Tutorial Week 8

Topics: normed vector spaces, inner product spaces.

8.1. Let V be a normed vector space. Prove that $(V, +)$ is a topological group.

8.2. Let $\big(V, \|\cdot\|\big)$ be a normed space and let $S \subseteq V$ be a subset. Prove that the closure $\overline{\operatorname{Span}(S)}$ of the span of S is the smallest closed subspace of V that contains S.

8.3. Let v be a non-zero vector in a normed vector space V. Prove that the one-dimensional subspace $\mathbf{F}v := \operatorname{Span}(v)$ of V is isometric to $\mathbf{F}$.

8.4. Let W be a finite-dimensional subspace of a normed vector space V. Prove that W is a closed subset of V.

8.5. Prove that equivalence of norms is an equivalence relation.

8.6. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on a vector space V.

(a) Prove that the identity function $\mathrm{id}_V \colon (V, \|\cdot\|_1) \longrightarrow (V, \|\cdot\|_2)$ is uniformly continuous.

(b) Prove that $(V, \|\cdot\|_1)$ is Banach if and only if $(V, \|\cdot\|_2)$ is Banach.

8.7. Prove that the following norms on $\mathbf{R}^n$ are not defined by inner products:

(a) the ℓ^1-norm defined by

$$\big\|(x_1, \ldots, x_n)\big\|_1 = \sum_{i=1}^{n} |x_i|,$$

(b) the ℓ^∞-norm defined by

$$\big\|(x_1, \ldots, x_n)\big\|_\infty = \max\big\{|x_1|, \ldots, |x_n|\big\}.$$

8.8. Let $(V, \langle\cdot,\cdot\rangle)$ be a complex inner product space and let $T \colon V \longrightarrow V$ be a linear operator. Show that $T = 0$ if and only if $\langle Tv, v\rangle = 0$ for every vector v in V.

Is this true for real inner product spaces?