MAST30026 Assignment 1

Due Wednesday 4 September 2024 at 20:00 on Canvas and Gradescope

Name:

Student ID:

Some guidelines:

- Please write clear and detailed solutions in the boxes following each question or part of question. This can be done by printing this document and physically writing in the boxes, or by opening a copy of this document on a tablet or other device.
- The boxes should typically provide sufficient space for your solution, but if you find you need extra space please take an empty sheet and continue your solution there, clearly indicating which question this refers to. Also indicate in the corresponding box that the solution continues at the end.
- There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you write in the box is a complete explanation.

  The quality of the exposition will be assessed alongside the correctness of the approach.
- For technical reasons (since you will be uploading your solutions to GradeScope), please write legibly with a very readable writing implement.
- Results from the lectures, tutorials, exercises can be used (without having to re-prove them); make sure you say clearly what result you are using, though.
- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).
- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.
- You may assume that not all questions will have the same weight in the assessment.
- No Chegg or anything similar. At all. Please.

This assignment consists of 7 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

1. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$ be a subset. Let $\iota\colon Y \longrightarrow X$ denote the inclusion map, that is $\iota(y) = y$ for all $y \in Y$.

   (a) Prove that the subspace topology $\mathcal{T}|_Y$ on Y is the coarsest topology for which ι is continuous.

   (b) Prove that, given any topological space Z and any function $g\colon Z \longrightarrow Y$, g is continuous with respect to $\mathcal{T}|_Y$ if and only if $\iota \circ g\colon Z \longrightarrow X$ is continuous.

   (c) Assume now that Y has the subspace topology. Complete the statement:
   "The map $\iota\colon Y \longrightarrow X$ is open if and only if Y ______________________________"
   Prove your statement.

$\longrightarrow$☐ **Yes,** I would like feedback (comments) on my solution to this question.

2. Let X_1 and X_2 be topological spaces, $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ subsets.

   We can construct two topologies on the set $Y_1 \times Y_2$:

   - $\mathcal{T}_{\text{prod}}$ = the product of the subspace topologies on Y_1 and Y_2;
   - $\mathcal{T}_{\text{sub}}$ = the subspace topology of the subset $Y_1 \times Y_2 \subseteq X_1 \times X_2$, where $X_1 \times X_2$ has the product topology.

   Is $\mathcal{T}_{\text{prod}} = \mathcal{T}_{\text{sub}}$? If yes, give a proof. If no, give a counterexample.

   $\longrightarrow$ □ **Yes,** I would like feedback (comments) on my solution to this question.

3. Let $f\colon X \longrightarrow Y$ be a continuous function between topological spaces and consider the *graph* $\Gamma(f)$ of f, which is defined as

$$\Gamma(f) = \{(x, y) \in X \times Y : y = f(x)\}$$

and is equipped with the subspace topology induced from $X \times Y$. Prove that the function $g\colon X \longrightarrow \Gamma(f)$ defined by $g(x) = \big(x, f(x)\big)$ is a homeomorphism.

$\longrightarrow\Box$ **Yes,** I would like feedback (comments) on my solution to this question.

4. We temporarily say a topological space X has the H property if the following holds:

   H: for any topological space Y and any continuous functions $f, g \colon Y \longrightarrow X$ such that f and g agree on some dense subset D of Y, we have $f = g$.

   (a) Let $f, g \colon Y \longrightarrow X$ be continuous functions between topological spaces. Suppose that X has the H property and that f and g agree on some subset S of Y. Prove that f and g agree on the closure $\overline{S}$ of S in Y.

   (b) Prove that every Hausdorff topological space has the H property.

   (c) Prove that every topological space with the H property is Hausdorff.

   $\longrightarrow$ ☐ **Yes,** I would like feedback (comments) on my solution to this question.

5. Define $f\colon (0,1] \longrightarrow \mathbf{R}$ by

$$f(x) = \sin\left(\frac{2\pi}{x}\right)$$

and let $\Gamma(f)$ be the graph of f in $\mathbf{R}^2$; in other words,

$$\Gamma(f) = \{(x,y) \in \mathbf{R}^2 \colon y = f(x)\}.$$

Now let $X = \Gamma(f) \cup \{(0,0)\}$.

(a) Prove that X is connected.

(b) Prove that X is not compact.

(c) Prove that there is no continuous function $g\colon [0,1] \longrightarrow X$ such that $g(0) = (0,0)$ and $g(1) = (1,0)$.

$\longrightarrow$ ☐ **Yes,** I would like feedback (comments) on my solution to this question.

6. Let (X, d) be a metric space.

   We say that a subset $K \subseteq X$ is *sequentially compact* if every sequence (k_n) in K has some subsequence (k_{n_j}) that converges to some $k \in K$.

   (a) Prove that if (x_n) is a Cauchy sequence in X such that some subsequence (x_{n_j}) converges to some $x \in X$, then $(x_n) \longrightarrow x$.

   (b) Suppose there exists $r > 0$ such that for all $x \in X$, the closure of the open ball $\mathbf{B}_r(x)$ is a sequentially compact subset of X. Prove that X is complete.

   (c) Suppose that for all $x \in X$ there exists $r > 0$ such that the closure of the open ball $\mathbf{B}_r(x)$ is a sequentially compact subset of X. Does it follow that X must be complete?

   If yes, prove it. If no, give a counterexample.

   (If you find it useful, you may assume without proof that closed intervals in $\mathbf{R}$ are sequentially compact.)

$\longrightarrow \square$ **Yes,** I would like feedback (comments) on my solution to this question.

7. **(*)** Consider **R** as an abelian group under addition of real numbers, and let G be a subgroup. Define

$$r = \inf\{x \in G \colon x > 0\}.$$

   (a) Prove that G is dense in **R** if $r = 0$.

   (b) Prove that G is discrete in **R** if $r > 0$, that is: the subspace topology on $G \subseteq \mathbf{R}$ is the discrete topology on G.

   (c) Conclude that G is either dense or discrete in **R**. (Don't forget the case in which r does not exist.)

$\longrightarrow\square$ **Yes,** I would like feedback (comments) on my solution to this question.