Tutorial Week 6

Topics: Cauchy sequences, completeness, uniform continuity.

6.1. Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be uniformly continuous functions between metric spaces. Prove that $g \circ f\colon X \longrightarrow Z$ is uniformly continuous.

6.2. Let S be a subset of a metric space (X, d_X) and let d_S be the induced metric on S.

(a) Prove that the inclusion function $\iota_S\colon S \longrightarrow X$ is uniformly continuous.

(b) Prove that a function $f\colon (Y, d_Y) \longrightarrow (S, d_S)$ is uniformly continuous if and only if $\iota_S \circ f$ is uniformly continuous.

6.3. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces and let d be a metric on $Y \times Z$ such that

$$\max\{d_Y(y_1, y_2), d_Z(z_1, z_2)\} \leqslant d\big((y_1, z_1), (y_2, z_2)\big) \leqslant d_Y(y_1, y_2) + d_Z(z_1, z_2)$$

for every pair of points (y_1, z_1) and (y_2, z_2) in $Y \times Z$.

(a) Prove that the projections $\pi_Y\colon Y \times Z \longrightarrow Y$ and $\pi_Z\colon Y \times Z \longrightarrow Z$ are uniformly continuous.

(b) Prove that a function $f\colon X \longrightarrow Y \times Z$ is uniformly continuous if and only if both $\pi_Y \circ f$ and $\pi_Z \circ f$ are.

6.4. Let (X, d_X) and (Y, d_Y) be metric spaces and let d be the sup norm metric on $X \times Y$.

(a) Prove that the sequence $\big((x_n, y_n)\big)$ is Cauchy in $X \times Y$ if and only if (x_n) is Cauchy in X and (y_n) is Cauchy in Y.

(b) Prove that if X and Y are complete then $X \times Y$ is complete. Is the converse true?

6.5. Suppose $f\colon X \longrightarrow Y$ is a *uniform homeomorphism* between metric spaces; that is, a homeomorphism such that both f and its inverse are uniformly continuous.

(a) Prove that a sequence (x_n) is Cauchy in X if and only if $\big(f(x_n)\big)$ is Cauchy in Y.

(b) Prove that X is complete if and only if Y is complete.

(c) Prove that $f\colon \mathbf{R} \longrightarrow (-\pi/2, \pi/2)$ given by $f(x) = \arctan(x)$ is uniformly continuous and a homeomorphism, but it is not a uniform homeomorphism.

(d) Do you feel strongly that uniformly continuous functions ought to preserve completeness? (After all, they preserve Cauchy sequences, and completeness is defined in terms of Cauchy sequences.)

Prove that the function f defined in part (c) does not preserve completeness though it is uniformly continuous and a homeomorphism.

6.6. Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \longrightarrow Y$ a surjective continuous function. Suppose that X is complete and for all $x_1, x_2 \in X$ we have

$$d_X\big(x_1, x_2\big) \leqslant d_Y\big(f(x_1), f(x_2)\big).$$

Prove that Y is complete.

In particular, distance-preserving maps preserve completeness.

6.7. Let (X, d) be a metric space.

(a) Fix an arbitrary element $y \in X$ and consider the function $f\colon X \longrightarrow \mathbf{R}$ given by $f(x) = d(x, y)$. Prove that f is uniformly continuous.

(b) Prove that $d\colon X \times X \longrightarrow \mathbf{R}$ is uniformly continuous with respect to the sup metric D on $X \times X$.