MAST30026 Assignment 2

Due Wednesday 9 October 2024 at 20:00 on Canvas and Gradescope

Name:

Student ID:

Some guidelines:

- Please write clear and detailed solutions in the boxes following each question or part of question. This can be done by printing this document and physically writing in the boxes, or by opening a copy of this document on a tablet or other device.
- The boxes should typically provide sufficient space for your solution, but if you find you need extra space please take an empty sheet and continue your solution there, clearly indicating which question this refers to. Also indicate in the corresponding box that the solution continues at the end.
- There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you write in the box is a complete explanation.

  The quality of the exposition will be assessed alongside the correctness of the approach.
- For technical reasons (since you will be uploading your solutions to GradeScope), please write legibly with a very readable writing implement.
- Results from the lectures, tutorials, exercises can be used (without having to re-prove them); make sure you say clearly what result you are using, though.
- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).
- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.
- You may assume that not all questions will have the same weight in the assessment.
- No Chegg or anything similar. At all. Please.

This assignment consists of 5 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

1. **(*)**

Suppose $(V, \|\cdot\|)$ is a normed space over $\mathbf{F} = \mathbf{C}$ such that

$$\|v+w\|^2 + \|v-w\|^2 = 2\big(\|v\|^2 + \|w\|^2\big) \qquad \text{for all } v, w \in V.$$

Define $[\cdot,\cdot] \colon V \times V \longrightarrow \mathbf{R}$ by

$$4[v,w] := \|v+w\|^2 - \|v-w\|^2.$$

(a) Prove that, for all $v, w \in V$, we have

$$[w,v] = [v,w] \tag{1}$$
$$[iv, iw] = [v,w] \tag{2}$$
$$[iv, w] = -[v, iw]. \tag{3}$$

(b) Prove that

$$[2u,w] + [2v,w] = 2[u+v,w] \quad \text{for all } u,v,w \in V. \tag{4}$$

Conclude that

$$[2v,w] = 2[v,w] \quad \text{for all } v,w \in V, \tag{5}$$

and then that

$$[u,w] + [v,w] = [u+v,w] \quad \text{for all } u,v,w \in V. \tag{6}$$

(c) Prove that, for all $v, w \in V$, we have

$$[nv,w] = n[v,w] \qquad \text{for all } n \in \mathbf{N} \tag{7}$$
$$[nv,w] = n[v,w] \qquad \text{for all } n \in \mathbf{Z} \tag{8}$$
$$[qv,w] = q[v,w] \qquad \text{for all } q \in \mathbf{Q} \tag{9}$$
$$[xv,w] = x[v,w] \qquad \text{for all } x \in \mathbf{R}. \tag{10}$$

(d) Prove that

$$[v,v] \geqslant 0 \quad \text{for all } v \in V \quad \text{and } [v,v] = 0 \quad \text{if and only if } v = 0.$$

(e) Show that parts (a)–(d) imply that the function $\langle\cdot,\cdot\rangle \colon V \times V \longrightarrow \mathbf{C}$ given by

$$4\langle v,w\rangle := 4[v,w] + 4i[v,iw] = \|v+w\|^2 - \|v-w\|^2 + i\|v+iw\|^2 - i\|v-iw\|^2$$

is an inner product on V with associated norm $\|\cdot\|$.

$\longrightarrow$ ☐ **Yes,** I would like feedback (comments) on my solution to this question.

(a) Equation (1) is obvious from the definition. For Equation (2), we have

$$\begin{aligned}4[iv,iw] &= \|iv+iw\|^2 - \|iv-iw\|^2 = \|i(v+w)\|^2 - \|i(v-w)\|^2 \\ &= |i|^2\|v+w\|^2 - |i|^2\|v-w\|^2 = 4[v,w].\end{aligned}$$

For Equation (3):

$$\begin{aligned}4[v,iw] &= \|v+iw\|^2 - \|v-iw\|^2 = \|i(w-iv)\|^2 - \|-i(w+iv)\|^2 \\ &= |i|^2\|w-iv\|^2 - |-i|^2\|w+iv\|^2 = -4[w,iv].\end{aligned}$$

(b) Equation (4) follows from the calculation

$$\begin{aligned}4[2u,w]+4[2v,w] &= \|2u+w\|^2-\|2u-w\|^2+\|2v+w\|^2-\|2v-w\|^2\\ &= \Big(\big\|(u+v+w)+(u-v)\big\|^2+\big\|(u+v+w)-(u-v)\big\|^2\Big)\\ &\quad -\Big(\big\|(u+v-w)+(u-v)\big\|^2+\big\|(u+v-w)-(u-v)\big\|^2\Big)\\ &= 2\big(\|u+v+w\|^2+\|u-v\|^2\big)-2\big(\|u+v-w\|^2+\|u-v\|^2\big)\\ &= 8[u+v,w].\end{aligned}$$

In particular, setting $u=0$ we have $[2u,w]=0$ (from the definition of $[\cdot,\cdot]$), hence Equation (5).

Using this on the LHS of Equation (4) we get Equation (6).

(c) We already have Equation (7) for $n=0,1,2$. Clearly repeated application of part (b) gives us

$$[nv,w]=n[v,w] \qquad \text{for all } n\in\mathbf{N}.$$

For (-1) we have

$$4[-v,w]=\|-v+w\|^2-\|-v-w\|^2=\|v-w\|^2-\|v+w\|^2=-4[v,w],$$

hence

$$[nv,w]=n[v,w] \qquad \text{for all } n\in\mathbf{Z}.$$

For any $q\in\mathbf{Q}$, write $q=m/n$ with $\gcd(m,n)=1$:

$$n[qv,w]=[nqv,w]=[mv,w]=m[v,w],$$

therefore

$$[qv,w]=q[v,w] \qquad \text{for all } q\in\mathbf{Q}.$$

Finally, for any $x\in\mathbf{R}$ choose a rational sequence $(q_n)\longrightarrow x$:

$$\begin{aligned}[xv,w] &= \Big[\Big(\lim_{n\longrightarrow\infty} q_n\Big)v,w\Big]=\Big[\lim_{n\longrightarrow\infty}(q_nv),w\Big]=\lim_{n\longrightarrow\infty}[q_nv,w]\\ &= \lim_{n\longrightarrow\infty}(q_n[v,w])=\Big(\lim_{n\longrightarrow\infty} q_n\Big)[v,w]=x[v,w].\end{aligned}$$

Somewhere in the middle we used the fact that $[\cdot,\cdot]$ is continuous in the first variable (which follows easily from the definition of $[\cdot,\cdot]$ and the fact that the norm is continuous).

(d) Straightforward, as

$$4[v,v]=4\|v\|^2.$$

(e) We check the inner product properties:

$$\langle w, v\rangle = [w, v] + i[w, iv] = [v, w] - i[v, iw] = \overline{\langle v, w\rangle}.$$

$$\begin{aligned}\langle u+v, w\rangle &= [u+v, w] + i[u+v, iw] \\ &= [u, w] + [v, w] + i[u, iw] + i[v, iw] \\ &= \langle u, w\rangle + \langle v, w\rangle.\end{aligned}$$

Writing $\alpha = x + iy \in \mathbf{C}$, we have

$$\begin{aligned}\langle \alpha v, w\rangle &= [\alpha v, w] + i[\alpha v, iw] \\ &= [xv + iyv, w] + i[xv + iyv, iw] \\ &= [xv, w] + [iyv, w] + i[xv, iw] + i[iyv, iw] \\ &= x[v, w] + y[iv, w] + ix[v, iw] + iy[iv, iw] \\ &= (x+iy)[v, w] + i(x+iy)[v, iw] \\ &= \alpha\langle v, w\rangle.\end{aligned}$$

Here we used in order Equations (6), (10), (2), and (3).

Finally, note that for all $v \in V$ we can apply (3) with $w = v$ and then (1) with $w = iv$ to see that

$$[v, iv] = -[v, iv] \quad \text{so} \quad [v, iv] = 0.$$

Therefore

$$\langle v, v\rangle = [v, v] + i[v, iv] = [v, v],$$

so positive-definiteness follows from part (d). Moreover, we have seen in part (d) that $[v, v] = \|v\|^2$, so $\langle v, v\rangle = [v, v] = \|v\|^2$, which implies that the norm defined by $\langle \cdot, \cdot \rangle$ is $\|\cdot\|$.

2. In the province of Hcanab, there are 2024 towns. Every town has exactly one arch-rival among the other towns (a town cannot be its own arch-rival, and arch-rivals are not necessarily mutual). Prove there exist two towns t_1 and t_2 such that the distance between t_1 and t_2 is less than or equal to the distance between their arch-rivals.

   (You may assume that Hcanab is small and flat enough that the towns lie in $\mathbf{R}^2$.)

   $\longrightarrow$ ☐ **Yes,** I would like feedback (comments) on my solution to this question.

Let $S \subseteq \mathbf{R}^2$ be the set of towns, and give it the subspace metric. Since S is finite, it follows from Tutorial Question 4.6 that S is compact, hence S is a complete metric space by the Heine–Borel theorem.
Let $a\colon S \longrightarrow S$ denote the map assigning to any town $t \in S$ its arch-rival $a(t) \in S$. Then a is not a contraction: if it were a contraction, the Banach Fixed Point Theorem would imply that a has a fixed point t, so that $a(t) = t$ and t would be its own arch-rival, which is prohibited.
So a is not a contraction, hence for all $C \in [0, 1)$ there exist $x_1, x_2 \in S$ such that

$$Cd(x_1, x_2) < d(a(x_1), a(x_2)). \tag{11}$$

From here on we proceed by contradiction: suppose that for all $s_1, s_2 \in S$ such that $s_1 \neq s_2$ we have

$$d(s_1, s_2) > d(a(s_1), a(s_2)) \quad \text{so} \quad \frac{d(a(s_1), a(s_2))}{d(s_1, s_2)} < 1.$$

Then, since S is a finite set, we get

$$\max_{\substack{s_1, s_2 \in S \\ s_1 \neq s_2}} \frac{d(a(s_1), a(s_2))}{d(s_1, s_2)} < 1.$$

But Equation (11) means that for all $C \in [0, 1)$ we have

$$C < \max_{\substack{s_1, s_2 \in S \\ s_1 \neq s_2}} \frac{d(a(s_1), a(s_2))}{d(s_1, s_2)} < 1.$$

Taking the limit as $C \longrightarrow 1$ we get

$$1 \leqslant \max_{\substack{s_1, s_2 \in S \\ s_1 \neq s_2}} \frac{d(a(s_1), a(s_2))}{d(s_1, s_2)} < 1,$$

contradiction.
We conclude that there must exist $t_1, t_2 \in S$ such that $t_1 \neq t_2$ and

$$d(t_1, t_2) \leqslant d(a(t_1), a(t_2)).$$

3. Let V be a normed vector space.

   (a) Prove that V is connected.

   (b) Prove that V is compact if and only if it is 0-dimensional.

$\longrightarrow\square$ **Yes,** I would like feedback (comments) on my solution to this question.

(a) Let $f\colon V \longrightarrow \{0,1\}$ be a continuous function, where $\{0,1\}$ is given the discrete topology. If v is a non-zero vector in V, then $\mathbf{F}v$ is homeomorphic to $\mathbf{F}$ because of Tutorial Question 8.3, but $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$ is of course connected. It then follows from Proposition 2.29 that f is constant when restricted to $\mathbf{F}v$, and therefore $f(0) = f(v)$. We have proved that $f(v) = f(0)$ for every vector v in V; in other words, f is constant. Hence V is connected.

(b) If V is 0-dimensional, in other words, if $V = \{0\}$, then V is compact because of Tutorial Question 4.6.

Conversely, suppose V is compact. If V has a non-zero vector, then it follows from Tutorial Question 8.4 that the one-dimensional subspace $\mathbf{F}v$ is closed, and therefore compact by Proposition 2.36. However, it follows from Tutorial Question 8.3 that $\mathbf{F}v$ is homeomorphic to $\mathbf{F}$, which is not compact, a contradiction. Hence $V = \{0\}$.

4. Recall the set of "finite" sequences

$$c_{00} = \left\{ (a_n) \in \mathbf{F}^{\mathbf{N}} : \text{there exists } M \in \mathbf{N} \text{ such that } a_n = 0 \text{ for all } n \geqslant M \right\}.$$

We have seen that this is a vector subspace of $\mathbf{F}^{\mathbf{N}}$.

(a) Show that c_{00} is contained in ℓ^p for all $1 \leqslant p \leqslant \infty$.

(b) Show that c_{00} is dense in ℓ^p for all $1 \leqslant p < \infty$. (More precisely stated: let $1 \leqslant p < \infty$ and consider c_{00} as a subset of the normed space ℓ^p endowed with the $\|\cdot\|_{\ell^p}$ norm. Show that c_{00} is dense with respect to this norm.)

(c) Find the closure of c_{00} in ℓ^∞. (More precisely stated: consider c_{00} as a subset of the normed space ℓ^∞ endowed with the $\|\cdot\|_{\ell^\infty}$ norm. Find the closure of c_{00} with respect to this norm.)

⟶ ☐ **Yes,** I would like feedback (comments) on my solution to this question.

(a) This is obvious: if $(a_n) \in c_{00}$ then

$$\|(a_n)\|_{\ell^p}^p = \sum_{n=1}^{\infty} |a_n|^p = \sum_{n=1}^{M-1} |a_n|^p < \infty.$$

(b) Given an element a of ℓ^p, let $a^{(m)}$ denote the sequence in c_{00} defined by

$$a_n^{(m)} = \begin{cases} a_n & \text{if } n \leqslant m, \\ 0 & \text{otherwise.} \end{cases}$$

As $a = (a_n) \in \ell^p$ we have

$$\|a\|_{\ell^p}^p = \sum_{n=1}^{\infty} |a_n|^p < \infty,$$

Then consider

$$\|a^{(m)} - a\|_{\ell^p}^p = \sum_{n=m+1}^{\infty} |a_n|^p,$$

but this is the $(m+1)$-"tail" of a convergent series, so must go to 0 as $m \longrightarrow \infty$. Hence $(a^{(m)}) \longrightarrow a$ with respect to the ℓ^p norm.

(c) I claim that $\overline{c_{00}} = c_0$, the space of sequences that converge to 0. It suffices to prove that c_{00} is dense in c_0, as the latter is closed in ℓ^∞ by Tutorial Question 9.6.

Given an element a of c_0, let $a^{(m)}$ denote the sequence in c_{00} defined by

$$a_n^{(m)} = \begin{cases} a_n & \text{if } n \leqslant m, \\ 0 & \text{otherwise.} \end{cases}$$

We will show that $a^{(m)} \longrightarrow a$ as $m \longrightarrow \infty$.

Since $a_n \longrightarrow 0$ as $n \longrightarrow \infty$, for every positive real number ϵ there exists a positive integer N such that $n > N$ implies $|a_n| < \epsilon$. It follows that $m > N$ implies $\|a^{(m)} - a\| < \epsilon$ because

$$\left(a^{(m)} - a\right)_n = \begin{cases} 0 & \text{if } n \leqslant m, \\ a_n & \text{otherwise.} \end{cases}$$

5. All sequences in this question are over the field $\mathbf{F} = \mathbf{R}$.

   Consider the following set of sequences:

$$h = \left\{ (x_n) \in \mathbf{R}^{\mathbf{N}} : |x_n| \leqslant \frac{1}{n} \text{ for all } n \in \mathbf{N} \right\}.$$

   (a) Prove that h is a subset of ℓ^2, but not a subset of ℓ^1.

   For the rest of the question, consider h as a subset of ℓ^2 and endow it with the metric defined by the $\|\cdot\|_{\ell^2}$ norm.

   (b) Show that h is a totally bounded subset of ℓ^2.
   [*Hint*: Exercise 2.55.]

   (c) Show that h is a closed subset of ℓ^2, and conclude that it is compact.

   (d) Show that h is a convex subset of ℓ^2.

   (e) Show that h has empty interior.

   (f) Show that $\overline{\operatorname{Span}(h)} = \ell^2$.
   [*Hint*: Prove that $c_{00} \subseteq \operatorname{Span}(h)$.]

$\longrightarrow$ ☐ **Yes,** I would like feedback (comments) on my solution to this question.

(a) If $(x_n) \in h$ we have

$$\|(x_n)\|_{\ell^2}^2 = \sum_{n=1}^{\infty} |x_n|^2 \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Note however that $(1/n) \in h$ but

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, so $(1/n) \notin \ell^1$.

(b) For every $m \in \mathbf{N}_{\geqslant 2}$ we can split the first $m-1$ terms of $(x_n) \in h$ from the rest of the sequence and get an inclusion

$$h \subseteq \mathbf{B}_1(0_{\mathbf{R}^{m-1}}) \times h_m,$$

where

$$h_m = \{(0, \dots, 0, x_m, x_{m+1}, \dots) \in h\}.$$

For any $y \in h_m$ we have

$$\|y\|_{\ell^2}^2 \leqslant \sum_{n=m}^{\infty} \frac{1}{n^2},$$

and this upper bound converges to 0 as $m \longrightarrow \infty$, since the series $\sum_{n=1}^{\infty}(1/n^2)$ converges.

Given $\varepsilon > 0$, choose $m > 2/\varepsilon$, then $h_m \subseteq \mathbf{B}_{\varepsilon/2}(0)$. Bounded implies totally bounded in $\mathbf{R}^{m-1}$, so we can cover $\mathbf{B}_1(0_{\mathbf{R}^{m-1}})$ by finitely many open balls of radius $\varepsilon/2$, and conclude using the argument from Exercise 2.55.

(c) Let $a \in \overline{h}$, so that there is a sequence (x_n) in h such that $(x_n) \longrightarrow a$ in ℓ^2. Suppose $a \notin h$, so that there exists $m \in \mathbf{N}$ such that $|a_m| > 1/m$. Let $\varepsilon = |a_m| - 1/m$, then

$$|x_{n,m} - a_m| \geqslant \varepsilon \qquad \text{for all } n \in \mathbf{N}.$$

Then

$$\|x_n - a\|_{\ell^2} \geqslant |x_{n,m} - a_m| \geqslant \varepsilon \qquad \text{for all } n \in \mathbf{N},$$

contradicting the assumption that $\|x_n - a\|_{\ell^2} \longrightarrow 0$ as $n \longrightarrow \infty$.

Since h is a closed subset of the complete space ℓ^2, it is itself complete. Since it is also totally bounded, we conclude that it is compact.

(d) Let $x = (x_n), y = (y_n) \in h$, $a, b \in \mathbf{R}_{\geqslant 0}$ with $a + b = 1$, and let $z = ax + by$. We have

$$|z_n| = |ax_n + by_n| \leqslant a|x_n| + b|y_n| \leqslant \frac{a}{n} + \frac{b}{n} = \frac{1}{n}.$$

(e) Let $x \in h$, let $\varepsilon > 0$, and consider the open ball $\mathbf{B}_\varepsilon(x)$ in ℓ^2. Let $m \in \mathbf{N}$ be such that $m > 3/\varepsilon$. Let $x'_n = x_n$ for all $n \neq m$, and let $x'_m = 2/m$. Then certainly $x' = (x'_n) \notin h$, but

$$\|x - x'\|_{\ell^2} = |x_m - x'_m| \leqslant |x_m| + \frac{2}{m} \leqslant \frac{3}{m} < \varepsilon,$$

so $x' \in \mathbf{B}_\varepsilon(x) \setminus h$.

(f) For any $x \in c_{00}$ there exists $M \in \mathbf{N}$ such that

$$x = (x_1, \ldots, x_{M-1}, 0, 0, \ldots) \in \operatorname{Span}\left\{e_1, \frac{1}{2}\, e_2, \ldots, \frac{1}{M-1}\, e_{M-1}\right\} \subseteq \operatorname{Span}(h),$$

so $c_{00} \subseteq \operatorname{Span}(h)$.

But c_{00} is dense in ℓ^2, so

$$\ell^2 = \overline{c_{00}} \subseteq \overline{\operatorname{Span}(h)},$$

implying that $\overline{\operatorname{Span}(h)} = \ell^2$.

Alternative proof for (b): Let $\varepsilon > 0$. Then there exists $N > 0$ such that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{\varepsilon^2}{2},$$

so for all $x \in h$ we have

$$\sum_{n=N+1}^{\infty} |x_n|^2 < \frac{\varepsilon^2}{2}.$$

From Exercise 2.55, $[-1,1] \times [\frac{-1}{2}, \frac{1}{2}] \times \cdots \times [\frac{-1}{N}, \frac{1}{N}]$ is totally bounded, hence there exist finitely many open balls in $\mathbf{R}^N$ of radius $\sqrt{\varepsilon^2/2}$ covering it. Let the balls have centres $((y_1^{(j)}, \ldots, y_N^{(j)}))_{1 \leqslant j \leqslant k}$. This means that for all $x \in h$, there exists j such that

$$\sqrt{\left|x_1 - y_1^{(j)}\right|^2 + \cdots + \left|x_N - y_N^{(j)}\right|^2} < \sqrt{\frac{\varepsilon^2}{2}},$$

hence

$$\begin{aligned}\left\|x - \left(y_1^{(j)}, \ldots, y_N^{(j)}, 0, \ldots\right)\right\|_{\ell^2} &= \left(\sum_{n=1}^{N} \left|x_n - y_n^{(j)}\right|^2 + \sum_{n=N+1}^{\infty} |x_n|^2\right)^{1/2} \\ &< \left(\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}\right)^{1/2} \\ &= \varepsilon.\end{aligned}$$

So take the balls of radius ε with centres $((y_1^{(j)}, \ldots, y_N^{(j)}, 0, \ldots))_{1 \leqslant j \leqslant k}$ in $h \subseteq \ell^2$, and these cover h.