Tutorial Week 2

Topics: metrics, topologies, continuous functions.

2.1. Let X be a set and d the discrete metric on X, that is $d(x_1, x_2) = 1$ for all $x_1 \neq x_2$; see also Exercise 2.6. Prove that the topology defined by d is the discrete topology.

2.2. Is the word "finite" necessary in the statement of Proposition 2.12? If no, give a proof of the statement without "finite". If yes, give an example of an infinite collection of open sets whose intersection is not an open set.

2.3. Find all topologies on the set $\{0,1\}$ and determine which of them is metrisable.

2.4. Let X be a set and S a subset of $\mathcal{P}(X)$. Prove that the topology generated by S is the intersection of all topologies $\mathcal{T}$ on X containing S, and is thus the coarsest among such topologies.

2.5. Let X and Y be two topological spaces, where the topology on X is the discrete topology. Prove that every function from X to Y is continuous.

2.6. Let $f: X \longrightarrow Y$ be a function between topological spaces. Suppose the topology on Y is generated by a subset S of $\mathcal{P}(Y)$. Prove that the function f is continuous if and only if $f^{-1}(U)$ is open for every element U of S.

2.7. Let $f: X \longrightarrow Y$ be a function and $\mathcal{T}_Y$ a topology on Y. Define

$$\mathcal{T}_X = \left\{ f^{-1}(U) : U \in \mathcal{T}_Y \right\}.$$

(a) Prove that $\mathcal{T}_X$ is the coarsest topology on X such that f is continuous. (This topology is called the *initial topology* induced by f.)

(b) Let $\mathcal{T}$ be another topology on X. Prove that $f: (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous if and only if $\mathcal{T}$ is finer than $\mathcal{T}_X$.

(c) Use an example to prove that $\mathcal{T}_X$ need not be metrisable even when $\mathcal{T}_Y$ is a metric topology.

(d) Give an example in which $\mathcal{T}_X$ is metrisable but $\mathcal{T}_Y$ is not.

(e) Suppose $\mathcal{T}_Y$ is generated by a subset S of $\mathcal{P}(Y)$. Prove that $\mathcal{T}_X$ is generated by the set

$$\left\{ f^{-1}(U) : U \in S \right\}.$$

[*Hint*: For (c) and (d), consider using Question 2.3.]

Note: There is a "dual" setting where you start with a topology on X and look for the finest topology on Y such that f is continuous, see Exercise 2.14.

2.8. Prove that a function $f: X \longrightarrow Y$ between metric spaces is continuous if and only if it satisfies the usual ϵ–δ definition: for every point x of X and every positive real number ϵ, there exists a positive real number δ such that $d_X(x,y) < \delta$ implies $d_Y\big(f(x), f(x)\big) < \epsilon$.

2.9.

(a) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be functions, where X, Y, Z are sets, and let $S \subseteq Z$. Then

$$f^{-1}\big(g^{-1}(S)\big) = (g \circ f)^{-1}(S).$$

(b) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be continuous functions, where X, Y, Z are topological spaces. Prove that $g \circ f\colon X \longrightarrow Z$ is continuous.