

Tutorial Week 6

Topics: Cauchy sequences, completeness, uniform continuity.

6.1. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be uniformly continuous functions between metric spaces. Prove that $g \circ f: X \rightarrow Z$ is uniformly continuous.

6.2. Let S be a subset of a metric space (X, d_X) and let d_S be the induced metric on S .

- (a) Prove that the inclusion function $\iota_S: S \rightarrow X$ is uniformly continuous.
- (b) Prove that a function $f: (Y, d_Y) \rightarrow (S, d_S)$ is uniformly continuous if and only if $\iota_S \circ f$ is uniformly continuous.

6.3. Let (X, d_X) , (Y, d_Y) , and (Z, d_Z) be metric spaces and let d be a metric on $Y \times Z$ such that

$$\max\{d_Y(y_1, y_2), d_Z(z_1, z_2)\} \leq d((y_1, z_1), (y_2, z_2)) \leq d_Y(y_1, y_2) + d_Z(z_1, z_2)$$

for every pair of points (y_1, z_1) and (y_2, z_2) in $Y \times Z$.

- (a) Prove that the projections $\pi_Y: Y \times Z \rightarrow Y$ and $\pi_Z: Y \times Z \rightarrow Z$ are uniformly continuous.
- (b) Prove that a function $f: X \rightarrow Y \times Z$ is uniformly continuous if and only if both $\pi_Y \circ f$ and $\pi_Z \circ f$ are.

6.4. Let (X, d_X) and (Y, d_Y) be metric spaces and let d be the sup norm metric on $X \times Y$.

- (a) Prove that the sequence $((x_n, y_n))$ is Cauchy in $X \times Y$ if and only if (x_n) is Cauchy in X and (y_n) is Cauchy in Y .
- (b) Prove that if X and Y are complete then $X \times Y$ is complete. Is the converse true?

6.5. Suppose $f: X \rightarrow Y$ is a *uniform homeomorphism* between metric spaces; that is, a homeomorphism such that both f and its inverse are uniformly continuous.

- (a) Prove that a sequence (x_n) is Cauchy in X if and only if $(f(x_n))$ is Cauchy in Y .
- (b) Prove that X is complete if and only if Y is complete.
- (c) Prove that $f: \mathbf{R} \rightarrow (-\pi/2, \pi/2)$ given by $f(x) = \arctan(x)$ is uniformly continuous and a homeomorphism, but it is not a uniform homeomorphism.
- (d) Do you feel strongly that uniformly continuous functions ought to preserve completeness? (After all, they preserve Cauchy sequences, and completeness is defined in terms of Cauchy sequences.)

Prove that the function f defined in part (c) does not preserve completeness though it is uniformly continuous and a homeomorphism.

6.6. Let (X, d_X) and (Y, d_Y) be metric spaces and $f: X \rightarrow Y$ a surjective continuous function. Suppose that X is complete and for all $x_1, x_2 \in X$ we have

$$d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)).$$

Prove that Y is complete.

In particular, distance-preserving maps preserve completeness.

6.7. Let (X, d) be a metric space.

- (a) Fix an arbitrary element $y \in X$ and consider the function $f: X \rightarrow \mathbf{R}$ given by $f(x) = d(x, y)$. Prove that f is uniformly continuous.
- (b) Prove that $d: X \times X \rightarrow \mathbf{R}$ is uniformly continuous with respect to the sup metric D on $X \times X$.