COPY

The University of Melbourne

Department of Mathematics and Statistics

Semester Two Assessment 2010

620–329 METRIC AND HILBERT SPACES

Examination duration:	Three hours
Reading time allowed:	Fifteen minutes
Number of pages:	3 (including this page)
Common Content:	No common content

Authorized Materials: No materials are authorized.

Instructions to Invigilators: One 14 page script book is to be given to each student initially. No written or printed material related to the subject may be brought into the examination. No mathematical tables or calculators may be used.

Instructions to Students:

- This paper has **8 questions**. You should attempt *all* problems.
- Aim for clear, concise and complete answers.
- Write all your solutions in the booklets provided.
- Number the questions clearly, and start each question on a new page.
- Use the *left* pages for rough working. Write material you wish to be marked on *right* pages only.
- All problems are of *equal* value.

This paper may be held in the Baillieu library.

Question 1.

(a) State the definitions of a *Cauchy sequence* and a *complete* metric space.
(b) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be continuous with $f(X) = Y$. Assume that (X, d_X) is complete and for all $x, x' \in X$

$$d_X(x, x') \leq d_Y(f(x), f(x')).$$

GIve a brief proof that (Y, d_Y) is complete.

Question 2.

Let (X, d_X) and (Y, d_Y) be metric spaces, and let $\{f_n\}$ be a sequence of continuous functions $f_n : X \to Y$.

(a) Give the definition of *uniform convergence* of the sequence $\{f_n\}$ to a function $f : X \to Y$.

(b) Define $f_n : [0, 1] \to \mathbb{R}$ for each $n \in \mathbb{N}$ by

$$f_n(x) = \frac{x^2}{1 + nx} \text{ for } x \in [0, 1].$$

Find the pointwise limit f of the sequence $\{f_n\}$ and determine whether the sequence converges uniformly to f. Give brief justifications for your answers.

Question 3.

Let (X, d) be a metric space.
(a) Prove from the axioms for a metric, that if $x, x', y, y' \in X$ then

$$|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y').$$

(b) Let A be a non-empty compact subset of X. Prove that there exist points $a, b \in A$ such that

$$d(a, b) = \sup\{d(x, y) : x, y \in A\}.$$

Question 4.

Let l^2 denote the Hilbert space of square summable complex sequences $(a_1, a_2, \ldots)$ with inner product

$$< (a_1, a_2, \ldots), (b_1, b_2, \ldots) >= \sum_n a_n \overline{b_n}.$$

Let $\{\lambda_1, \lambda_2, \ldots\}$ be a bounded sequence of complex numbers.
Define $T : l^2 \to l^2$ by $T(a_1, a_2, \ldots) = (\lambda_1 a_2, \lambda_2 a_4, , \ldots, \lambda_n a_{2n}, \ldots)$
(a) Show that T is a bounded linear operator.
(b) Compute $\|T\|$.
(c) Find the adjoint operator T^*.

Question 5.

(a) Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and let $f : X \to Y$ be a continuous map. Prove that if X is connected then so is $f(X)$.

(b) Consider the following pairs of spaces. Are these spaces homeomorphic or not? Give brief explanations.

(1) The circle $\{(x, y) : x^2 + y^2 = 1\}$ and the interval $[0, 1]$ with the topologies induced from $\mathbb{R}^2, \mathbb{R}$ respectively;

(2) The intervals $[0, 1]$ and $(0, 1)$ with the topology induced from $\mathbb{R}$.

(3) The intervals $[0, 1]$ and $[0, 2]$ with the topology induced from $\mathbb{R}$.

Question 6.

Suppose that $(H, < \cdot >)$ is a real Hilbert space and fix an element $v \in H$.

(a) Prove that the functional $f : H \to \mathbb{R}$ given by $f(x) =< x, v >$ is a bounded linear operator. Compute $\|f\|$ for this functional.

(b) State the Riesz representation theorem and use it to prove that the dual space H' is isometric to H, viewed as a Banach space.

Question 7.

(a) Give the definition of a compact self adjoint linear operator $T : V \to W$ where V, W are Hilbert spaces.

(b) Prove that any eigenspace of a bounded linear operator is a closed subspace.

(c) Prove that any sequence of different non-zero eigenvalues of a compact self adjoint operator must have a subsequence converging to zero.

Question 8.

(a) State the Banach fixed point theorem.

(b) Verify that the mapping

$$f(x) = -\frac{1}{12}x^3 + x + \frac{1}{4}$$

satisfies the conditions of the Banach fixed point theorem on the metric space $([1, 2], d)$, where d is the usual Euclidean metric.

(c) Find directly the unique fixed point for f.

End of the examination paper