# MAST30026 Assignment 1

Due Wednesday 4 September 2024 at 20:00 on Canvas and Gradescope

**Name:**

**Student ID:**

**Some guidelines:**

- Please write clear and detailed solutions in the boxes following each question or part of question. This can be done by printing this document and physically writing in the boxes, or by opening a copy of this document on a tablet or other device.
- The boxes should typically provide sufficient space for your solution, but if you find you need extra space please take an empty sheet and continue your solution there, clearly indicating which question this refers to. Also indicate in the corresponding box that the solution continues at the end.
- There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you write in the box is a complete explanation.

  The quality of the exposition will be assessed alongside the correctness of the approach.
- For technical reasons (since you will be uploading your solutions to GradeScope), please write legibly with a very readable writing implement.
- Results from the lectures, tutorials, exercises can be used (without having to re-prove them); make sure you say clearly what result you are using, though.
- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).
- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.
- You may assume that not all questions will have the same weight in the assessment.
- No Chegg or anything similar. At all. Please.

**This assignment consists of 7 questions. Please scan your answer pages and upload them to GradeScope in the correct order.**

1. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$ be a subset. Let $\iota\colon Y \longrightarrow X$ denote the inclusion map, that is $\iota(y) = y$ for all $y \in Y$.

   (a) Prove that the subspace topology $\mathcal{T}|_Y$ on $Y$ is the coarsest topology for which $\iota$ is continuous.

   (b) Prove that, given any topological space $Z$ and any function $g\colon Z \longrightarrow Y$, $g$ is continuous with respect to $\mathcal{T}|_Y$ if and only if $\iota \circ g\colon Z \longrightarrow X$ is continuous.

   (c) Assume now that $Y$ has the subspace topology. Complete the statement:
   "The map $\iota\colon Y \longrightarrow X$ is open if and only if $Y$ ______________________________"
   Prove your statement.

$\longrightarrow$ ☐ **Yes,** I would like feedback (comments) on my solution to this question.

(a) For any subset $U \subseteq X$, we have $\iota^{-1}(U) = Y \cap U$. Therefore

$$\mathcal{T}|_Y = \{Y \cap U\colon U \in \mathcal{T}\} = \{\iota^{-1}(U)\colon U \in \mathcal{T}\},$$

meaning that $\iota$ is continuous with respect to some topology $\mathcal{T}'$ on $Y$ if and only if $\mathcal{T}|_Y \subseteq \mathcal{T}'$.

(b) For any $U \subseteq X$ we have

$$(\iota \circ g)^{-1}(U) = g^{-1}\big(\iota^{-1}(U)\big) = g^{-1}(Y \cap U).$$

Therefore $g$ is continuous if and only if $g^{-1}(Y \cap U)$ is open for all $U \subseteq X$ open, if and only if $(\iota \circ g)^{-1}(U)$ is open for all $U \subseteq X$ open, if and only if $\iota \circ g$ is continuous.

(c) "The map $\iota\colon Y \longrightarrow X$ is open if and only if $Y$ is an open subset of $X$."

If the map is open, then since $Y$ is open in $Y$, we must have $Y$ is open in $X$. Conversely, suppose $Y$ is open in $X$. Given any $U' \subseteq Y$ open, we have $U' = Y \cap U$ for some $U \subseteq X$ open, but then $U'$ is the intersection of two open subsets of $X$, hence is itself open in $X$.

2. Let $X_1$ and $X_2$ be topological spaces, $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ subsets.

   We can construct two topologies on the set $Y_1 \times Y_2$:

   - $\mathcal{T}_{\text{prod}}$ = the product of the subspace topologies on $Y_1$ and $Y_2$;
   - $\mathcal{T}_{\text{sub}}$ = the subspace topology of the subset $Y_1 \times Y_2 \subseteq X_1 \times X_2$, where $X_1 \times X_2$ has the product topology.

   Is $\mathcal{T}_{\text{prod}} = \mathcal{T}_{\text{sub}}$? If yes, give a proof. If no, give a counterexample.

   $\longrightarrow$ ☐ **Yes,** I would like feedback (comments) on my solution to this question.

Yes, the two topologies are equal.
Note that $\mathcal{T}_{\text{prod}}$ is generated by the set

$$S_{\text{prod}} = \{(U_1 \cap Y_1) \times (U_2 \cap Y_2) \colon U_1 \text{ open in } X_1, U_2 \text{ open in } X_2\}.$$

On the other hand, the topology on $X_1 \times X_2$ is generated by

$$\{U_1 \times U_2 \colon U_1 \text{ open in } X_1, U_2 \text{ open in } X_2\},$$

so $\mathcal{T}_{\text{sub}}$ is generated by

$$S_{\text{sub}} = \{(U_1 \times U_2) \cap (Y_1 \times Y_2) \colon U_1 \text{ open in } X_1, U_2 \text{ open in } X_2\}.$$

However by Example 2.18 we have

$$(U_1 \times U_2) \cap (Y_1 \times Y_2) = (U_1 \cap Y_1) \times (U_2 \cap Y_2),$$

which implies that $S_{\text{sub}} = S_{\text{prod}}$, so that $\mathcal{T}_{\text{sub}} = \mathcal{T}_{\text{prod}}$.

3. Let $f\colon X \longrightarrow Y$ be a continuous function between topological spaces and consider the *graph* $\Gamma(f)$ of $f$, which is defined as

$$\Gamma(f) = \{(x,y) \in X \times Y : y = f(x)\}$$

and is equipped with the subspace topology induced from $X \times Y$. Prove that the function $g\colon X \longrightarrow \Gamma(f)$ defined by $g(x) = \big(x, f(x)\big)$ is a homeomorphism.

$\longrightarrow\square$ **Yes,** I would like feedback (comments) on my solution to this question.

Let $\iota\colon \Gamma(f) \longrightarrow X \times Y$ be the inclusion function and let $\pi_X\colon X \times Y \longrightarrow X$ and $\pi_Y\colon X \times Y \longrightarrow Y$ be the projections.
It is straightforward to see that the composite function $\pi_x \circ \iota$ is the inverse of $g$, so it remains to prove the continuity of $g$ and $\pi_X \circ \iota$.
We start with proving the continuity of $g$. It is straightforward to verify that $\pi_X \circ \iota \circ g$ is the identity function of $X$, which is continuous by Exercise 2.23, while $\pi_Y \circ \iota \circ g = f$, which is also continuous. Tutorial Question 3.7 then implies $\iota \circ g$ is continuous. Since $\Gamma(f)$ is given the subspace topology, it follows from part (b) of Question 1 that $g$ is continuous.
For $\pi_X \circ \iota$, we note that $\pi_X$ is continuous by Proposition 2.19, while $\iota$ is continuous by Exercise 2.23. It follows from Tutorial Question 2.9 that $\pi_X \circ \iota$ is continuous.

4. We temporarily say a topological space $X$ has the $H$ property if the following holds:

**H:** for any topological space $Y$ and any continuous functions $f, g\colon Y \longrightarrow X$ such that $f$ and $g$ agree on some dense subset $D$ of $Y$, we have $f = g$.

(a) Let $f, g\colon Y \longrightarrow X$ be continuous functions between topological spaces. Suppose that $X$ has the $H$ property and that $f$ and $g$ agree on some subset $S$ of $Y$. Prove that $f$ and $g$ agree on the closure $\overline{S}$ of $S$ in $Y$.

(b) Prove that every Hausdorff topological space has the $H$ property.

(c) Prove that every topological space with the $H$ property is Hausdorff.

$\longrightarrow$☐ **Yes,** I would like feedback (comments) on my solution to this question.

(a) Let $f', g'\colon \overline{D} \longrightarrow X$ denote the restrictions of $f$ and $g$ to $\overline{D}$ respectively. Since $D$ is dense in $\overline{D}$, and the functions $f'$ and $g'$ agree on $D$, the $H$ property of $X$ implies that $f' = g'$. Hence the result follows.

(b) Let $X$ be a topological space, $Y$ a Hausdorff topological space, $D$ a dense subset of $Y$. Consider two continuous functions $f, g\colon Y \longrightarrow X$ that agree on $D$.

Let $(f, g)\colon Y \longrightarrow X \times X$ be the function defined by $(f, g)(y) = (f(y), g(y))$ (see Tutorial Question 3.7). Since both $f$ and $g$ are continuous, it follows from Tutorial Question 3.7 that $(f, g)$ is continuous.

Let $\Delta$ denote the diagonal function of $X$, defined in Tutorial Question 3.9. By part (c) of Tutorial Question 3.9, the Hausdorffness of $X$ implies that $\Delta(X)$ is closed in $X \times X$. It then follows from Exercise 2.13 that $(f, g)^{-1}\big(\Delta(X)\big)$ is closed.

Since $f(y) = g(y)$ for every element of $y$ of $D$, it follows that $(f, g)(D) \subseteq \Delta(X)$, and thus $D \subseteq (f, g)^{-1}\big(\Delta(X)\big)$. However, $(f, g)^{-1}\big(\Delta(X)\big)$ is closed, so it contains the closure $Y$ of $D$. This implies that $(f(y), g(y)) \in \Delta(X)$ for every element $y$ of $Y$; in other words, $f(y) = g(y)$ for every element $y$ of $Y$.

(c) In part (b), we have shown that Hausdorffness implies the $H$ property, so it suffices to prove the other direction.

Let $X$ be a topological space with the $H$ property and $\Delta$ the diagonal function of $X$, defined in Tutorial Question 3.9. Define two projections $\pi_1\colon X \times X \longrightarrow X$ and $\pi_2\colon X \times X \longrightarrow X$ by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. It follows from Proposition 2.19 that $\pi_1$ and $\pi_2$ are both continuous. The projections $\pi_1$ and $\pi_2$ agree on $\Delta(X)$ by definition, so they agree on the closure $\overline{\Delta(X)}$ of $\Delta(X)$ by part (a). Therefore, if $(x, y) \in \overline{\Delta(X)}$, then

$$x = \pi_1(x, y) = \pi_2(x, y) = y,$$

so $(x, y) \in \Delta(X)$. It follows that $\overline{\Delta(X)} = \Delta(X)$.

5. Define $f\colon (0,1] \longrightarrow \mathbf{R}$ by

$$f(x) = \sin\left(\frac{2\pi}{x}\right)$$

and let $\Gamma(f)$ be the graph of $f$ in $\mathbf{R}^2$; in other words,

$$\Gamma(f) = \{(x,y) \in \mathbf{R}^2 \colon y = f(x)\}.$$

Now let $X = \Gamma(f) \cup \{(0,0)\}$.

(a) Prove that $X$ is connected.

(b) Prove that $X$ is not compact.

(c) Prove that there is no continuous function $g\colon [0,1] \longrightarrow X$ such that $g(0) = (0,0)$ and $g(1) = (1,0)$.

$\longrightarrow\square$ **Yes,** I would like feedback (comments) on my solution to this question.

We will use a few times the fact that intervals in $\mathbf{R}$ are connected, see Example 2.33.

(a) By Question 3, $\Gamma(f)$ is homeomorphic to $(0,1]$, which is connected. Since the sequence $\big((\frac{1}{n},0)\big)$ in $\Gamma(f)$ converges to $(0,0)$, it follows that $\Gamma(f)$ is dense in $X$. By Tutorial Question 4.1, this implies that $X$ is connected.

(b) Since the sequence $\big((\frac{1}{n+\frac{1}{4}},1)\big)$ in $\Gamma(f)$ converges to $(0,1)$ but $(0,1) \notin X$, it follows that $X$ is not closed in $\mathbf{R}^2$, so $X$ cannot by compact because of Proposition 2.35.

(c) Suppose such a continuous function $g\colon [0,1] \longrightarrow X$ exists.

Denote by $\pi_1\colon X \longrightarrow \mathbf{R}$ the (restriction of the) projection onto the first coordinate in $\mathbf{R}^2$. Note that $\pi_1((0,0)) = 0$ and $\pi_1((x,f(x))) = x$ for all $x \in (0,1]$, so $\pi_1$ gives a bijection that we also denote $\pi_1\colon X \longrightarrow [0,1]$.

Consider the composition $\pi_1 \circ g\colon [0,1] \longrightarrow [0,1]$. We have $(\pi_1 \circ g)(0) = 0$ and $(\pi_1 \circ g)(1) = 1$, so by the Intermediate Value Theorem (Theorem 2.34) the function $\pi_1 \circ g$ is surjective:

$$(\pi_1 \circ g)([0,1]) = [0,1].$$

But $\pi_1\colon X \longrightarrow [0,1]$ is bijective, so $g$ must be surjective.

It then follows from Proposition 2.37 that $X = g([0,1])$ is compact, which contradicts part (b). Hence there is no such function $g$.

6. Let $(X, d)$ be a metric space.

   We say that a subset $K \subseteq X$ is *sequentially compact* if every sequence $(k_n)$ in $K$ has some subsequence $(k_{n_j})$ that converges to some $k \in K$.

   (a) Prove that if $(x_n)$ is a Cauchy sequence in $X$ such that some subsequence $(x_{n_j})$ converges to some $x \in X$, then $(x_n) \longrightarrow x$.

   (b) Suppose there exists $r > 0$ such that for all $x \in X$, the closure of the open ball $\mathbf{B}_r(x)$ is a sequentially compact subset of $X$. Prove that $X$ is complete.

   (c) Suppose that for all $x \in X$ there exists $r > 0$ such that the closure of the open ball $\mathbf{B}_r(x)$ is a sequentially compact subset of $X$. Does it follow that $X$ must be complete?

   If yes, prove it. If no, give a counterexample.

   (If you find it useful, you may assume without proof that closed intervals in $\mathbf{R}$ are sequentially compact.)

$\longrightarrow\square$ **Yes,** I would like feedback (comments) on my solution to this question.

(a) Let $\varepsilon > 0$.

Since $(x_n)$ is Cauchy, there exists $N \in \mathbf{N}$ such that

$$d(x_m, x_n) < \frac{\varepsilon}{2} \qquad \text{for all } m, n \geqslant N.$$

I claim that $d(x_n, x) < \varepsilon$ for all $n \geqslant N$.

To show this, let $n \geqslant N$. Since $(x_{n_j}) \longrightarrow x$ as $j \longrightarrow \infty$, there exists $J \in \mathbf{N}$ such that

$$d(x_{n_j}, x) < \frac{\varepsilon}{2} \qquad \text{for all } j \geqslant J.$$

Let $j \geqslant J$ be such that $n_j \geqslant N$. (This can be done as $(n_j)$ is an increasing sequence of natural numbers.) Then

$$d(x_n, x) \leqslant d(x_n, x_{n_j}) + d(x_{n_j}, x) < \varepsilon.$$

(b) Let $(x_n)$ be a Cauchy sequence in $X$. There exists $N \in \mathbf{N}$ such that

$$d(x_n, x_m) < r \qquad \text{for all } n, m \geqslant N.$$

In particular, for all $n \geqslant N$ we have $x_n \in \mathbf{B}_r(x_N)$. This gives us a sequence $(x_n)_{n \geqslant N}$ in the compact set $\overline{\mathbf{B}_r(x_N)}$, hence by part (a) there must be a subsequence $(x_{n_j})$ that converges to some $x \in X$. But this is also a subsequence of the original Cauchy sequence $(x_n)$ which by part (b) must therefore also converge to $x$.

(c) It does not follow that $X$ must be complete.

For a counterexample, take $X = (0,1) \subseteq \mathbf{R}$. This is certainly not complete since $(1/n)$ is a Cauchy sequence that does not converge in $X$. But for any $x \in X$ we can take $r = \frac{1}{2}\min\{x, 1-x\}$ and then $\overline{\mathbf{B}_r(x)} = [x-r, x+r]$, which is a closed interval in $\mathbf{R}$, hence sequentially compact.

7. Consider $\mathbf{R}$ as an abelian group under addition of real numbers, and let $G$ be a subgroup. Define
$$r = \inf\{x \in G \colon x > 0\}.$$
   (a) Prove that $G$ is dense in $\mathbf{R}$ if $r = 0$.
   (b) Prove that $G$ is discrete in $\mathbf{R}$ if $r > 0$, that is: the subspace topology on $G \subseteq \mathbf{R}$ is the discrete topology on $G$.
   (c) Conclude that $G$ is either dense or discrete in $\mathbf{R}$. (Don't forget the case in which $r$ does not exist.)

$\longrightarrow\square$ **Yes,** I would like feedback (comments) on my solution to this question.

(a) If $U$ is an non-empty open subset of $\mathbf{R}$, then it contains some open interval $(a,b)$. Since $\inf\{x \in G \colon x > 0\} = r = 0$, there exists an element $x$ of $G$ such that $0 < x < b - a$, so $\frac{a}{x} + 1 < \frac{b}{x}$. This means that there exists an integer $n$ such that $\frac{a}{x} < n < \frac{b}{x}$, in other words $nx \in (a,b)$. It follows that $G \cap (a,b)$ is non-empty, so $G \cap U$ is non-empty. Hence $G$ intersects every non-empty open subset of $\mathbf{R}$, and is therefore dense in $\mathbf{R}$.

(b) If $x \in G$, then
$$\{x\} = G \cap (x - r, x + r),$$
otherwise there exist two elements of $G$ whose difference is smaller than $r$. We conclude that all singletons are open in $G$, hence $G$ is discrete in $\mathbf{R}$.

(c) We have considered the cases when $r = 0$ and $r > 0$. The only case remaining is the case in which $r$ does not exist, which happens only when $\{x \in G \colon x > 0\} = \varnothing$. This implies that $G = \{0\}$, which is a discrete subset of $\mathbf{R}$, so the result follows.