THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assessment, 2022

School of Mathematics and Statistics

MAST30026 Metric and Hilbert Spaces

Reading time: 15 minutes — Writing time: 3 hours

This exam consists of 2 pages (including this page)

Permitted Materials

- No books, notes or other printed or handwritten material are permitted.
- No calculators are permitted.
- Any mobile phones or internet-enabled devices brought into the exam room must be **turned off** and placed on the floor under your table.

Instructions to Students

- The marking scheme will include marks allocated/deducted for mathematical exposition. Which deifinitions/facts can be used without statement/justification depends on the context of the question. Make your exposition decisions wisely based on what you have learned about quality mathematical writing over the course of this class.
- Write your answers in the booklet or booklets provided. Start each question on a new page. Include the question number at the top of each page. Write on the lined pages only.
- You must NOT remove this question paper, or any booklets provided to you, at the conclusion of the examination.

Instructions to Invigilators

- Supply answer booklets to students.
- Collect this exam paper, and all answer booklets, at the conclusion of the examination.
- This is writing intensive exam. Supply TWO answer booklets initially and additional answer booklets as needed.

Question 1 Let (X, d_X) be a metric space and let $(a_1, a_2, \ldots)$ be a sequence in X. Prove that if $(a_1, a_2, \ldots)$ is convergent then $(a_1, a_2, \ldots)$ is Cauchy. (Be sure to carefully state the definitions!)

Question 2

(a) Let $a, b \in \mathbb{R}$ with $a < b$. Let $C([a, b])$ be the Banach space of continuous functions $f\colon [a, b] \to \mathbb{R}$ with the supremum norm. Let $t_0 \in [a, b]$. Define $A\colon C([a, b]) \to \mathbb{R}$ by

$$Af = f(t_0).$$

Show that A is a bounded linear functional with $\|A\| = 1$.

(b) Let $V = C([0, 1])$ be the vector space of continuous functions $f\colon [0, 1] \to \mathbb{R}$ with norm given by

$$\|f\| = \int_0^1 |f(t)|dt.$$

Let $T\colon V \to \mathbb{R}$ be given by $T(f) = f(0)$. Show that V is infinite dimensional and that T is not bounded.

Question 3 Let (X, d_X) and (Y, d_Y) be metric spaces and let $f\colon X \to Y$ be a continuous function. Show that if X is compact then $f\colon X \to Y$ is uniformly continuous.

Question 4 Prove that $\mathbb{C}$ with the standard metric is a metric space. (Be sure to carefully state the definitions!)

Question 5 Carefully state and prove the Reisz representation theorem (the theorem which establishes how a Hilbert space is "the same" as its dual).

Question 6 A topological space $(X, \mathcal{T}_X)$ is *first countable* if it satisfies:

If $a \in X$ then there exist $N_1, N_2, \ldots$ in $\mathcal{N}(a)$ such that
if $N \in \mathcal{N}(a)$ then there exists $r \in \mathbb{Z}_{>0}$ with $N \supseteq N_r$.

(In English: A countable number of neighborhoods of each point is enough to determine the topology. The notation $\mathcal{N}(a)$ indicates the collection of neighborhoods of a.)

Let

$$\mathcal{T}_{\mathbb{R}} = \{U \subseteq \mathbb{R} \mid U^c \text{ is finite}\} \cup \{\emptyset, \mathbb{R}\}.$$

Show that $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ is not a first countable topological space.

Question 7 Let $(V, \|\ \|)$ be a normed vector space and let $d\colon V \times V \to \mathbb{R}_{\geq 0}$ be the metric on V given by $d(x, y) = \|x - y\|$. Prove that V is a Banach space if and only if V satisfies

$$\begin{gathered}\text{If } (a_1, a_2, \ldots) \text{ is a sequence in } V \text{ and } \sum_{i \in \mathbb{Z}_{>0}} \|a_i\| \text{ converges} \\ \text{then } \sum_{i \in \mathbb{Z}_{>0}} a_i \text{ converges.}\end{gathered} \qquad (*)$$

End of Exam