Tutorial Week 5

Topics: Topological groups, sequences.

5.1. Let d_1 and d_2 be equivalent metrics (they define the same topology) on a set X. Prove that a sequence converges to a point x in (X, d_1) if and only if it converges to x in (X, d_2).

5.2. Let (x_n) be a sequence in a metric space X, let $\varphi\colon \mathbf{N} \longrightarrow \mathbf{N}$ be an injective function, and consider the sequence $(y_n) = \big(x_{\varphi(n)}\big)$ in X. Prove that if (x_n) converges to x, then so does (y_n).

Does the converse hold?

5.3. (*) Let $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$ and define

$$\mathcal{T} = \mathcal{P}(\mathbf{N}) \cup \{U \in \mathcal{P}(\mathbf{N}^*)\colon \infty \in U \text{ and } \mathbf{N}^* \setminus U \text{ is finite}\}.$$

(a) Prove that $\mathcal{T}$ is a topology on $\mathbf{N}^*$.

(b) Prove that $(\mathbf{N}^*, \mathcal{T})$ is compact.

(c) Let X be a metric space and $f\colon (\mathbf{N}^*, \mathcal{T}) \longrightarrow X$. Prove that f is continuous if and only if $\big(f(n)\big)$ converges to $f(\infty)$. (In other words, convergent sequences in X are exactly continuous functions from $(\mathbf{N}^*, \mathcal{T})$ to X.)

(d) Let X be a metric space and let (x_n) be a sequence in X that converges to a point x in X. Prove that $\{x\} \cup \{x_n\colon n \in \mathbf{N}\}$ is compact.

5.4. Let (X, d_X) and (Y, d_Y) be metric spaces and let d be the sup norm metric on $X \times Y$:

$$d\big((x_1, y_1), (x_2, y_2)\big) = \max\big(d_X(x_1, x_2), d_Y(y_1, y_2)\big).$$

Prove that $\big((x_n, y_n)\big) \longrightarrow (x, y) \in X \times Y$ if and only if $(x_n) \longrightarrow x \in X$ and $(y_n) \longrightarrow y \in Y$.

5.5. (*) Let G be a topological group and let H be a subgroup of G.

(a) Prove that H is closed if it is open. Does the converse hold?

(b) Prove that H is open if it is closed and has finite index. Does the converse hold?

(c) Suppose G is compact and H is open. Prove that H has finite index.

(d) Is the compactness of G necessary in part (c)?

5.6. (*) Let S and T be subsets of a topological group G. Define

$$ST = \{st\colon s \in S \text{ and } t \in T\}.$$

(a) Suppose S and T are open. Prove that ST is open.

(b) Suppose S and T are connected. Prove that ST is connected.

(c) Suppose S and T are compact. Prove that ST is compact.

(d) Suppose S is compact and T is closed. Prove that ST is closed.

[*Hint*: Use Theorem 2.39 after checking that

$$ST = \pi_2\Big(j^{-1}\big(m^{-1}(T)\big)\Big),$$

where $m\colon G \times G \longrightarrow G$ is the multiplication map of G, j is the inclusion of $S^{-1} \times G$ into $G \times G$, and $\pi_2\colon S^{-1} \times G \longrightarrow G$ is the projection onto the second factor.]

(e) Assuming without proof the fact that $\mathbf{Z} + \pi\mathbf{Z}$ is dense in $\mathbf{R}$, convince yourself that ST need not be closed even if both S and T are.