

Tutorial Week 3

Topics: Equivalent metrics, local continuity, closure, interior, denseness, product, Hausdorff.

3.1. Let X be a topological space. Prove that a subset U of X is open if and only if it is a neighbourhood of every element of itself.

3.2. Let (X, d) be a metric space.

(a) Prove that the metric topology on (X, d) is generated by open balls of radii smaller than 1.

(b) Define $d': X \times X \rightarrow \mathbf{R}_{\geq 0}$ by

$$d'(x, y) = \min \{d(x, y), 1\}.$$

Prove that d' is a metric.

(c) Prove that d and d' are equivalent (that is, they give rise to the same topology on X).

3.3. Let $f: X \rightarrow Y$ be a function between topological spaces. Given $x \in X$, we say that f is *continuous at x* if the inverse image $f^{-1}(N)$ of every neighbourhood N of $f(x)$ is a neighbourhood of x . Prove that f is continuous if and only if it is continuous at every $x \in X$.

3.4. Let A and B be subsets of a topological space X .

(a) Suppose $A \subseteq B$. Prove that $\overline{A} \subseteq \overline{B}$ and $A^\circ \subseteq B^\circ$.

(b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(c) Prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Find an example in which $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

(d) Prove that $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$. Find an example in which $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.

[Hint: For (c) and (d), think of some subsets of \mathbf{R} .]

3.5. Prove that \mathbf{Z} is a nowhere dense subset of \mathbf{R} .

3.6. Let X be a topological space.

(a) Prove that any subset of a nowhere dense subset of X is nowhere dense in X .

(b) Prove that a subset $N \subseteq X$ is nowhere dense if and only if $X \setminus \overline{N}$ is dense in X .

(c) Prove that the union of any finite collection of nowhere dense subsets of X is nowhere dense in X .

3.7. Let $X, Y_1,$ and Y_2 be topological spaces, and $\pi_1: Y_1 \times Y_2 \rightarrow Y_1$ and $\pi_2: Y_1 \times Y_2 \rightarrow Y_2$ be the projections. Prove that a function $f: X \rightarrow Y_1 \times Y_2$ is continuous if and only if both $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

3.8. Let X and Y be topological spaces and let A and B be subsets of X and Y respectively.

(a) Suppose A and B are closed in X and Y respectively. Prove that if A and B are closed, then $A \times B$ is closed.

(b) Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.

3.9. Given a set X , define the *diagonal function*

$$\Delta: X \longrightarrow X \times X, \quad x \longmapsto (x, x).$$

- (a) Prove that two subsets A and B of X are disjoint if and only if $\Delta(X)$ and $A \times B$ are disjoint.
- (b) If X is a topological space, prove that Δ is continuous.
- (c) Prove that a topological space X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.