Assignment 1

Experimental Mathematics 2020

Due 7 May 2020 by 5pm*

Exercise 1. Please resist the urge to Google this one, you'll spoil your fun.

You read in the paper that a baseball player's batting average is .334. What is the smallest number of times the player must have been at bat? What is the second smallest number of times the player must have been at bat?

(For those of you not into baseball, the batting average of a player is the rational number $0 \leq h/b \leq 1$ rounded to three decimal places, where h is the number of hits and b is the number of battings.)

Exercise 2. Consider the numbers

$$\alpha_j = \sum_{k=0}^{\infty} \frac{1}{16^k(8k+j)}$$

for $j = 1, 2, \ldots, 7$.

Find an integer relation involving the α_j's and π.

Exercise 3. Given real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, a *multiplicative relation* is an equality of the form

$$\alpha_1^{m_1} \alpha_2^{m_2} \ldots \alpha_n^{m_n} = 1$$

with $m_1, m_2, \ldots, m_n \in \mathbb{Z}$.

Explain how you can reduce the question of finding multiplicative relations to solving the integer relation problem.

Use your method to find a multiplicative relation involving the numbers

$$2, 3, 5, 7, 11, 13, 17, \pi, \zeta(14).$$

Restate the relation in a form conducive to the use of continued fractions, and use the continued fraction method to obtain further evidence.

Exercise 4. Using Mathematica, write a function `IntRel` that does the same thing as the Sage function `intrel` from the 3 April computer lab. The main ingredient is the Mathematica function `LatticeReduce`.

Run your function on Example 3.2 from the lecture notes.

See also the builtin Mathematica function `FindIntegerNullVector`.

*I will give instructions on how to submit your solutions soon. For now, start playing with it and keep all your working.

Exercise 5. Lattice reduction isn't the only integer relation algorithm in town. There are other contenders, such as the PSLQ algorithm.

An implementation is present in `mpmath`, a library that is accessible in Sage. Read the documentation for its function `pslq` at

http://mpmath.org/doc/1.1.0/identification.html?highlight=pslqn#mpmath.pslq

Adapt its examples to find all the Machin-like formulas of the type

$$\pi = c_1 \operatorname{arccot}(a_1) + c_2 \operatorname{arccot}(a_2)$$

with $2 \le a_1 \le a_2 \le 1000$ and $c_1, c_2 \in \mathbb{Q}$.

Exercise 6. For $n \in \mathbb{N}$, let α_n be the number defined by

$$\alpha_n = 1 + \sqrt{1 + \sqrt{1 + \ldots \sqrt{2}}},$$

where the expression contains n ones and n square roots.

(So $\alpha_1 = 1 + \sqrt{2}$, $\alpha_2 = 1 + \sqrt{1 + \sqrt{2}}$, $\alpha_3 = 1 + \sqrt{1 + \sqrt{1 + \sqrt{2}}}$, etc.)

Tell me as much as you can figure out about the minimal polynomial p_n of α_n.