THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assessment, 2023

School of Mathematics and Statistics

MAST30026 Metric and Hilbert Spaces

Reading time: 15 minutes — Writing time: 3 hours

This exam consists of 42 pages (including this page) with 9 questions and 80 total marks

Permitted Materials

- No books, notes or other printed or handwritten material are permitted.
- No calculators are permitted.
- Any mobile phones or internet-enabled devices brought into the exam room must be **turned off** and placed on the floor under your table.

Instructions to Students

- Write your answers in the boxes provided on the exam. There is extra space you can use for answers to any question commencing on page 39. If you still do not have enough space, tick the box near the bottom of page 42 and request a booklet from an invigilator—include the question number at the top of each page in the booklet.
- You must NOT remove this question paper, or any booklets provided to you, at the conclusion of the examination.

Instructions to Invigilators

- Students are to write their answers on the paper. They may request a booklet if they run out of space on the exam paper.
- This exam paper contains examinable material and must be collected, together with answer booklets if any, at the conclusion of the examination.

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Question 1 (8 marks)

Let (X, d) be a metric space.

(a) Define the concept "D is a dense subset of X".

(b) Show that $D \subseteq X$ is a dense subset of X if and only if $D \cap U \neq \emptyset$ for all nonempty open sets U in X.

(c) Prove that the intersection of two dense open sets U_1 and U_2 is dense.

(a) **(1 marks)** $X = \overline{D}$.

(b) **(2 marks)** Suppose D is dense and let U be nonempty open. Let $x \in U$. As U is open, there exists $\mathbb{B}_r(x) \subseteq U$ with $r > 0$. If $x \in D$, we are done. Otherwise, $x \in X \setminus D = \overline{D} \setminus D$, so it is a limit point of D, so there exists $a \in \mathbb{B}_r(x) \cap D$ such that $a \neq x$, hence $a \in U \cap D$.

(2 marks) Conversely, suppose $D \cap U$ is nonempty for any nonempty open U. Let $x \in X \setminus D$. For every $r > 0$, $U := \mathbb{B}_r(x)$ is open so $D \cap \mathbb{B}_r(x)$ is nonempty, and x is not in this intersection so there must be a point distinct from x in it, hence $x \in \overline{D}$.

(c) **(3 marks)** Let $U_{12} = U_1 \cap U_2$.

To show that U_{12} is dense, we use the previous part and show that $U_{12} \cap U \neq \emptyset$ for all nonempty open U:

$$U_{12} \cap U = \big(U_1 \cap U_2\big) \cap U = U_1 \cap \big(U_2 \cap U\big).$$

Since U_2 is dense and open, $U_2 \cap U$ is nonempty and open. Since U_1 is dense, $U_1 \cap \big(U_2 \cap U\big)$ is nonempty. So $U_{12} \cap U \neq \emptyset$, hence U_{12} is dense.

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Question 2 (10 marks)
Let (X, d) be a metric space.

(a) Define the concept "D is a disconnected subset of X".

(b) Prove that a subset D of X is disconnected if and only if there exists a surjective continuous function $g : D \longrightarrow \{0, 1\}$, where $\{0, 1\}$ is given the discrete metric.

(c) Suppose $A \subseteq X$ is a connected subset and $\{C_i : i \in I\}$ is an arbitrary collection of connected subsets of X such that $A \cap C_i \neq \emptyset$ for all $i \in I$. Prove that

$$B := A \cup \bigcup_{i \in I} C_i$$

is a connected subset of X.

(a) **(2 marks)** There exist open subsets U and V of D such that

$$D = U \cup V, \qquad U \cap V = \emptyset, \qquad U \neq \emptyset, \qquad V \neq \emptyset.$$

(b) **(2 marks)** If such a function g exists, let $U = g^{-1}(0)$ and $V = g^{-1}(1)$, then $U \neq \emptyset$, $V \neq \emptyset$ since g is surjective. As $\{0\} \cap \{1\} = \emptyset$, we have $U \cap V = \emptyset$. Clearly $D = U \cup V$, and both U and V are open since $\{0\}$ and $\{1\}$ are open. This implies that D is disconnected.

(2 marks) For the other direction, suppose that D is disconnected and write $D = U \cup V$ with U, V as in the definition. Define $g : X \longrightarrow \{0, 1\}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in V. \end{cases}$$

This is well-defined since $U \cap V = \emptyset$. It is continuous as $g^{-1}(0) = U$ and $g^{-1}(1) = V$ are open. It is surjective since it takes both values 0 and 1 (as both U and V are nonempty).

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(c) **(4 marks)** Let $g: B \longrightarrow \{0,1\}$ be an arbitrary continuous function.

Its restriction $g|_A: A \longrightarrow \{0,1\}$ cannot be surjective, since A is connected. So $g|_A$ is constant, let's say 0 for concreteness.

Now let $i \in I$. The restriction $g|_{C_i}: C_i \longrightarrow \{0,1\}$ must be constant, for the same reason as before. But $A \cap C_i \neq \emptyset$ and g is zero on A, so g must be zero on C_i.

As this holds for all $i \in I$, we conclude that g is zero on B.

So there is no surjective continuous map $B \longrightarrow \{0,1\}$, hence B must be connected.

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Question 3 (9 marks)
Let (X, d) be a metric space.

(a) Define the concept "K is a compact subset of X".

(b) Let C be a closed subset of a compact subset K of X. Prove that C is compact.

(c) Let K and L be compact subsets of X. Prove that $K \cup L$ is compact.

(a) **(2 marks)** Given any open cover of K:

$$K \subseteq \bigcup_{i \in I} U_i,$$

there exists a finite subset $\{i_1, \ldots, i_n\} \subseteq I$ such that

$$K \subseteq \bigcup_{j=1}^{n} U_{i_j}.$$

(b) **(4 marks)** Consider an arbitrary open cover of C:

$$C \subseteq \bigcup_{i \in I} U_i.$$

Then we have

$$K \subseteq X = C \cup (X \setminus C) \subseteq \left(\bigcup_{i \in I} U_i \right) \cup (X \setminus C),$$

which is an open cover of K. As K is compact, there is a finite subcover, so that

$$K \subseteq \left(\bigcup_{n=1}^{N} U_{i_n} \right) \cup (X \setminus C), \qquad i_n \in I,$$

hence

$$C \subseteq \bigcup_{n=1}^{N} U_{i_n}.$$

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(c) **(3 marks)** Consider an arbitrary open cover of $K \cup L$:

$$K \cup L \subseteq \bigcup_{i \in I} U_i.$$

This is also an open cover of K, so there is a finite subcover that still covers K:

$$K \subseteq \bigcup_{n=1}^{N} U_{i_n}, \qquad i_n \in I.$$

Similarly, we get a finite subcover that covers L:

$$L \subseteq \bigcup_{m=1}^{M} U_{j_m}, \qquad j_m \in I.$$

Letting $S = \{i_1, \ldots, i_N\} \cup \{j_1, \ldots, j_M\}$, we get a finite subcover that covers $K \cup L$:

$$K \cup L \subseteq \bigcup_{s \in S} U_s.$$

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Question 4 (9 marks)
Consider the equation

$$x^3 - x - 1 = 0. \tag{1}$$

(a) Show that Equation (1) must have **at least one solution** in the interval $[1, 2]$.

(b) Show that the function $f: [1, 2] \longrightarrow [1, 2]$ given by

$$f(x) = (1 + x)^{1/3}$$

is a contraction.

(c) Show that Equation (1) has **a unique solution** ξ in the interval $[1, 2]$ and describe a sequence of real numbers that converges to ξ.

(a) **(2 marks)** We can use the Intermediate Value Theorem, as $x^3 - x - 1$ is clearly continuous. At $x = 1$, $x^3 - x - 1 = -1 < 0$, while at $x = 2$, $x^3 - x - 1 = 5 > 0$, so there must be at least one point x in $[1, 2]$ such that $x^3 - x - 1 = 0$.

(b) **(4 marks)** The derivative of f is

$$f'(x) = \frac{1}{3}(1 + x)^{-2/3} = \frac{1}{3}\frac{1}{(1 + x)^{2/3}}.$$

As $x \in [1, 2]$, we have $f'(x) > 0$ and

$$1 \leqslant x \Rightarrow 2 \leqslant 1 + x \Rightarrow \frac{1}{1 + x} \leqslant \frac{1}{2} \Rightarrow \frac{1}{(1 + x)^{2/3}} \leqslant \frac{1}{2^{2/3}} \leqslant 1,$$

so that

$$f'(x) \leqslant \frac{1}{3}.$$

Now let x, y be such that $1 \leqslant x < y \leqslant 2$ and apply the Mean Value Theorem to f on $[x, y]$ to deduce that there exists $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \Rightarrow |f(y) - f(x)| = |f'(c)|\,|y - x| \leqslant \frac{1}{3}|y - x|.$$

We conclude that f is a contraction.

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(c) **(3 marks)** Observe that $x^3 - x - 1 = 0$ is equivalent to $f(x) = x$, so the solutions of Equation (1) are precisely the fixed points of f. As f is a contraction and $[1, 2]$ is complete, the Banach Fixed Point Theorem says that there is a unique fixed point ξ in $[1, 2]$. It also tells us that we can start with any $x_1 \in [1, 2]$, for instance $x_1 = 1$, and iteratively apply f to get a sequence (x_n) converging to ξ:

$$x_1 = 1, \qquad x_2 = f(x_1) = 2^{1/3}, \qquad x_3 = f(x_2) = \left(1 + 2^{1/3}\right)^{1/3}, \ldots$$

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Question 5 (7 marks)

(a) Let $f \in L(V, W)$ be a continuous linear map between normed spaces. Prove that if U is a closed subspace of W, then its preimage $f^{-1}(U)$ is a closed subspace of V.

(b) Prove that the following set of sequences

$$S = \left\{ (a_n) \in \ell^1 : \sum_{n=1}^{\infty} a_n = 0 \right\}$$

is a closed subspace of the Banach space ℓ^1:

$$\ell^1 = \left\{ (a_n) \in \mathbb{F}^{\mathbb{N}} : \sum_{n=1}^{\infty} |a_n| < \infty \right\}.$$

(a) **(2 marks)** Clear since f is linear so the inverse image of a subspace is a subspace; and f is continuous so the inverse image of a closed set is a closed set.

(b) **(5 marks)** Consider the function $f : \ell^1 \longrightarrow \mathbb{F}$ given by

$$f\big((a_n)\big) = \sum_{n=1}^{\infty} a_n.$$

First note that this is a reasonable definition, because the infinite series on the right hand side converges in $\mathbb{F}$:

$$\left| \sum_{n=1}^{N} a_n \right| \leqslant \sum_{n=1}^{N} |a_n|,$$

and the latter converges as $N \longrightarrow \infty$ since $(a_n) \in \ell^1$.

The function f is linear. It is also continuous, because as we have just seen:

$$\big|f\big((a_n)\big)\big| = \left| \sum_{n=1}^{\infty} a_n \right| \leqslant \sum_{n=1}^{\infty} |a_n| = \|(a_n)\|_{\ell^1}.$$

Hence $f \in L(\ell^1, \mathbb{F}) = (\ell^1)^\vee$ and its kernel is S, so S is a closed subspace of ℓ^1.

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Question 6 (9 marks)
Let $\big(V, \langle \cdot, \cdot \rangle\big)$ be an inner product space.

(a) Given a subset S of V, define the concept "the orthogonal complement $S^\perp$ of S".

(b) Prove that $S \subseteq \big(S^\perp\big)^\perp$.

(c) Prove that if V is a Hilbert space and W is a closed subspace of V, then $\big(W^\perp\big)^\perp = W$.

(a) **(2 marks)** $S^\perp = \{v \in V : \langle v, s \rangle = 0 \text{ for all } s \in S\}$.

(b) **(3 marks)** Let $s \in S$. For any $x \in S^\perp$, we have

$$\langle s, x \rangle = \overline{\langle x, s \rangle} = 0,$$

so $s \in \big(S^\perp\big)^\perp$.

(c) **(4 marks)** We have seen above that $W \subseteq \big(W^\perp\big)^\perp$.

Let $x \in \big(W^\perp\big)^\perp$. By the Hilbert Projection Theorem, we can decompose

$$H = W \oplus W^\perp.$$

So we have $x = y + z$ with $y \in W$ and $z \in W^\perp$. Then

$$0 = \langle x, z \rangle = \langle y + z, z \rangle = \langle y, z \rangle + \langle z, z \rangle = 0 + \|z\|^2,$$

implying that $z = 0$ and $x = y \in W$.

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Question 7 (9 marks)

(a) State the Cauchy–Schwarz Inequality for inner product spaces.

(b) Let V be an inner product space. Prove that for any $u \in V$ we have

$$\|u\| = \sup_{\|v\|=1} \big|\langle u, v\rangle\big|.$$

(c) Now let W be a second inner product space and let $f \in L(V, W)$ be a continuous linear map. Prove that

$$\|f\| = \sup_{\|v\|_V=1=\|w\|_W} \big|\langle f(v), w\rangle_W\big|.$$

(a) **(2 marks)** For any u, v in an inner product space V we have

$$\big|\langle u, v\rangle\big| \leqslant \|u\|\,\|v\|.$$

Equality holds if and only if u and v are linearly dependent.

(b) **(3 marks)** If $u = 0$ then the equality is obvious.

So assume now that $u \neq 0$. Applying Cauchy–Schwarz with $v \in V$ such that $\|v\| = 1$, we have

$$\big|\langle u, v\rangle\big| \leqslant \|u\|,$$

so that

$$\sup_{\|v\|=1} \big|\langle u, v\rangle\big| \leqslant \|u\|.$$

To get equality, take $v = \frac{1}{\|u\|} u$ and see that the LHS is indeed $\|u\|$.

(c) **(4 marks)** From the previous part:

$$\|u\|_W = \sup_{\|w\|_W=1} \big|\langle u, w\rangle_W\big| \qquad \text{for all } u \in W.$$

Setting $u = f(v)$ for some $v \in V$, we get

$$\|f(v)\|_W = \sup_{\|w\|_W=1} \big|\langle f(v), w\rangle_W\big| \qquad \text{for all } v \in V.$$

Therefore

$$\|f\| = \sup_{\|v\|_V=1} \|f(v)\|_W = \sup_{\|v\|_V=1=\|w\|_W} \big|\langle f(v), w\rangle_W\big|.$$

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Question 8 (7 marks)
Consider the function $g : \ell^2 \longrightarrow \mathbb{F}$ given by

$$g(x) = \sum_{n=1}^{\infty} \frac{x_n}{n^2}.$$

(a) Find $y \in \ell^2$ such that

$$g(x) = \langle x, y \rangle \qquad \text{for all } x \in \ell^2.$$

(b) Deduce that g is linear and continuous and find its norm $\|g\|$.

[*Hint*: You may use without proof the fact that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.]

(a) **(4 marks)** Setting $y = (y_n)$ with

$$y_n = \frac{1}{n^2},$$

we certainly have for all $x = (x_n) \in \ell^2$:

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y}_n = \sum_{n=1}^{\infty} \frac{x_n}{n^2} = g(x).$$

We should check that $y \in \ell^2$:

$$\|y\|_{\ell^2}^2 = \sum_{n=1}^{\infty} y_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(b) **(3 marks)** From the previous part we know that $g = y^{\vee}$, so certainly g is linear and continuous. We also have

$$\|g\| = \|y^{\vee}\| = \|y\|_{\ell^2} = \frac{\pi^2}{3\sqrt{10}},$$

as we have seen in the previous part.

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Question 9 (12 marks)

Let (a_n) be a decreasing sequence of non-negative real numbers. Consider $f: \ell^2 \longrightarrow \mathbb{F}^{\mathbb{N}}$ given by

$$f(x) = \big(a_1 x_1, a_2 x_2, \ldots, a_n x_n, \ldots\big).$$

(a) Prove that the image of f is contained in ℓ^2 and that $f: \ell^2 \longrightarrow \ell^2$ is linear and continuous.

(b) Find the norm $\|f\|$.

(c) Find the adjoint f^* of f.

(d) How much can you relax the conditions on the sequence (a_n) and still retain the statement in part (a)? Make an educated guess and describe briefly how/if the answers to parts (b) and (c) change.

(a) **(2 marks)** We have

$$\|f(x)\|_{\ell^2}^2 = \sum_{n=1}^{\infty} a_n^2 \, |x_n|^2 \leqslant a_1^2 \sum_{n=1}^{\infty} |x_n|^2 = a_1^2 \, \|x\|_{\ell^2}^2,$$

so if $x \in \ell^2$ then $f(x) \in \ell^2$.

It is straightforward that f is linear. It is clear that f is continuous from the inequality above.

(b) **(3 marks)** We have

$$\|f\| = \sup_{\|x\|=1} \big\|f(x)\big\| \leqslant a_1$$

from the previous part.

Taking $x = e_1 = (1, 0, 0, \ldots)$ we have $\|e_1\| = 1$ and $f(e_1) = (a_1, 0, 0, \ldots)$ so $\|f(e_1)\| = a_1$, therefore $\|f\| = a_1$.

(c) **(3 marks)** We have

$$\big\langle f(x), y \big\rangle = \sum_{n=1}^{\infty} a_n \, x_n \, \overline{y}_n = \sum_{n=1}^{\infty} x_n \, \overline{(a_n \, y_n)} = \big\langle x, f(y) \big\rangle,$$

where we used the fact that $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

Therefore $f^* = f$.

(d) **(4 marks)** We can take (a_n) to be any bounded sequence of complex numbers and (a) still holds. In (b) we get $\|f\| = \|(a_n)\|_{\ell^\infty}$, and in (c) we get

$$f^*(y) = \big(\overline{a}_1 y_1, \overline{a}_2 y_2, \ldots, \overline{a}_n y_n, \ldots\big).$$

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End of Exam — Total Available Marks = 80

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