Tutorial Week 7

Topics: Contractions, (total) boundedness, uniform convergence.

7.1. Find a non-empty metric space X and a contraction $f\colon X \longrightarrow X$ such that f has no fixed points.

7.2. Find a bounded subset of a metric space that is not totally bounded.

7.3.

(a) Prove that every subspace of a totally bounded space is totally bounded.

(b) Suppose a metric space X has a totally bounded dense subset D. Prove that X is totally bounded.

(c) Prove that a metric space X is totally bounded if and only if it is isometric to a subspace of a compact metric space. [*Hint*: Completion.]

7.4. For each $n \in \mathbf{N}$, consider the function $f_n\colon [0,1] \longrightarrow \mathbf{R}$ given by

$$f_n(x) = \frac{x^2}{1+nx}.$$

(a) Prove that f_n is bounded, for all $n \in \mathbf{N}$.

(b) Find the pointwise limit f of the sequence (f_n).

(c) For any $n \in \mathbf{N}$, compute the uniform distance $d_\infty(f_n, f)$.

(d) Does the sequence (f_n) converge uniformly to f?

7.5. Let $f_0\colon \mathbf{R} \longrightarrow \mathbf{R}$ be the function defined by

$$f_0(x) = \begin{cases} 1+x & \text{if } -1 \leqslant x \leqslant 0, \\ 1-x & \text{if } 0 < x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each positive integer n, define $f_n\colon \mathbf{R} \longrightarrow \mathbf{R}$ by

$$f_n(x) = f_0(x-n).$$

(a) Prove that f_n is bounded, for all $n \in \mathbf{N}$.

(b) Find the pointwise limit f of the sequence (f_n).

(c) For any $n \in \mathbf{N}$, compute the uniform distance $d_\infty(f_n, f)$.

(d) Does the sequence (f_n) converge uniformly to f?

7.6.

(a) Prove that every closed interval on $\mathbf{R}$ is compact.

(b) Prove that every closed ball in $\mathbf{R}^n$ is compact.

(c) (*The classical Heine–Borel theorem*) Prove that a subset of $\mathbf{R}^n$ is compact if and only if it is bounded and closed.

(d) Prove that every bounded subset of $\mathbf{R}^n$ is totally bounded.

7.7. (*) Let $A = (a_{ij})$ be an $n \times n$ real matrix with all $|a_{ij}| < 1$. Prove that any real eigenvalue λ of A satisfies $|\lambda| < n$.

[*Hint*: Show that if $|\lambda| \geqslant n$ then the function $f\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ given by $f(v) = \frac{1}{\lambda} Av$ is a contraction for the sup metric topology on $\mathbf{R}^n$; then use the Banach Fixed Point Theorem.]