THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assessment, 2023

School of Mathematics and Statistics

MAST30026 Metric and Hilbert Spaces

Reading time: 15 minutes — Writing time: 3 hours

This exam consists of 42 pages (including this page) with 9 questions and 80 total marks

Permitted Materials

- No books, notes or other printed or handwritten material are permitted.
- No calculators are permitted.
- Any mobile phones or internet-enabled devices brought into the exam room must be **turned off** and placed on the floor under your table.

Instructions to Students

- Write your answers in the boxes provided on the exam. There is extra space you can use for answers to any question commencing on page 39. If you still do not have enough space, tick the box near the bottom of page 42 and request a booklet from an invigilator—include the question number at the top of each page in the booklet.
- You must NOT remove this question paper, or any booklets provided to you, at the conclusion of the examination.

Instructions to Invigilators

- Students are to write their answers on the paper. They may request a booklet if they run out of space on the exam paper.
- This exam paper contains examinable material and must be collected, together with answer booklets if any, at the conclusion of the examination.

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Question 1 (8 marks)
Let (X, d) be a metric space.

(a) Define the concept "D is a dense subset of X".

(b) Show that $D \subseteq X$ is a dense subset of X if and only if $D \cap U \neq \emptyset$ for all nonempty open sets U in X.

(c) Prove that the intersection of two dense open sets U_1 and U_2 is dense.

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Question 2 (10 marks)
Let (X, d) be a metric space.

(a) Define the concept "D is a disconnected subset of X".

(b) Prove that a subset D of X is disconnected if and only if there exists a surjective continuous function $g:\ D \longrightarrow \{0,1\}$, where $\{0,1\}$ is given the discrete metric.

(c) Suppose $A \subseteq X$ is a connected subset and $\{C_i :\ i \in I\}$ is an arbitrary collection of connected subsets of X such that $A \cap C_i \neq \emptyset$ for all $i \in I$. Prove that

$$B := A \cup \bigcup_{i \in I} C_i$$

is a connected subset of X.

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Question 3 (9 marks)
Let (X, d) be a metric space.

(a) Define the concept "K is a compact subset of X".

(b) Let C be a closed subset of a compact subset K of X. Prove that C is compact.

(c) Let K and L be compact subsets of X. Prove that $K \cup L$ is compact.

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Question 4 (9 marks)
Consider the equation

$$x^3 - x - 1 = 0. \tag{1}$$

(a) Show that Equation (1) must have **at least one solution** in the interval $[1, 2]$.

(b) Show that the function $f : [1, 2] \longrightarrow [1, 2]$ given by

$$f(x) = (1 + x)^{1/3}$$

is a contraction.

(c) Show that Equation (1) has **a unique solution** ξ in the interval $[1, 2]$ and describe a sequence of real numbers that converges to ξ.

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Question 5 (7 marks)

(a) Let $f \in L(V, W)$ be a continuous linear map between normed spaces. Prove that if U is a closed subspace of W, then its preimage $f^{-1}(U)$ is a closed subspace of V.

(b) Prove that the following set of sequences

$$S = \left\{ (a_n) \in \ell^1 : \sum_{n=1}^{\infty} a_n = 0 \right\}$$

is a closed subspace of the Banach space ℓ^1:

$$\ell^1 = \left\{ (a_n) \in \mathbb{F}^{\mathbb{N}} : \sum_{n=1}^{\infty} |a_n| < \infty \right\}.$$

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Question 6 (9 marks)
Let $\big(V, \langle\cdot,\cdot\rangle\big)$ be an inner product space.

(a) Given a subset S of V, define the concept "the orthogonal complement $S^\perp$ of S".

(b) Prove that $S \subseteq \big(S^\perp\big)^\perp$.

(c) Prove that if V is a Hilbert space and W is a closed subspace of V, then $\big(W^\perp\big)^\perp = W$.

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Question 7 (9 marks)

(a) State the Cauchy–Schwarz Inequality for inner product spaces.

(b) Let V be an inner product space. Prove that for any $u \in V$ we have

$$\|u\| = \sup_{\|v\|=1} \big|\langle u, v\rangle\big|.$$

(c) Now let W be a second inner product space and let $f \in L(V, W)$ be a continuous linear map. Prove that

$$\|f\| = \sup_{\|v\|_V=1=\|w\|_W} \big|\langle f(v), w\rangle_W\big|.$$

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Question 8 (7 marks)
Consider the function $g: \ell^2 \longrightarrow \mathbb{F}$ given by

$$g(x) = \sum_{n=1}^{\infty} \frac{x_n}{n^2}.$$

(a) Find $y \in \ell^2$ such that

$$g(x) = \langle x, y \rangle \qquad \text{for all } x \in \ell^2.$$

(b) Deduce that g is linear and continuous and find its norm $\|g\|$.

[*Hint*: You may use without proof the fact that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.]

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Question 9 (12 marks)
Let (a_n) be a decreasing sequence of non-negative real numbers. Consider $f:\ \ell^2 \longrightarrow \mathbb{F}^{\mathbb{N}}$ given by
$$f(x) = \bigl(a_1x_1, a_2x_2, \dots, a_nx_n, \dots\bigr).$$

(a) Prove that the image of f is contained in ℓ^2 and that $f:\ \ell^2 \longrightarrow \ell^2$ is linear and continuous.

(b) Find the norm $\|f\|$.

(c) Find the adjoint f^* of f.

(d) How much can you relax the conditions on the sequence (a_n) and still retain the statement in part (a)? Make an educated guess and describe briefly how/if the answers to parts (b) and (c) change.

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End of Exam — Total Available Marks = 80

You must tick this box if you have used extra booklets ☐