MAST30026 Assignment 2

Due Wednesday 9 October 2024 at 20:00 on Canvas and Gradescope

Name:

Student ID:

Some guidelines:

- Please write clear and detailed solutions in the boxes following each question or part of question. This can be done by printing this document and physically writing in the boxes, or by opening a copy of this document on a tablet or other device.
- The boxes should typically provide sufficient space for your solution, but if you find you need extra space please take an empty sheet and continue your solution there, clearly indicating which question this refers to. Also indicate in the corresponding box that the solution continues at the end.
- There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you write in the box is a complete explanation.

  The quality of the exposition will be assessed alongside the correctness of the approach.
- For technical reasons (since you will be uploading your solutions to GradeScope), please write legibly with a very readable writing implement.
- Results from the lectures, tutorials, exercises can be used (without having to re-prove them); make sure you say clearly what result you are using, though.
- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).
- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.
- You may assume that not all questions will have the same weight in the assessment.
- No Chegg or anything similar. At all. Please.

This assignment consists of 5 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

1. **(*)** Suppose $\big(V, \|\cdot\|\big)$ is a normed space over $\mathbf{F} = \mathbf{C}$ such that
$$\|v+w\|^2 + \|v-w\|^2 = 2\big(\|v\|^2 + \|w\|^2\big) \qquad \text{for all } v, w \in V.$$
Define $[\cdot,\cdot]\colon V \times V \longrightarrow \mathbf{R}$ by
$$4[v,w] := \|v+w\|^2 - \|v-w\|^2.$$

(a) Prove that, for all $v, w \in V$, we have
$$[w,v] = [v,w] \tag{1}$$
$$[iv,iw] = [v,w] \tag{2}$$
$$[iv,w] = -[v,iw]. \tag{3}$$

(b) Prove that
$$[2u,w] + [2v,w] = 2[u+v,w] \quad \text{for all } u, v, w \in V. \tag{4}$$
Conclude that
$$[2v,w] = 2[v,w] \quad \text{for all } v, w \in V, \tag{5}$$
and then that
$$[u,w] + [v,w] = [u+v,w] \quad \text{for all } u, v, w \in V. \tag{6}$$

(c) Prove that, for all $v, w \in V$, we have
$$[nv,w] = n[v,w] \qquad \text{for all } n \in \mathbf{N} \tag{7}$$
$$[nv,w] = n[v,w] \qquad \text{for all } n \in \mathbf{Z} \tag{8}$$
$$[qv,w] = q[v,w] \qquad \text{for all } q \in \mathbf{Q} \tag{9}$$
$$[xv,w] = x[v,w] \qquad \text{for all } x \in \mathbf{R}. \tag{10}$$

(d) Prove that
$$[v,v] \geqslant 0 \quad \text{for all } v \in V \quad \text{and } [v,v] = 0 \quad \text{if and only if } v = 0.$$

(e) Show that parts (a)–(d) imply that the function $\langle\cdot,\cdot\rangle\colon V \times V \longrightarrow \mathbf{C}$ given by
$$4\langle v,w\rangle := 4[v,w] + 4i[v,iw] = \|v+w\|^2 - \|v-w\|^2 + i\|v+iw\|^2 - i\|v-iw\|^2$$
is an inner product on V with associated norm $\|\cdot\|$.

$\longrightarrow$☐ **Yes,** I would like feedback (comments) on my solution to this question.

2. In the province of Hcanab, there are 2024 towns. Every town has exactly one arch-rival among the other towns (a town cannot be its own arch-rival, and arch-rivals are not necessarily mutual). Prove there exist two towns t_1 and t_2 such that the distance between t_1 and t_2 is less than or equal to the distance between their arch-rivals.

   (You may assume that Hcanab is small and flat enough that the towns lie in $\mathbf{R}^2$.)

   $\longrightarrow\square$ **Yes,** I would like feedback (comments) on my solution to this question.

3. Let V be a normed vector space.

   (a) Prove that V is connected.

   (b) Prove that V is compact if and only if it is 0-dimensional.

   $\longrightarrow \square$ **Yes,** I would like feedback (comments) on my solution to this question.

4. Recall the set of "finite" sequences

$$c_{00} = \left\{(a_n) \in \mathbf{F}^{\mathbf{N}} : \text{there exists } M \in \mathbf{N} \text{ such that } a_n = 0 \text{ for all } n \geqslant M\right\}.$$

We have seen that this is a vector subspace of $\mathbf{F}^{\mathbf{N}}$.

(a) Show that c_{00} is contained in ℓ^p for all $1 \leqslant p \leqslant \infty$.

(b) Show that c_{00} is dense in ℓ^p for all $1 \leqslant p < \infty$. (More precisely stated: let $1 \leqslant p < \infty$ and consider c_{00} as a subset of the normed space ℓ^p endowed with the $\|\cdot\|_{\ell^p}$ norm. Show that c_{00} is dense with respect to this norm.)

(c) Find the closure of c_{00} in ℓ^∞. (More precisely stated: consider c_{00} as a subset of the normed space ℓ^∞ endowed with the $\|\cdot\|_{\ell^\infty}$ norm. Find the closure of c_{00} with respect to this norm.)

$\longrightarrow\square$ **Yes,** I would like feedback (comments) on my solution to this question.

5. All sequences in this question are over the field $\mathbf{F} = \mathbf{R}$.

   Consider the following set of sequences:

$$h = \left\{ (x_n) \in \mathbf{R}^{\mathbf{N}} : |x_n| \leqslant \frac{1}{n} \text{ for all } n \in \mathbf{N} \right\}.$$

   (a) Prove that h is a subset of ℓ^2, but not a subset of ℓ^1.

   For the rest of the question, consider h as a subset of ℓ^2 and endow it with the metric defined by the $\|\cdot\|_{\ell^2}$ norm.

   (b) Show that h is a totally bounded subset of ℓ^2.
   [*Hint*: Exercise 2.55.]

   (c) Show that h is a closed subset of ℓ^2, and conclude that it is compact.

   (d) Show that h is a convex subset of ℓ^2.

   (e) Show that h has empty interior.

   (f) Show that $\overline{\mathrm{Span}(h)} = \ell^2$.
   [*Hint*: Prove that $c_{00} \subseteq \mathrm{Span}(h)$.]

   $\longrightarrow\square$ **Yes,** I would like feedback (comments) on my solution to this question.