

Tutorial Week 7

Topics: Contractions, (total) boundedness, uniform convergence.

7.1. Find a non-empty metric space X and a contraction $f: X \rightarrow X$ such that f has no fixed points.

7.2. Find a bounded subset of a metric space that is not totally bounded.

7.3.

- (a) Prove that every subspace of a totally bounded space is totally bounded.
- (b) Suppose a metric space X has a totally bounded dense subset D . Prove that X is totally bounded.
- (c) Prove that a metric space X is totally bounded if and only if it is isometric to a subspace of a compact metric space. [*Hint:* Completion.]

7.4. For each $n \in \mathbf{N}$, consider the function $f_n: [0, 1] \rightarrow \mathbf{R}$ given by

$$f_n(x) = \frac{x^2}{1 + nx}.$$

- (a) Prove that f_n is bounded, for all $n \in \mathbf{N}$.
- (b) Find the pointwise limit f of the sequence (f_n) .
- (c) For any $n \in \mathbf{N}$, compute the uniform distance $d_\infty(f_n, f)$.
- (d) Does the sequence (f_n) converge uniformly to f ?

7.5. Let $f_0: \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by

$$f_0(x) = \begin{cases} 1 + x & \text{if } -1 \leq x \leq 0, \\ 1 - x & \text{if } 0 < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each positive integer n , define $f_n: \mathbf{R} \rightarrow \mathbf{R}$ by

$$f_n(x) = f_0(x - n).$$

- (a) Prove that f_n is bounded, for all $n \in \mathbf{N}$.
- (b) Find the pointwise limit f of the sequence (f_n) .
- (c) For any $n \in \mathbf{N}$, compute the uniform distance $d_\infty(f_n, f)$.
- (d) Does the sequence (f_n) converge uniformly to f ?

7.6.

- (a) Prove that every closed interval on \mathbf{R} is compact.
- (b) Prove that every closed ball in \mathbf{R}^n is compact.
- (c) (*The classical Heine–Borel theorem*) Prove that a subset of \mathbf{R}^n is compact if and only if it is bounded and closed.
- (d) Prove that every bounded subset of \mathbf{R}^n is totally bounded.

7.7. Let $A = (a_{ij})$ be an $n \times n$ real matrix with all $|a_{ij}| < 1$. Prove that any real eigenvalue λ of A satisfies $|\lambda| < n$.

[*Hint:* Show that if $|\lambda| \geq n$ then the function $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$ given by $f(v) = \frac{1}{\lambda} Av$ is a contraction for the sup metric topology on \mathbf{R}^n ; then use the Banach Fixed Point Theorem.]