Tutorial Week 4

Topics: Connectedness, closed functions, compactness.

4.1. Prove that if a topological space X admits a connected dense subset D, then X is itself connected.

4.2. Let C_1 and C_2 be two connected subsets of a topological space X such that $C_1 \cap C_2 \neq \varnothing$. Prove that $C_1 \cup C_2$ is connected.

4.3. Let X be a topological space. Suppose A is a connected subset of X and $\{C_i : i \in I\}$ is an arbitrary collection of connected subsets of X such that $A \cap C_i \neq \varnothing$ for all $i \in I$. Then

$$A \cup \bigcup_{i \in I} C_i$$

is a connected subset of X.

4.4. Let X and Y be non-empty topological spaces. Prove that $X \times Y$ is connected if and only if both X and Y are connected.

4.5. (a) Prove that the composition of two closed maps is a closed map.

(b) Prove that a continuous bijection between topological spaces is a homeomorphism if and only if it is closed.

4.6. Prove that every finite topological space is compact.

4.7. Let X be a topological space and let K be a subset of X. We will say that K is a *compact subspace* of X if the subspace topology on $K \subseteq X$ makes K into a compact topological space.

Prove that K is a compact subset of X (as defined at the start of Section 2.5 in the lecture notes) if and only if it is a compact subspace of X (as defined above).

(In other words, compactness is an intrinsic property of topological spaces: it does not depend on the ambient topological space.)

4.8. Let K and L be compact subsets of a topological space X. Prove that $K \cup L$ is compact.

4.9. Let X be a discrete topological space.

(a) Prove that X is compact if and only if X is finite.

(b) Prove that X is connected if and only if X is empty or is a singleton.

4.10. Let X be a compact topological space and let Y be a Hausdorff topological space. Prove that every continuous bijection from X to Y is a homeomorphism.

Recall that a topological space is *totally disconnected* if its only connected subsets are $\varnothing$ and the singletons.

4.11. (*) A topological space X is called *totally separated* if for every pair (x, y) of distinct points in X there exist disjoint clopen neighbourhoods U and V of x and y respectively. Prove that every totally separated space is totally disconnected.

4.12. (*) Prove that the following are totally disconnected:

(a) $\mathbf{Q}$ equipped with the Euclidean topology;

(b) every discrete topological space.