mathrm{univ},N}$.

Proof. This is clear from inspection of the equivariant charts ψ we constructed in the proof of the previous lemma. $\square$

We now consider the pullback bundles $\varepsilon_N^* T_N^\vee$ and $\varepsilon^* T^\vee$ along the zero sections $\varepsilon_N : \mathfrak{S}_{g,N} \to A_{\mathrm{univ},N}$ and $\varepsilon : \mathfrak{S}_g \to A_{\mathrm{univ}}$.

Lemma 4.4.4. *The isomorphism* (4.3) *pulls back along the zero sections to an isomorphism of vector bundles on* $\mathfrak{S}_{g,N}$*:*

$$\Gamma_g(N)\backslash \varepsilon^* T^\vee_{A_{\mathrm{univ}}/\mathfrak{S}_g} \cong \varepsilon_N^* T^\vee_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}}.$$

Proof. Note that ε and ε_N are holomorphic embeddings, so we can identify $\varepsilon^* T^\vee$ with the restricted bundle $T^\vee_{\varepsilon(\mathfrak{S}_g)}$, and $\varepsilon_N^* T_N^\vee$ with $T_N^\vee|_{\varepsilon_N(\mathfrak{S}_{g,N})}$. Since the subbundle $T^\vee_{\varepsilon(\mathfrak{S}_g)}$ is preserved by the action of $\Gamma_g(N)$, the quotient bundle $\Gamma_g(N)\backslash T^\vee_{\varepsilon(\mathfrak{S}_g)}$ is a subbundle of $\Gamma_g(N)\backslash T^\vee$. It is clear that (4.3) identifies this quotient with $T_N^\vee|_{\varepsilon_N(\mathfrak{S}_{g,N})}$. $\square$

Definition 4.4.5. Suppose we are in the situation of Lemma 4.4.1. We say that a section $\sigma : X \to E$ of E is Γ-*invariant* if $\sigma(\gamma x) = \gamma\sigma(x)$ for all $\gamma \in \Gamma$ and $x \in X$.

The point of this notion is that Γ-invariant sections of E descend to sections of $\Gamma\backslash E$, and every section of $\Gamma\backslash E$ arises from a Γ-invariant section of E in this way.

Proposition 4.4.6. *The sheaves of holomorphic and smooth sections of the vector bundle* $\varepsilon_N^* T^\vee_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}}$ *are characterised in terms of the action of* $\Gamma_g(N)$ *as follows:*

(i) The sheaf of holomorphic sections $\varepsilon_N^*\Omega^1_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}}$ *is isomorphic to the* $\mathcal{O}_{\mathfrak{S}_{g,N}}$*-submodule of* $\varepsilon^*\Omega^1_{A_{\mathrm{univ}}/\mathfrak{S}_g}$ *consisting of the* $\Gamma_g(N)$*-invariant holomorphic sections.*

(ii) The sheaf of smooth sections $\varepsilon_N^*\mathcal{A}^{1,0}_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}}$ *is isomorphic to the* $\mathcal{C}^\infty_{\mathfrak{S}_{g,N}}$*-submodule of* $\varepsilon^*\mathcal{A}^{1,0}_{A_{\mathrm{univ}}/\mathfrak{S}_g}$ *consisting of the* $\Gamma_g(N)$*-invariant smooth sections.*

Proof. This follows from Lemma 4.4.4 and [22, Proposition 3.1.4]. $\square$

Corollary 4.4.7. *We have an isomorphism of* $\mathcal{C}^\infty_{\mathfrak{S}_{g,N}}$*-modules*

$$\varepsilon_N^*\mathcal{A}^{1,0}_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}} \cong \mathcal{C}^\infty_{\mathfrak{S}_{g,N}} \otimes_{\mathcal{O}_{\mathfrak{S}_{g,N}}} \varepsilon_N^*\Omega^1_{A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}}.$$

Proof. Note that $\varepsilon^*\mathcal{A}^{1,0} := \varepsilon^*\mathcal{A}^{1,0}_{A_{\mathrm{univ}}/\mathfrak{S}_g}$ is the free $\mathcal{C}^\infty_{\mathfrak{S}_g}$-module on the basis of global sections $du_1 \circ \varepsilon, \dots, du_g \circ \varepsilon$, while $\varepsilon^*\Omega^1$ is the free $\mathcal{O}_{\mathfrak{S}_g}$-module on the same basis. Hence, $\varepsilon^*\mathcal{A}^{1,0} = \mathcal{C}^\infty_{\mathfrak{S}_g} \otimes_{\mathcal{O}_{\mathfrak{S}_g}} \varepsilon^*\Omega^1$. Now use Proposition 4.4.6. $\square$

Since $\varepsilon^*T^\vee$ is trivial, its $\Gamma_g(N)$-invariant sections are naturally identified with functions $f : \mathfrak{S}_g \to \mathbb{C}^g$ satisfying

$$f(\gamma\tau) = {}^t(c\tau + d)f(\tau) \tag{4.4}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_g(N)$ and all $\tau \in \mathfrak{S}_g$. This functional equation is related to the functional equation for Siegel modular forms.

Definition 4.4.8. Let $k \in \mathbb{Z}$. A *Siegel modular form* of *weight* k is a holomorphic function $f : \mathfrak{S}_g \to \mathbb{C}$ such that

$$f(\gamma\tau) = \det(c\tau + d)^k f(\tau) \tag{4.5}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_g(N)$ and all $\tau \in \mathfrak{S}_g$. We say that f has *degree* g and *level* N. [2]

Definition 4.4.9. Let $k \in \mathbb{Z}$. A C^∞ *Siegel modular form* of *degree* g, *weight* k and *level* N is a smooth function $f : \mathfrak{S}_g \to \mathbb{C}$ satisfying (4.5). [3]

Remark. From Proposition 4.4.6, we see that Siegel modular forms are global sections of the line bundle $(\bigwedge^g \varepsilon_N^* T_N^\vee)^{\otimes k}$.

The next lemma connects holomorphic functions $f : \mathfrak{S}_g \to \mathbb{C}^g$ satisfying (4.4) and sections of the de Rham cohomology sheaf $\mathcal{H}^1_{dR}(A_{univ,N}/\mathfrak{S}_{g,N})$.

Lemma 4.4.10. *There is an isomorphism of $\mathcal{O}_{\mathfrak{S}_{g,N}}$-modules*

$$\pi_{N*}\Omega^1_{A_{univ,N}/\mathfrak{S}_{g,N}} \cong \varepsilon_N^*\Omega^1_{A_{univ,N}/\mathfrak{S}_{g,N}}.$$

Proof. We first give an isomorphism $\pi_*\Omega^1 \cong \varepsilon^*\Omega^1$. Note that $\pi_*\Omega^1$ is free on the sections $du_1, \dots, du_g$ of the bundle $T^\vee$, and that the action of $\mathcal{O}_{\mathfrak{S}_g}$ is given by $f\,du_i := f \circ \pi\, du_i$. On the other hand $\varepsilon^*\Omega^1$ is the sheaf of sections of the pullback bundle $\varepsilon^*T^\vee$. Since the fibre of this bundle above a point $\tau \in \mathfrak{S}_g$ is the cotangent space of A_τ at 0, a section $\sigma : \mathfrak{S}_g \to \varepsilon^*T^\vee$ can be written as an $\mathcal{O}_{\mathfrak{S}_g}$-linear combination

$$\sigma = f_1\, du_1 \circ \varepsilon + \cdots + f_g\, du_g \circ \varepsilon.$$

Hence, the assignment $du_i \mapsto du_i \circ \varepsilon$ gives an $\mathcal{O}_{\mathfrak{S}_g}$-linear isomorphism $\pi_*\Omega^1 \cong \varepsilon^*\Omega^1$. Since this isomorphism preserves $\Gamma_g(N)$-invariant sections, it lifts to the desired isomorphism $\pi_{N*}\Omega^1_N \cong \varepsilon_N^*\Omega^1_N$. □

We will write

$$\omega := \bigwedge^g \pi_*\Omega^1_{A_{univ}/\mathfrak{S}_g}, \qquad \omega_N := \bigwedge^g \pi_{N*}\Omega^1_{A_{univ,N}/\mathfrak{S}_{g,N}}$$

for the determinants of the Hodge sheaves of the families $A_{univ}/\mathfrak{S}_g$ and $A_{univ,N}/\mathfrak{S}_{g,N}$, and

$$\omega_\infty := \mathcal{C}^\infty_{\mathfrak{S}_g} \otimes_{\mathcal{O}_{\mathfrak{S}_g}} \omega, \qquad \omega_{N,\infty} := \mathcal{C}^\infty_{\mathfrak{S}_{g,N}} \otimes_{\mathcal{O}_{\mathfrak{S}_{g,N}}} \omega_N$$

for the sheaves obtained from ω and ω_N by tensoring with the smooth structure sheaves.

[2] If $g = 1$, we should also require that f is holomorphic at ∞. (Holomorphicity at ∞ is automatically satisfied if $g > 1$, by the Koecher principle.)

[3] One normally imposes a growth condition at ∞. Following Harris [13, Section 1.1], we will ignore this condition, since it won't play a role in our construction.

Theorem 4.4.11. *We have bijective correspondences between the following pairs of sets:*

(i) global sections of $\omega_N^{\otimes k}$ *and Siegel modular forms of weight* k *and level* N*;*

(ii) global sections of $\omega_{N,\infty}^{\otimes k}$ *and* C^∞ *Siegel modular forms of weight* k *and level* N*.*

Proof. For (i), note that $\det {}^t(c\tau + d) = \det(c\tau + d)$, and combine Proposition 4.4.6 with Lemma 4.4.10. Part (ii) follows from the same considerations and Corollary 4.4.7. □

Remark. An interesting consequence of this result is that every C^∞ Siegel modular form can be written as a $\mathcal{C}^\infty_{\mathfrak{S}_{g,N}}(\mathfrak{S}_{g,N})$-linear combination of holomorphic forms.

Chapter 5

The C^∞ theta operator

We assemble everything we have developed in the previous chapters to define an operator

$$\Theta_N : \omega_{N,\infty}^{\otimes k} \to \omega_{N,\infty}^{\otimes(k+2)}.$$

The key to the construction of Θ_N is the Gauss-Manin connection ∇_∞, which provides a means of differentiating relative de Rham cohomology classes: since the sheaf of Siegel modular forms $\omega_{N,\infty}^{\otimes k}$ is a subsheaf of the twisted cohomology sheaf $\mathcal{H}^1_\infty(A_{\mathrm{univ},N}/\mathfrak{S}_{g,N})^{\det^{\otimes k}}$, we can apply a twist of ∇_∞ to differentiate its sections. Postcomposing this by a projection onto the holomorphic part of cohomology defines a map

$$\vartheta^{0,k} : \omega_{N,\infty}^{\otimes k} \to \operatorname{Sym}^2 \mathbb{E}_{N,\infty} \otimes \omega_{N,\infty}^{\otimes k},$$

where $\mathbb{E}_{N,\infty} := \mathcal{H}^{1,0}_\infty(A_{\mathrm{univ},N}/\mathfrak{S}_{g,N})$. Iterating this procedure, we define further maps $\vartheta^{1,k}, \dots, \vartheta^{g-1,k}$, such that composing them, we get a map

$$\vartheta^{g-1,k} \circ \cdots \vartheta^{1,k} \circ \vartheta^{0,k} : \omega_{N,\infty}^{\otimes k} \to (\operatorname{Sym}^2 \mathbb{E}_{N,\infty})^{\otimes g} \otimes \omega_{N,\infty}^{\otimes k}.$$

Now the representation $(\operatorname{Sym}^2 \mathrm{std})^{\otimes g}$ is reducible and contains a copy of $(\det \mathrm{std})^{\otimes 2}$, so $(\operatorname{Sym}^2 \mathbb{E}_{N,\infty})^{\otimes g}$ contains a copy of $\omega_{N,\infty}^{\otimes 2}$. Projecting onto this copy defines Θ_N.

In Section 5.1, we show that if k is a field of characteristic 0 or of prime characteristic $p \geq g+2$, then the representation $(\operatorname{Sym}^2 \mathrm{std})^{\otimes g}$ of $\mathrm{GL}_g(k)$ contains $(\det \mathrm{std})^{\otimes 2}$ as an irreducible factor. In Section 5.2, we define the Gauss-Manin connection on $\mathcal{H}^1_{\mathrm{dR}}(X/S)$, and show how to twist it by certain representations. In Section 5.3, we discuss the analytic Kodaira-Spencer isomorphism and give the construction of Θ_N. Finally, in Section 5.4, we calculate Θ_N in the case $g = 2$.

5.1 A one-dimensional factor of $(\operatorname{Sym}^2 \mathrm{std})^{\otimes g}$

Let k be a field, and let E be a vector space over k of dimension $g \geq 1$. Consider the linear representation $(\operatorname{Sym}^2 E)^{\otimes g}$. We want find a one-dimensional subrepresentation that is isomorphic to $(\det E)^{\otimes 2} = \bigwedge^g(E) \otimes \bigwedge^g(E)$. We start by defining a map

$$f : (E \otimes E)^{\otimes g} \to \bigwedge^g(E) \otimes \bigwedge^g(E)$$

by

$$(u_1 \otimes v_1) \otimes \cdots \otimes (u_g \otimes v_g) \mapsto u_1 \wedge \cdots \wedge u_g \otimes v_1 \wedge \cdots \wedge v_g.$$

Consider the action of the symmetric group S_2 on $E \otimes E$, which is given on pure tensors by permuting the two factors:

$$(12) : u \otimes v \mapsto v \otimes u.$$

This action extends to an action of $S_2^{\times g}$ on $(E \otimes E)^{\otimes g}$, via

$$(\kappa_1, \ldots, \kappa_g) \cdot x_1 \otimes \cdots \otimes x_g := \kappa_1 \cdot x_1 \otimes \cdots \otimes \kappa_g \cdot x_g.$$

Given $\omega \in (E \otimes E)^{\otimes g}$, we define

$$\varphi(\omega) := \sum_{(\kappa_1,\ldots,\kappa_g) \in S_2^{\times g}} f((\kappa_1, \ldots, \kappa_g) \cdot \omega).$$

This map φ is clearly $GL_g(k)$-equivariant. Moreover, it is symmetric in u_i and v_i for each i, so it factors through $(\operatorname{Sym}^2 E)^{\otimes g}$:

$$\begin{array}{ccc} (\operatorname{Sym}^2 E)^{\otimes g} & \overset{\exists!\, \widetilde{\varphi}}{\dashrightarrow} & \bigwedge^g(E) \otimes \bigwedge^g(E) \\ \uparrow & \nearrow_{\varphi} & \\ (E \otimes E)^{\otimes g}. & & \end{array}$$

We will show below that $\widetilde{\varphi}$ is surjective, so that we have a short exact sequence

$$0 \longrightarrow \ker \widetilde{\varphi} \longrightarrow (\operatorname{Sym}^2 E)^{\otimes g} \xrightarrow{\widetilde{\varphi}} \bigwedge^g(E) \otimes \bigwedge^g(E) \longrightarrow 0. \tag{5.1}$$

To show that this sequence splits, we will exhibit a right-inverse for $\widetilde{\varphi}$. As a first approximation, define a $GL_g(k)$-equivariant map $\psi : E^{\otimes g} \otimes E^{\otimes g} \to (\operatorname{Sym}^2 E)^{\otimes g}$ by

$$\psi((u_1 \otimes \cdots \otimes u_g) \otimes (v_1 \otimes \cdots \otimes v_g)) := \sum_{(\sigma,\tau) \in S_g \times S_g} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)\, u_{\sigma 1} \cdot v_{\tau 1} \otimes \cdots \otimes u_{\sigma g} \cdot v_{\tau g}.$$

By construction, ψ is alternating both in the u_i and in the v_i, so it factors through $\bigwedge^g(E) \otimes \bigwedge^g(E)$:

$$\begin{array}{ccc} \bigwedge^g(E) \otimes \bigwedge^g(E) & \overset{\exists!\, \widetilde{\psi}}{\dashrightarrow} & (\operatorname{Sym}^2 E)^{\otimes g} \\ \uparrow & \nearrow_{\psi} & \\ E^{\otimes g} \otimes E^{\otimes g}. & & \end{array}$$

Choose a basis $e_1, \ldots, e_g$ for E, so that $(e_1 \wedge \cdots \wedge e_g)^{\otimes 2}$ is a basis for $\bigwedge^g(E)^{\otimes 2}$. Its image under $\widetilde{\psi}$ is a sum of terms of the form

$$\operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)\, e_{\sigma 1} \cdot e_{\tau 1} \otimes \cdots \otimes e_{\sigma g} \cdot e_{\tau g},$$

with $\sigma, \tau \in S_g$. To what multiple of $(e_1 \wedge \cdots \wedge e_g)^{\otimes 2}$ does $\widetilde{\varphi}$ send these summands? We first consider the case where $\sigma = (1)$.

Lemma 5.1.1. *Let $\tau \in S_g$. Then*

$$\widetilde{\varphi}(\operatorname{sgn}(\tau)\, e_1 \cdot e_{\tau 1} \otimes \cdots \otimes e_g \cdot e_{\tau g}) = 2^r\, (e_1 \wedge \cdots \wedge e_g)^{\otimes 2},$$

where r is the number of disjoint cycles in the cycle decomposition of τ.

Proof. Let $\omega = (e_1 \otimes e_{\tau 1}) \otimes \cdots \otimes (e_g \otimes e_{\tau g})$, and note that each of the 2^g summands of $\widetilde{\varphi}(e_1 \cdot e_{\tau 1} \otimes \cdots \otimes e_g \cdot e_{\tau g})$ is obtained by acting on ω by $S_2^{\times g}$, then applying the map f. We record the indices of ω in an array:

$$\begin{pmatrix} 1 & 2 & \cdots & j & \cdots & g \\ \tau 1 & \tau 2 & \cdots & \tau j & \cdots & \tau g \end{pmatrix},$$

where the i, j-entry is the index of the basis element located in ith factor of E, within the jth copy of $E \otimes E$. Now if $(\kappa_1, \ldots, \kappa_g) \in S_2^{\times g}$, then the index array of $(\kappa_1, \ldots, \kappa_g) \cdot \omega$ is obtained from that of ω by permuting within the columns:

$$\begin{pmatrix} \kappa_1(1) & \cdots & \kappa_j(j) & \cdots & \kappa_g(g) \\ \kappa_1(\tau 1) & \cdots & \kappa_j(\tau j) & \cdots & \kappa_g(\tau g) \end{pmatrix}.$$

We can then read off $f((\kappa_1, \ldots, \kappa_g) \cdot \omega)$ from the rows:

$$f((\kappa_1, \ldots, \kappa_g) \cdot \omega) = e_{\kappa_1(1)} \wedge \ldots \wedge e_{\kappa_g(g)} \otimes e_{\kappa_1(\tau 1)} \wedge \cdots \wedge e_{\kappa_g(\tau g)}.$$

In particular, note that f sends $(\kappa_1, \ldots, \kappa_g) \cdot \omega$ to 0 exactly when either row of its array contains a repeated index.

In fact, there are precisely 2^r elements of $S_2^{\times g}$ which, when they act on the index array of ω, do not produce a row-repeated index. To see this, suppose that such an element has exchanged i_0 in the top row with $i_1 = \tau(i_0)$ in the bottom row. Let $(i_0\, i_1 \ldots i_s)$ be the cycle appearing in the decomposition of τ to which i_0 belongs. We claim that for each $0 \le j \le s$, the i_j which was originally in the top row has been swapped with $\tau(i_j)$ in the bottom row. If $s = 0$, the claim is true. Otherwise, the i_1 in the top row must have been exchanged with $\tau(i_1)$ in the bottom row, since if not, the top row would contain two i_1's. Continuing like this establishes the claim.

Finally, we need to know that

$$\operatorname{sgn}(\tau)\, e_{\kappa_1(1)} \wedge \ldots \wedge e_{\kappa_g(g)} \otimes e_{\kappa_1(\tau 1)} \wedge \cdots \wedge e_{\kappa_g(\tau g)} = (e_1 \wedge \cdots \wedge e_g)^{\otimes 2}.$$

If we let ρ_1, ρ_2 denote the two rows of the array

$$\begin{pmatrix} \kappa_1(1) & \cdots & \kappa_j(j) & \cdots & \kappa_g(g) \\ \kappa_1(\tau 1) & \cdots & \kappa_j(\tau j) & \cdots & \kappa_g(\tau g) \end{pmatrix}$$

thought of as permutations of $\{1, 2, \ldots, g\}$ in one-line notation, then what we have to show is that $\operatorname{sgn}(\rho_1)\operatorname{sgn}(\rho_2) = \operatorname{sgn}(\tau)$. In fact, $\tau = \rho_1 \rho_2$. To see this, just note that if τ factors into disjoint cycles as $\tau = \tau_1 \cdots \tau_r$, and the action of $S_2^{\times g}$ exchanges the indices belonging to the cycles $\tau_{j_1}, \ldots, \tau_{j_l}$, then ρ_2 fixes each of these indices while agreeing with τ outside of them, while $\rho_1 = \tau_{j_1} \cdots \tau_{j_l}$. □

Lemma 5.1.2. *Let* $\sigma, \tau \in S_g$. *Then*

$$\widetilde{\varphi}(\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\, e_{\sigma 1} \cdot e_{\tau 1} \otimes \cdots \otimes e_{\sigma g} \cdot e_{\tau g}) = 2^r\,(e_1 \wedge \cdots \wedge e_g)^{\otimes 2},$$

where r *is the number of disjoint cycles in the cycle decomposition of* $\sigma^{-1}\tau$.

Proof. Note that if $(\kappa_1, \ldots, \kappa_g) \in S_2^{\times g}$, we have

$$\begin{aligned} &\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\, e_{\kappa_1(\sigma 1)} \wedge \ldots \wedge e_{\kappa_g(\sigma g)} \otimes e_{\kappa_1(\tau 1)} \wedge \cdots \wedge e_{\kappa_g(\tau g)} \\ = \quad &\operatorname{sgn}(\sigma^{-1}\tau)\, e_{\kappa_1(1)} \wedge \ldots \wedge e_{\kappa_g(g)} \otimes e_{\kappa_1(\sigma^{-1}\tau 1)} \wedge \cdots \wedge e_{\kappa_g(\sigma^{-1}\tau g)}. \end{aligned}$$

Then follow the proof of the previous lemma. □

Recall that the number of permutations of $\{1, 2, \ldots, g\}$ with exactly r disjoint cycles is counted by the *Stirling cycle number* $\left[{g \atop r}\right]$.

Lemma 5.1.3. *Let* $g \geq 1$ *be an integer. We have*

$$\sum_{1 \leq r \leq g} \begin{bmatrix} g \\ r \end{bmatrix} 2^r = (g+1)!\ .$$

Proof. We argue inductively. The result clearly holds for $g = 1$, so suppose it is true for some $g \geq 1$. Using the recurrence relation of the Stirling cycle numbers, we obtain

$$\begin{aligned} \sum_{1 \leq r \leq g+1} \begin{bmatrix} g+1 \\ r \end{bmatrix} 2^r &= \sum_{1 \leq r \leq g+1} \left(g \begin{bmatrix} g \\ r \end{bmatrix} + \begin{bmatrix} g \\ r-1 \end{bmatrix} \right) 2^r \\ &= g \sum_{1 \leq r \leq g+1} \begin{bmatrix} g \\ r \end{bmatrix} + \sum_{0 \leq r \leq g} \begin{bmatrix} g \\ r \end{bmatrix} 2^{r+1} \\ &= g \sum_{1 \leq r \leq g} \begin{bmatrix} g \\ r \end{bmatrix} + 2 \sum_{1 \leq r \leq g} \begin{bmatrix} g \\ r \end{bmatrix} 2^r \\ &= g(g+1)! + 2(g+1)! \\ &= (g+2)!\ . \end{aligned}$$

□

Lemma 5.1.4. *We have* $\widetilde{\varphi} \circ \widetilde{\psi} = g!(g+1)! \cdot \mathrm{id}$.

Proof. We write $r(\sigma)$ for the number of disjoint cycles in the decomposition of a permutation σ. We have

$$\begin{aligned} \widetilde{\varphi} \circ \widetilde{\psi}\left((e_1 \wedge \cdots \wedge e_g)^{\otimes 2}\right) &= \widetilde{\varphi}\left(\sum_{\sigma, \tau \in S_g} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\, e_{\sigma 1} \cdot e_{\tau 1} \otimes \cdots \otimes e_{\sigma g} \cdot e_{\tau g} \right) \\ &= \sum_{\sigma \in S_g} \sum_{\tau \in S_g} 2^{r(\sigma^{-1}\tau)} (e_1 \wedge \cdots \wedge e_g)^{\otimes 2} \\ &= \sum_{\sigma \in S_g} \sum_{\tau \in S_g} 2^{r(\tau)} (e_1 \wedge \cdots \wedge e_g)^{\otimes 2} \\ &= g! \left(\sum_{1 \leq r \leq g} \begin{bmatrix} g \\ r \end{bmatrix} 2^r \right) (e_1 \wedge \cdots \wedge e_g)^{\otimes 2} \\ &= g!(g+1)!(e_1 \wedge \cdots \wedge e_g)^{\otimes 2}. \end{aligned}$$

□

Theorem 5.1.5. *If the characteristic of* k *is* 0 *or a prime* $p \geq g+2$, *then* $(\mathrm{Sym}^2 E)^{\otimes g}$ *has a subrepresentation isomorphic to* $(\det E)^{\otimes 2}$. *The formula for projection onto this factor is given by* $\widetilde{\varphi}$.

Proof. In this case, $\frac{1}{g!(g+1)!}\widetilde{\psi}$ is a right-inverse for $\widetilde{\varphi}$, so it splits the short exact sequence (5.1). $\square$

Remark. Compare the results in this section with [10], in particular the map β on page 96.

5.2 The twisted Gauss-Manin connection

In this section, we discuss the parts of the construction of Θ that apply to a general family of complex manifolds X/S. We assume that the fibres of X/S have a Hodge decomposition, so that by Proposition 3.3.1 and Theorem 3.3.5, we have a splitting

$$\mathcal{H}^1_{\mathrm{dR}}(X/S) = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1},$$

such that $\mathcal{H}^{1,0}$ is identified with the Hodge sheaf $f_*\Omega^1_{X/S}$.

We begin by defining the Gauss-Manin connection on $\mathcal{H}^1(X/S)$, following [27, Definition 9.13] and [1, Section 2.A]. Conceptually, this provides a means of differentiating cohomology classes with respect to coordinates on the base S.

Definition 5.2.1. Let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module. A *connection* on $\mathcal{E}$ is a $\mathbb{C}$-linear map of sheaves

$$\nabla : \mathcal{E} \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{E},$$

satisfying the Leibniz rule

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla(\sigma),$$

where $f \in \mathcal{O}_S(U)$ and $\sigma \in \mathcal{E}(U)$ are sections over some open $U \subseteq S$.

Remark. A similar definition applies to $\mathcal{C}^\infty_S$-modules.

For example, the exterior derivative $\partial : \mathcal{O}_S \to \Omega^1_S$ is a connection on $\mathcal{O}_S$. More generally, if $\mathcal{V}$ is a local system of vector spaces on S, then $\nabla = \partial \otimes 1$ is a connection on the associated $\mathcal{O}_S$-module $\mathcal{O}_S \otimes_{\underline{\mathbb{C}}} \mathcal{V}$. Note that $\ker \nabla = \mathcal{V}$.

Definition 5.2.2. The connection

$$\nabla := \partial \otimes 1 : \mathcal{H}^1(X/S) \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{H}^1(X/S)$$

on $\mathcal{H}^1(X/S)$ is called the *Gauss-Manin connection*.

We need a smooth extension of ∇ to the $\mathcal{C}^\infty_S$-module $\mathcal{H}^1_\infty(X/S) := \mathcal{C}^\infty_S \otimes_{\underline{\mathbb{C}}} R^1 f_*\underline{\mathbb{C}}$.

Definition 5.2.3. The connection

$$\nabla_\infty := \partial \otimes 1 : \mathcal{H}^1_\infty(X/S) \to \mathcal{A}^{1,0}_S \otimes_{\mathcal{C}^\infty_S} \mathcal{H}^1_\infty(X/S)$$

on $\mathcal{H}^1_\infty(X/S)$ is called the C^∞ *Gauss-Manin connection*.

By definition, the operator $\partial : \mathcal{C}_S^\infty \to \mathcal{A}_S^{1,0}$ restricts to $\partial : \mathcal{O}_S \to \Omega_S^1$, so the connections ∇ and ∇_∞ agree on $\mathcal{H}^1(X/S)$. Note that we can use the relative de Rham isomorphism to get connections on $\mathcal{H}^1_{dR}(X/S)$ and $\mathcal{C}_S^\infty \otimes_{\mathcal{O}_S} \mathcal{H}^1_{dR}(X/S)$.

Consider a representation of the form

$$\kappa = \kappa^{j,k} := (\mathrm{Sym}^2 \mathrm{std})^{\otimes j} \otimes (\det \mathrm{std})^{\otimes k}, \tag{5.2}$$

where $j \geq 1$ and $k \geq 1$. We will 'twist' the sequence

$$\mathcal{H}^{1,0} \xrightarrow{\ i\ } \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1} \xrightarrow{\ p\ } \mathcal{H}^{1,0}$$

by κ. Taking determinants gives

$$\textstyle\bigwedge^g \mathcal{H}^{1,0} \xrightarrow{i^{\det}} \bigoplus_{\mu+\nu=g} \bigwedge^\mu \mathcal{H}^{1,0} \otimes \bigwedge^\nu \mathcal{H}^{0,1} \xrightarrow{p^{\det}} \bigwedge^g \mathcal{H}^{1,0},$$

while taking symmetric squares gives

$$\mathrm{Sym}^2 \mathcal{H}^{1,0} \xrightarrow{i^{\mathrm{Sym}^2}} \bigoplus_{\mu+\nu=2} \mathrm{Sym}^\mu \mathcal{H}^{1,0} \otimes \mathrm{Sym}^\nu \mathcal{H}^{0,1} \xrightarrow{p^{\mathrm{Sym}^2}} \mathrm{Sym}^2 \mathcal{H}^{1,0}.$$

Taking tensor powers then gives maps $i^{\det^{\otimes k}}$, $p^{\det^{\otimes k}}$ and $i^{(\mathrm{Sym}^2)^{\otimes j}}$, $p^{(\mathrm{Sym}^2)^{\otimes j}}$. Tensoring these maps together, we obtain a sequence which we will denote by

$$\left(\mathcal{H}^{1,0}\right)^\kappa \xrightarrow{i^\kappa} \left(\mathcal{H}^1_{dR}\right)^\kappa \xrightarrow{p^\kappa} \left(\mathcal{H}^{1,0}\right)^\kappa.$$

We also have a smooth version of this sequence:

$$\left(\mathcal{H}^{1,0}_\infty\right)^\kappa \xrightarrow{i^\kappa} \left(\mathcal{H}^1_\infty\right)^\kappa \xrightarrow{p^\kappa} \left(\mathcal{H}^{1,0}_\infty\right)^\kappa,$$

which we get by tensoring with $\mathcal{C}_S^\infty$.

Note that we can extend ∇_∞ to a connection $\nabla_\infty^{\otimes k}$ on $(\mathcal{H}^1_\infty)^{\otimes k}$ via the Leibniz rule:

$$\nabla_\infty^{\otimes k}(v_1 \otimes \cdots \otimes v_k) := \sum_{1 \leq i \leq g} \sigma_i\big(v_1 \otimes \cdots \otimes \nabla_\infty(v_i) \otimes \cdots \otimes v_g\big),$$

where $v_1, \ldots, v_g$ are sections of $(\mathcal{H}^1_\infty)^{\otimes k}$ over some open subset of S, and

$$\sigma_i : \left(\mathcal{H}^1_\infty\right)^{\otimes(i-1)} \otimes \mathcal{A}_S^{1,0} \otimes \left(\mathcal{H}^1_\infty\right)^{\otimes(k-i+1)} \xrightarrow{\sim} \mathcal{A}_S^{1,0} \otimes \left(\mathcal{H}^1_\infty\right)^{\otimes k}$$

is the isomorphism that swaps the 1st and ith factors. In a similar way, we can extend ∇_∞ to symmetric and exterior powers of $\mathcal{H}^1_\infty$. This allows us to define a connection

$$\nabla_\infty^\kappa : \left(\mathcal{H}^1_\infty\right)^\kappa \to \mathcal{A}_S^{1,0} \otimes_{\mathcal{C}_S^\infty} \left(\mathcal{H}^1_\infty\right)^\kappa,$$

which we refer to as the *twist* of ∇_∞ by κ. (For a more detailed discussion of this twisting operation, see [9].)

The following diagram is the basic component of the operator Θ:

$$\begin{array}{ccc} \left(\mathcal{H}^{1,0}_\infty\right)^\kappa & \xrightarrow{\quad i^\kappa \quad} & \left(\mathcal{H}^1_\infty\right)^\kappa \\ \downarrow & & \downarrow \nabla_\infty^\kappa \\ \mathcal{A}_S^{1,0} \otimes \left(\mathcal{H}^{1,0}_\infty\right)^\kappa & \xleftarrow[\mathrm{id} \otimes p^\kappa]{} & \mathcal{A}_S^{1,0} \otimes \left(\mathcal{H}^1_\infty\right)^\kappa. \end{array} \tag{5.3}$$

5.3 Definition of Θ

We now specialise the constructions of the previous section to our families of abelian varieties $A_{\text{univ}}/\mathfrak{S}_g$ and $A_{\text{univ},N}/\mathfrak{S}_{g,N}$. We denote their Hodge sheaves by

$$\mathbb{E} := \pi_* \Omega^1_{A_{\text{univ}}/\mathfrak{S}_g} \qquad \text{and} \qquad \mathbb{E}_N := \pi_{N*} \Omega^1_{A_{\text{univ},N}/\mathfrak{S}_{g,N}},$$

and we write

$$\mathbb{E}_\infty := \mathcal{C}^\infty_{\mathfrak{S}_g} \otimes_{\mathcal{O}_{\mathfrak{S}_g}} \mathbb{E} \qquad \text{and} \qquad \mathbb{E}_{N,\infty} := \mathcal{C}^\infty_{\mathfrak{S}_{g,N}} \otimes_{\mathcal{O}_{\mathfrak{S}_{g,N}}} \mathbb{E}_N$$

for the smooth versions of these sheaves. If $\kappa = \kappa^{j,k}$ is a representation of the form (5.2), then we denote the twists of $\mathbb{E}_\infty$ and $\mathbb{E}_{N,\infty}$ by $\mathbb{E}^\kappa_\infty$ and $(\mathbb{E}_{N,\infty})^\kappa$. Note that we have $\omega_\infty = \mathbb{E}^{\det}_\infty$ and $\omega_{N,\infty} = (\mathbb{E}_{N,\infty})^{\det}$ in the notation of the previous chapter.

If we consider Diagram (5.3) with respect to the family $A_{\text{univ}}/\mathfrak{S}_g$, we see that the dashed arrow lands in the space $\mathcal{A}^{1,0}_{\mathfrak{S}_g} \otimes \mathbb{E}^\kappa_\infty$. Since our operator Θ is defined by iterations, we need to interpret $\mathcal{A}^{1,0}_{\mathfrak{S}_g} \otimes \mathbb{E}^\kappa_\infty$ as $\mathbb{E}^{\kappa'}_\infty$ for some representation κ'. Note that if $z_{ij} = z_{ji}$ denotes the standard coordinates on $\mathfrak{S}_g$, then the sheaf $\Omega^1_{\mathfrak{S}_g}$ is free on the basis $dz_{ij} = dz_{ji}$. On the other hand, if $u_1, \ldots, u_g$ are the usual coordinates on $\mathbb{C}^g$, then $\mathbb{E}$ is free on the basis $du_1, \ldots, du_g$. The map

$$\mathrm{KS} : \mathrm{Sym}^2 \mathbb{E} \to \Omega^1_{\mathfrak{S}_g}, \qquad du_i \cdot du_j \mapsto dz_{ij}$$

is an isomorphism of $\mathcal{O}_{\mathfrak{S}_g}$-modules, known as the *Kodaira-Spencer isomorphism*. Extending its inverse to a C^∞ isomorphism $\mathrm{KS}^{-1} : \mathcal{A}^{1,0}_{\mathfrak{S}_g} \cong \mathrm{Sym}^2 \mathbb{E}_\infty$, we define a map $\vartheta^{j,k} : \mathbb{E}^{\kappa^{j,k}}_\infty \to \mathbb{E}^{\kappa^{j+1,k}}_\infty$ by the diagram

$$\begin{array}{ccc} \mathbb{E}^\kappa_\infty & \xrightarrow{\quad i^\kappa \quad} & \left(\mathcal{H}^1_\infty\right)^\kappa \\ \vdots & & \big\downarrow \nabla^\kappa_\infty \\ \vdots\ \vartheta^{j,k} & & \mathcal{A}^{1,0}_{\mathfrak{S}_g} \otimes \left(\mathcal{H}^1_\infty\right)^\kappa \\ \vdots & & \big\downarrow \mathrm{KS}^{-1} \otimes \mathrm{id} \\ \mathrm{Sym}^2 \mathbb{E}_\infty \otimes \mathbb{E}^\kappa_\infty & \xleftarrow[\mathrm{id} \otimes p^\kappa]{} & \mathrm{Sym}^2 \mathbb{E}_\infty \otimes \left(\mathcal{H}^1_\infty\right)^\kappa . \end{array} \tag{5.4}$$

The next step is to show that $\vartheta^{j,k}$ descends to a map

$$\vartheta^{j,k}_N : (\mathbb{E}_{N,\infty})^{\kappa^{j,k}} \to (\mathbb{E}_{N,\infty})^{\kappa^{j+1,k}}.$$

Now both $\Omega^1_{\mathfrak{S}_g}$ and $\mathrm{Sym}^2 \mathbb{E}$ have actions of $\Gamma_g(N)$, such that $\Omega^1_{\mathfrak{S}_{g,N}} \cong \Gamma_g(N) \backslash \Omega^1_{\mathfrak{S}_g}$ and $\mathrm{Sym}^2 \mathbb{E}_N \cong \Gamma_g(N) \backslash \mathrm{Sym}^2 \mathbb{E}$. Since KS is equivariant with respect to these actions (see [11, Section 14]), it descends to an isomorphism $\mathrm{KS} : \mathrm{Sym}^2 \mathbb{E}_N \cong \Omega^1_{\mathfrak{S}_{g,N}}$.

We define $\vartheta_N^{j,k}$ by the diagram analogous to (5.4) for the family $A_{\mathrm{univ},N}/\mathfrak{S}_{g,N}$:

$$\begin{array}{ccc}
(\mathbb{E}_{N,\infty})^{\kappa} & \xrightarrow{\ i^{\kappa}\ } & (\mathcal{H}^1_{\infty})^{\kappa} \\
 & & \big\downarrow \nabla^{\kappa}_{\infty} \\
\Big\downarrow \vartheta_N^{j,k} & & \mathcal{A}^{1,0}_{\mathfrak{S}_{g,N}} \otimes (\mathcal{H}^1_{\infty})^{\kappa} \\
 & & \big\downarrow \mathrm{KS}^{-1} \otimes \mathrm{id} \\
\mathrm{Sym}^2\, \mathbb{E}_{N,\infty} \otimes (\mathbb{E}_{N,\infty})^{\kappa} & \xleftarrow[\mathrm{id} \otimes p^{\kappa}]{} & \mathrm{Sym}^2\, \mathbb{E}_{N,\infty} \otimes (\mathcal{H}^1_{\infty})^{\kappa}.
\end{array}$$

By construction, $\vartheta^{j,k}$ is a lift of $\vartheta_N^{j,k}$.

Let $\mathrm{proj} : (\mathrm{Sym}^2\, \mathbb{E}_{N,\infty})^{\otimes g} \to \omega_{N,\infty}^{\otimes 2}$ be the projection induced from Theorem 5.1.5.

Definition 5.3.1. The C^{∞} *theta operator* $\Theta_N : \omega_{N,\infty}^{\otimes k} \to \omega_{N,\infty}^{\otimes k+2}$ is defined to be the composition

$$\Theta_N := (\mathrm{proj} \otimes \mathrm{id}) \circ \vartheta_N^{g-1,k} \circ \cdots \circ \vartheta_N^{1,k} \circ \vartheta_N^{0,k}.$$

To compute Θ_N, we lift it to a map $\Theta : \omega_{\infty}^{\otimes k} \to \omega_{\infty}^{\otimes k+2}$, defined by

$$\Theta := (\mathrm{proj} \otimes \mathrm{id}) \circ \vartheta^{g-1,k} \circ \cdots \circ \vartheta^{1,k} \circ \vartheta^{0,k}.$$

Here proj is the projection of $(\mathrm{Sym}^2\, \mathbb{E}_{\infty})^{\otimes g}$ onto $\omega_{\infty}^{\otimes 2}$. The computational problem is to find a formula for $\Theta(f\, du_1 \wedge \cdots \wedge du_g)$, where f is some smooth $\mathbb{C}$-valued function on $\mathfrak{S}_g$. Consider the 'untwisted' theta operator

$$\begin{array}{ccc}
\mathbb{E}_{\infty} & \xrightarrow{\ i\ } & \mathcal{H}^1_{\infty} \\
 & & \big\downarrow \nabla_{\infty} \\
\Big\downarrow \vartheta & & \mathcal{A}^{1,0}_{\mathfrak{S}_g} \otimes \mathcal{H}^1_{\infty} \\
 & & \big\downarrow \mathrm{KS}^{-1} \otimes \mathrm{id} \\
\mathrm{Sym}^2\, \mathbb{E}_{\infty} \otimes \mathbb{E}_{\infty} & \xleftarrow[\mathrm{id} \otimes p]{} & \mathrm{Sym}^2\, \mathbb{E}_{\infty} \otimes \mathcal{H}^1_{\infty};
\end{array}$$

this ϑ is the basic operator introduced and calculated by Harris in [13, Section 4.3]. Let $\mathbf{z} = (z_{ij})$ be the standard coordinates on $\mathfrak{S}_g$. We want to know how ϑ acts on the basis $du_1, \ldots, du_g$ of $\mathbb{E}_{\infty}$. Recall the Equations (3.13):

$$du_l = \alpha_l + \textstyle\sum_{1 \le \nu \le g} z_{l\nu} \beta_{\nu}, \qquad d\bar{u}_l = \alpha_l + \textstyle\sum_{1 \le \nu \le g} \bar{z}_{l\nu} \beta_{\nu}.$$

Writing these equations in terms of matrices

$$d\mathbf{u} = \alpha + \mathbf{z}\beta, \qquad d\bar{\mathbf{u}} = \alpha + \bar{\mathbf{z}}\beta$$

(here $d\mathbf{u}$, $d\bar{\mathbf{u}}$, α and β are column vectors), we see that

$$d\mathbf{u} - d\bar{\mathbf{u}} = (\mathbf{z} - \bar{\mathbf{z}})\beta \implies \beta = \frac{1}{2i}\mathbf{y}^{-1}(d\mathbf{u} - d\bar{\mathbf{u}}),$$

where $\mathbf{y} = \operatorname{Im}\mathbf{z}$. So the projection of β onto $\mathbb{E}_\infty$ is $\frac{1}{2i}\mathbf{y}^{-1}d\mathbf{u}$. Now by definition, $\nabla_\infty(du_l) = \sum_{1\le\nu\le g} dz_{l\nu}\beta_\nu$, so the formula for ϑ is

$$\vartheta(du_l) = \frac{1}{2i}\sum_{1\le\nu\le g} du_\nu \cdot du_l \otimes (\mathbf{y}^{-1}d\mathbf{u})_\nu, \tag{5.5}$$

where $(\mathbf{y}^{-1}d\mathbf{u})_\nu$ is the νth entry of the column vector $\mathbf{y}^{-1}d\mathbf{u}$.

5.4 Computation of Θ for $g = 2$

In this section, we compute $\Theta = (\operatorname{proj}\otimes\operatorname{id})\circ\vartheta^{1,k}\circ\vartheta^{0,k}$. We proceed in stages: first we compute $\vartheta^{0,k}$, then $(\operatorname{proj}\otimes\operatorname{id})\circ\vartheta^{1,k}$. Note that when $g = 2$, Equation (5.5) becomes

$$\vartheta(du_l) = \frac{1}{2i\det\mathbf{y}}\big(y_{22}\,du_1\cdot du_l\otimes du_1 - y_{12}\,du_1\cdot du_l\otimes du_2 \\ - y_{12}\,du_2\cdot du_l\otimes du_1 + y_{11}\,du_2\cdot du_l\otimes du_2\big). \tag{5.6}$$

Lemma 5.4.1. *If* $f : \mathfrak{S}_2 \to \mathbb{C}$ *is a smooth function, then*

$$\vartheta^{0,k}\big(f\,(du_1\wedge du_2)^{\otimes k}\big) = \left[\left(\frac{\partial f}{\partial z_{11}} + \frac{k}{2i}\frac{y_{22}}{\det\mathbf{y}}f\right)du_1\cdot du_1 + \left(\frac{\partial f}{\partial z_{12}} - \frac{k}{i}\frac{y_{12}}{\det\mathbf{y}}f\right)du_1\cdot du_2 \\ + \left(\frac{\partial f}{\partial z_{22}} + \frac{k}{2i}\frac{y_{11}}{\det\mathbf{y}}f\right)du_2\cdot du_2\right](du_1\wedge du_2)^{\otimes k}.$$

Proof. Note that

$$\vartheta^{0,k}\big(f\,(du_1\wedge du_2)^{\otimes k}\big) = \partial f\otimes(du_1\wedge du_2)^{\otimes k} \\ + kf\,\vartheta^{0,1}(du_1\wedge du_2)\otimes(du_1\wedge du_2)^{\otimes(k-1)},$$

where $\partial f = \frac{\partial f}{\partial z_{11}}du_1\cdot du_1 + \frac{\partial f}{\partial z_{12}}du_1\cdot du_2 + \frac{\partial f}{\partial z_{22}}du_2\cdot du_2$. Then use (5.6) to compute that

$$\vartheta^{0,1}(du_1\wedge du_2) = \frac{1}{2i\det\mathbf{y}}\big(y_{22}\,du_1\cdot du_1 - 2y_{12}\,du_1\cdot du_2 + y_{11}\,du_2\cdot du_2\big) \\ \otimes du_1\wedge du_2.$$

□

We now look at $\vartheta^{1,k}$. Suppose $F, G, H : \mathfrak{S}_2 \to \mathbb{C}$ are smooth functions. By definition of $\vartheta^{1,k}$, we have

$$\vartheta^{1,k}\big((F\,du_1\cdot du_1 + G\,du_1\cdot du_2 + H\,du_2\cdot du_2)(du_1\wedge du_2)^{\otimes k}\big) = \\ \big[\partial F\otimes du_1\cdot du_1 + \partial G\otimes du_1\cdot du_2 + \partial H\otimes du_2\cdot du_2\big](du_1\wedge du_2)^{\otimes k} \\ + \big[F\,\vartheta^{1,0}(du_1\cdot du_1) + G\,\vartheta^{1,0}(du_1\cdot du_2) + H\,\vartheta^{1,0}(du_2\cdot du_2)\big](du_1\wedge du_2)^{\otimes k} \\ + \big[F\,du_1\cdot du_1 + G\,du_1\cdot du_2 + H\,du_2\cdot du_2\big]\vartheta^{0,k}\big((du_1\wedge du_2)^{\otimes k}\big). \tag{5.7}$$

Instead of expanding everything out to obtain a formula for $\vartheta^{1,k}$, we will apply $\operatorname{proj}\otimes\operatorname{id}$ to (5.7) directly. First note the following:

Lemma 5.4.2. *Suppose* $du_{i_1} \cdot du_{i_2} \otimes du_{j_1} \cdot du_{j_2} \in \mathbb{E}^{\mathrm{Sym}^2}(\mathfrak{S}_g)$. *Assuming without loss of generality that* $i_1 \leq i_2$ *and* $j_1 \leq j_2$, *we have*

$$\operatorname{proj}(du_{i_1} \cdot du_{i_2} \otimes du_{j_1} \cdot du_{j_2}) = \begin{cases} 4\,(du_1 \wedge du_2)^{\otimes 2} & i_1 = i_2 = 1 \text{ and } j_1 = j_2 = 2 \\ 4\,(du_1 \wedge du_2)^{\otimes 2} & i_2 = i_2 = 2 \text{ and } j_1 = j_2 = 1 \\ -2\,(du_1 \wedge du_2)^{\otimes 2} & i_1 = j_1 = 1 \text{ and } i_2 = j_2 = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Immediate from the definition of proj. □

Lemma 5.4.3. *If we apply* proj *to the coefficient of the first* $(du_1 \wedge du_2)^{\otimes k}$ *on the RHS of* (5.7), *we get*

$$\operatorname{proj}(\partial F \otimes du_1 \cdot du_1 + \partial G \otimes du_1 \cdot du_2 + \partial H \otimes du_2 \cdot du_2) = \left(4\frac{\partial H}{\partial z_{11}} - 2\frac{\partial G}{\partial z_{12}} + 4\frac{\partial F}{\partial z_{22}}\right)(du_1 \wedge du_2)^{\otimes 2}.$$

Proof. Note that

$$\partial F = \frac{\partial F}{\partial z_{11}} du_1 \cdot du_1 + \frac{\partial F}{\partial z_{12}} du_1 \cdot du_2 + \frac{\partial F}{\partial z_{22}} du_2 \cdot du_2,$$

and similarly for ∂G and ∂H. Then apply the previous lemma. □

Lemma 5.4.4. *If we apply* proj *to the coefficient of the second* $(du_1 \wedge du_2)^{\otimes k}$ *on the RHS of* (5.7), *we get*

$$\operatorname{proj}(F\,\vartheta^{1,0}(du_1 \cdot du_1) + G\,\vartheta^{1,0}(du_1 \cdot du_2) + H\,\vartheta^{1,0}(du_2 \cdot du_2)) = + \frac{2i}{\det \mathbf{y}}(y_{11}F + y_{12}G + y_{22}H)\,(du_1 \wedge du_2)^{\otimes 2}.$$

Proof. Using (5.6), we find that

$$\begin{aligned} \vartheta^{1,0}(du_1 \cdot du_1) &= 2\,\vartheta(du_1) \cdot du_1 \\ &= \frac{1}{i \det \mathbf{y}}\big(y_{22}\, du_1 \cdot du_1 \otimes du_1 \cdot du_1 - y_{12}\, du_1 \cdot du_1 \otimes du_1 \cdot du_2 \\ &\quad - y_{12}\, du_1 \cdot du_2 \otimes du_1 \cdot du_1 + y_{11}\, du_1 \cdot du_2 \otimes du_1 \cdot du_2\big), \end{aligned}$$

and

$$\begin{aligned} \vartheta^{1,0}(du_2 \cdot du_2) &= 2\,\vartheta(du_2) \cdot du_2 \\ &= \frac{1}{i \det \mathbf{y}}\big(y_{22}\, du_1 \cdot du_2 \otimes du_1 \cdot du_2 - y_{12}\, du_1 \cdot du_2 \otimes du_2 \cdot du_2 \\ &\quad - y_{12}\, du_2 \cdot du_2 \otimes du_1 \cdot du_2 + y_{11}\, du_2 \cdot du_2 \otimes du_2 \cdot du_2\big), \end{aligned}$$

and

$$\begin{aligned} \vartheta^{1,0}(du_1 \cdot du_2) &= \vartheta(du_1) \cdot du_2 + du_1 \cdot \vartheta(du_2) \\ &= \frac{1}{2i \det \mathbf{y}}\big(y_{22}\, du_1 \cdot du_1 \otimes du_1 \cdot du_2 - y_{12}\, du_1 \cdot du_1 \otimes du_2 \cdot du_2 \\ &\quad - y_{12}\, du_1 \cdot du_2 \otimes du_1 \cdot du_2 + y_{11}\, du_1 \cdot du_2 \otimes du_2 \cdot du_2 \\ &\quad + y_{22}\, du_1 \cdot du_2 \otimes du_1 \cdot du_1 - y_{12}\, du_1 \cdot du_2 \otimes du_1 \cdot du_2 \\ &\quad - y_{12}\, du_2 \cdot du_2 \otimes du_1 \cdot du_1 + y_{11}\, du_2 \cdot du_2 \otimes du_1 \cdot du_2\big). \end{aligned}$$

Hence, we have

$$\begin{aligned}
\operatorname{proj}(\vartheta^{1,0}(du_1 \cdot du_1)) &= -2 \cdot \frac{1}{i}\frac{y_{11}}{\det \mathbf{y}} = 2i\frac{y_{11}}{\det \mathbf{y}} \\
\operatorname{proj}(\vartheta^{1,0}(du_1 \cdot du_2)) &= (-4+2+2-4) \cdot \frac{1}{2i}\frac{y_{12}}{\det \mathbf{y}} = 2i\frac{y_{12}}{\det \mathbf{y}} \\
\operatorname{proj}(\vartheta^{1,0}(du_2 \cdot du_2)) &= -2 \cdot \frac{1}{i}\frac{y_{22}}{\det \mathbf{y}} = 2i\frac{y_{22}}{\det \mathbf{y}}.
\end{aligned}$$

□

Lemma 5.4.5. *If we apply* $\operatorname{proj} \otimes \operatorname{id}$ *to the third term on the RHS of* (5.7)*, we get*

$$\operatorname{proj} \otimes \operatorname{id}\Big(\big[F\, du_1 \cdot du_1 + G\, du_1 \cdot du_2 + H\, du_2 \cdot du_2\big]\vartheta^{0,k}\big((du_1 \wedge du_2)^{\otimes k}\big)\Big) = -\frac{2ki}{\det \mathbf{y}}\left(y_{11}F + y_{12}G + y_{22}H\right)(du_1 \wedge du_2)^{\otimes(k+2)}.$$

Proof. Just take $f = 1$ in Lemma 5.4.1, and project. □

Combining the previous three lemmas, we get:

Lemma 5.4.6. *A formula for* $(\operatorname{proj} \otimes \operatorname{id}) \circ \vartheta^{1,k}$ *is given by*

$$(\operatorname{proj} \otimes \operatorname{id}) \circ \vartheta^{1,k}\big((F\, du_1 \cdot du_1 + G\, du_1 \cdot du_2 + H\, du_2 \cdot du_2)(du_1 \wedge du_2)^{\otimes k}\big) = \left(4\frac{\partial H}{\partial z_{11}} - 2\frac{\partial G}{\partial z_{12}} + 4\frac{\partial F}{\partial z_{22}} - \frac{2(k-1)i}{\det \mathbf{y}}(y_{11}F + y_{12}G + y_{22}H)\right)(du_1 \wedge du_2)^{\otimes(k+2)}. \quad (5.8)$$

Finally, to compute $\Theta(f\,(du_1 \wedge du_2)^{\otimes k})$, we substitute

$$F = \frac{\partial f}{\partial z_{11}} + \frac{k}{2i}\frac{y_{22}}{\det \mathbf{y}}f, \quad G = \frac{\partial f}{\partial z_{12}} - \frac{k}{i}\frac{y_{12}}{\det \mathbf{y}}f, \quad H = \frac{\partial f}{\partial z_{22}} + \frac{k}{2i}\frac{y_{11}}{\det \mathbf{y}}f.$$

into Equation (5.8).

Lemma 5.4.7. *With these values of* F, G *and* H, *we have*

$$4\frac{\partial H}{\partial z_{11}} - 2\frac{\partial G}{\partial z_{12}} + 4\frac{\partial F}{\partial z_{22}} = 8\frac{\partial^2 f}{\partial z_{11}\partial z_{22}} - 2\frac{\partial^2 f}{\partial z_{12}\partial z_{12}} - \frac{2ki}{\det \mathbf{y}}\left(y_{11}\frac{\partial f}{\partial z_{11}} + y_{12}\frac{\partial f}{\partial z_{12}} + y_{22}\frac{\partial f}{\partial z_{22}}\right) - \frac{k}{\det \mathbf{y}}f.$$

Proof. By the product rule,

$$\begin{aligned}
4\frac{\partial H}{\partial z_{11}} - 2\frac{\partial G}{\partial z_{12}} + 4\frac{\partial F}{\partial z_{22}} &= 4\frac{\partial^2 f}{\partial z_{11}\partial z_{22}} - 2\frac{\partial^2 f}{\partial z_{12}^2} + 4\frac{\partial^2 f}{\partial z_{22}\partial z_{11}} \\
&\quad - \frac{2ki}{\det \mathbf{y}}\left(y_{11}\frac{\partial f}{\partial z_{11}} + y_{12}\frac{\partial f}{\partial z_{12}} + y_{22}\frac{\partial f}{\partial z_{22}}\right) \\
&\quad - 2ki\left(\frac{\partial}{\partial z_{11}}\frac{y_{11}}{\det \mathbf{y}} + \frac{\partial}{\partial z_{12}}\frac{y_{12}}{\det \mathbf{y}} + \frac{\partial}{\partial z_{22}}\frac{y_{22}}{\det \mathbf{y}}\right)f.
\end{aligned}$$

To simplify the last term, recall that

$$\frac{\partial}{\partial z_{ij}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{ij}} - i\frac{\partial}{\partial y_{ij}}\right),$$

so

$$-2ki\left(\frac{\partial}{\partial z_{11}}\frac{y_{11}}{\det \mathbf{y}} + \frac{\partial}{\partial z_{12}}\frac{y_{12}}{\det \mathbf{y}} + \frac{\partial}{\partial z_{22}}\frac{y_{22}}{\det \mathbf{y}}\right) = \\ -k\left(\frac{\partial}{\partial y_{11}}\frac{y_{11}}{\det \mathbf{y}} + \frac{\partial}{\partial y_{12}}\frac{y_{12}}{\det \mathbf{y}} + \frac{\partial}{\partial y_{22}}\frac{y_{22}}{\det \mathbf{y}}\right).$$

Using the quotient rule, one computes easily that

$$-k\left(\frac{\partial}{\partial y_{11}}\frac{y_{11}}{\det \mathbf{y}} + \frac{\partial}{\partial y_{12}}\frac{y_{12}}{\det \mathbf{y}} + \frac{\partial}{\partial y_{22}}\frac{y_{22}}{\det \mathbf{y}}\right) = \frac{-k}{\det \mathbf{y}}.$$

□

Theorem 5.4.8. *For $g = 2$, the theta operator $\Theta_N : \omega_{N,\infty}^{\otimes k} \to \omega_{N,\infty}^{\otimes(k+2)}$ is given by the formula*

$$\Theta(f\,(du_1 \wedge du_2)^{\otimes k}) = \left[8\frac{\partial^2 f}{\partial z_{11}\partial z_{22}} - 2\frac{\partial^2 f}{\partial z_{12}^2} - \frac{2(2k-1)i}{\det \mathbf{y}}\left(y_{11}\frac{\partial f}{\partial z_{11}} + y_{12}\frac{\partial f}{\partial z_{12}} + y_{22}\frac{\partial f}{\partial z_{22}}\right) \\ - \frac{k(2k-1)}{\det \mathbf{y}} f\right](du_1 \wedge du_2)^{\otimes(k+2)}.$$

Proof. With F, G and H as above, it's easy to see that

$$y_{11}F + y_{12}G + y_{22}H = y_{11}\frac{\partial f}{\partial z_{11}} + y_{12}\frac{\partial f}{\partial z_{12}} + y_{22}\frac{\partial f}{\partial z_{22}} - ki\,f.$$

Combining this with the previous lemma and Lemma 5.4.6 establishes the result.

□

This is the formula for Maass' operator δ_2; see [21, Section 19].

Index of notation

Here is a table of the most important notation occurring in this thesis, roughly organised by chapter. In this table, X denotes a complex manifold, and X/S denotes a family of complex manifolds.

Chapter 2	
$\pi_1(X, x)$	the fundamental group of X at a point $x \in X$
$H^k(X; \mathbb{C})$	the kth singular cohomology group with coefficients in $\mathbb{C}$
$H^k(X, \mathcal{F})$	the kth cohomology group with coefficients in an abelian sheaf $\mathcal{F}$
$\mathcal{H}^k(X/S)$	the kth relative cohomology sheaf of X/S
Chapter 3	
$\mathcal{O}_X$	the sheaf of holomorphic functions on X
$\mathcal{C}_X^\infty$	the sheaf of smooth complex-valued functions on X
$T_{X,\mathbb{R}}$	the tangent bundle of X, considered as a smooth manifold
$T_{X,\mathbb{C}}$	the complexified tangent bundle of X
T_X	the holomorphic tangent bundle of X
$T_{X,\mathbb{C}}^\vee$	the complexified cotangent bundle of X
$T_X^\vee$	the holomorphic cotangent bundle of X
$\mathcal{A}_X^k$	the sheaf of smooth complex-valued k-forms on X
$\mathcal{A}_X^{p,q}$	the subsheaf of $\mathcal{A}_X^{p+q}$ consisting of forms of type (p, q)
Ω_X^p	the sheaf of holomorphic p-forms on X
$A^k(X)$	$\Gamma(X, \mathcal{A}_X^k)$
d	the exterior derivative of the de Rham complex
$\partial, \bar{\partial}$	the holomorphic and anti-holomorphic components of d
$H_{dR}^k(X)$	the kth de Rham cohomology group of X (with complex coefficients)
$T_{X/S,\mathbb{R}}$	the relative tangent bundle of X/S, considered as a smooth family
$T_{X/S,\mathbb{C}}$	the complexified relative tangent bundle of X/S
$T_{X/S}$	the holomorphic relative tangent bundle of X/S
$T_{X/S,\mathbb{C}}^\vee$	the complexified relative cotangent bundle of X/S
$T_{X/S}^\vee$	the holomorphic relative cotangent bundle of X/S
$\mathcal{A}_{X/S}^k$	the sheaf of C^∞ complex-valued relative k-forms on X/S
$\mathcal{A}_{X/S}^{p,q}$	the subsheaf of $\mathcal{A}_{X/S}^{p+q}$ consisting of forms of type (p, q)
$(\mathcal{A}_{X/S}^\bullet, d_{X/S})$	the smooth relative de Rham complex
$\Omega_{X/S}^p$	the sheaf of holomorphic relative p-forms on X/S
$(\Omega_{X/S}^\bullet, \partial_{X/S})$	the holomorphic relative de Rham complex
$\mathcal{H}_{dR}^k(X/S)$	the kth relative de Rham cohomology sheaf of X/S

$\mathcal{H}^{1,0}$, $\mathcal{H}^{0,1}$	holomorphic and anti-holomorphic summands of $\mathcal{H}^1_{dR}(X/S)$
ι	the de Rham isomorphism $H^1_{dR}(X) \xrightarrow{\sim} H^1(X;\mathbb{C})$

Chapter 4

$Sp_{2g}(\mathbb{R})$	the real symplectic group
$Sp_{2g}(\mathbb{Z})$	the Siegel modular group
$\Gamma_g(N)$	the level N principal congruence subgroup of $Sp_{2g}(\mathbb{Z})$
$\mathfrak{S}_g$	the Siegel upper half space
$\mathfrak{S}_{g,N}$	the Siegel modular variety of level N
A_{univ}	a universal family of abelian varieties over $\mathfrak{S}_g$
$A_{univ,N}$	a universal family of abelian varieties over $\mathfrak{S}_{g,N}$
ε	the zero section of $\pi : A_{univ} \to \mathfrak{S}_g$
ε_N	the zero section of $\pi_N : A_{univ,N} \to \mathfrak{S}_{g,N}$
ω	the determinant of $\pi_* \Omega_{A_{univ}/\mathfrak{S}_g}$
ω_N	the determinant of $\pi_{N*} \Omega_{A_{univ,N}/\mathfrak{S}_{g,N}}$
ω_∞	$\mathcal{C}^\infty_{\mathfrak{S}_g} \otimes_{\mathcal{O}_{\mathfrak{S}_g}} \omega$
$\omega_{N,\infty}$	$\mathcal{C}^\infty_{\mathfrak{S}_{g,N}} \otimes_{\mathcal{O}_{\mathfrak{S}_{g,N}}} \omega_N$

Chapter 5

std	the standard representation of $GL_g(\mathbb{C})$ on $\mathbb{C}^g$
Sym^2	the symmetric square of the standard representation
det	the determinant representation of $GL_g(\mathbb{C})$ on $\mathbb{C}^g$
sgn	the sign of a permutation
∇	the Gauss-Manin connection
∇_∞	the C^∞ Gauss-Manin connection
$\kappa^{j,k}$	the representation $(Sym^2 \, std)^{\otimes j} \otimes (\det std)^{\otimes k}$
$\mathcal{H}^1_\infty(X/S) = \mathcal{H}^1_\infty$	either $\mathcal{C}^\infty_S \otimes_{\mathcal{O}_S} \mathcal{H}^1(X/S)$ or $\mathcal{C}^\infty_S \otimes_{\mathcal{O}_S} \mathcal{H}^1_{dR}(X/S)$
$\mathcal{H}^{1,0}_\infty$	$\mathcal{C}^\infty_S \otimes_{\mathcal{O}_S} \mathcal{H}^{1,0}$
$(\mathcal{H}^1_\infty)^\kappa$, $(\mathcal{H}^{1,0}_\infty)^\kappa$	twists of $\mathcal{H}^1_\infty$, $\mathcal{H}^{1,0}_\infty$ by $\kappa = \kappa^{j,k}$
∇^κ_∞	the C^∞ Gauss-Manin connection twisted by $\kappa = \kappa^{j,k}$
$\mathbb{E}$	the Hodge sheaf $\pi_* \Omega_{A_{univ}/\mathfrak{S}_g}$
$\mathbb{E}_N$	the Hodge sheaf $\pi_{N*} \Omega_{A_{univ,N}/\mathfrak{S}_{g,N}}$
$\mathbb{E}_\infty$	$\mathcal{C}^\infty_{\mathfrak{S}_g} \otimes_{\mathcal{O}_{\mathfrak{S}_g}} \mathbb{E}$
$\mathbb{E}_{N,\infty}$	$\mathcal{C}^\infty_{\mathfrak{S}_{g,N}} \otimes_{\mathcal{O}_{\mathfrak{S}_{g,N}}} \mathbb{E}_N$
$\mathbb{E}^\kappa_\infty$, $(\mathbb{E}_{N,\infty})^\kappa$	twists of $\mathbb{E}_\infty$, $\mathbb{E}_{N,\infty}$ by $\kappa = \kappa^{j,k}$
KS	the Kodaira-Spencer isomorphism
Θ_N	the C^∞ theta operator $\omega^{\otimes k}_{N,\infty} \to \omega^{\otimes(k+2)}_{N,\infty}$
Θ	the canonical lift of Θ_N to $\omega^{\otimes k}_\infty$
$\vartheta^{j,k}_N$	the jth component of Θ_N
$\vartheta^{j,k}$	the jth component of Θ
ϑ	the untwisted theta operator $\mathbb{E}_\infty \to Sym^2 \, \mathbb{E}_\infty \otimes \mathbb{E}_\infty$
proj	the projection $(Sym^2 \, \mathbb{E}_{N,\infty})^{\otimes g} \to \omega^{\otimes 2}_{N,\infty}$, or its lift to $(Sym^2 \, \mathbb{E}_\infty)^{\otimes g}$

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