# Tutorial Week 9

**Topics:** sequence spaces, series.

**9.1** (Hölder's Inequality for $\ell^1$ and $\ell^\infty$)**.** Prove that if $u = \big(u_n\big) \in \ell^\infty$ and $v = \big(v_n\big) \in \ell^1$, then

$$\sum_{n=1}^{\infty} |u_n v_n| \leqslant \|u\|_{\ell^\infty} \|v\|_{\ell^1}.$$

**9.2.** Prove that the norms on the sequence spaces $\ell^\infty$ and $\ell^p$ for $p \neq 2$ cannot defined by inner products.

**9.3.** Suppose $1 \leqslant p \leqslant q$. Prove that

$$\ell^p \subseteq \ell^q.$$

Show that if $p < q$ then the inclusion is strict: $\ell^p \subsetneq \ell^q$.

**9.4.** Prove that every finite-dimensional normed vector space is separable.

**9.5.** Let $c_{00}$ be the space of sequences with only finitely many nonzero terms (see Example 3.23), and consider it as a subspace of $\ell^\infty$. Prove that $c_{00}$ is separable.

**9.6.** Consider the subset $c_0 \subseteq \mathbf{F}^{\mathbf{N}}$ of all sequences with limit 0:

$$c_0 = \big\{\big(a_n\big) \in \mathbf{F}^{\mathbf{N}} : \big(a_n\big) \longrightarrow 0\big\}.$$

(a) Prove that $c_0$ is a closed subspace of $\ell^\infty$.

(b) Conclude that $c_0$ is a Banach space.

(c) Prove that $c_0$ is separable.

**9.7.** Consider the space $\ell^\infty$ of bounded sequences.

(a) Let $S \subseteq \ell^\infty$ be the subset of sequences $\big(a_n\big)$ such that $a_n \in \{0, 1\}$ for all $n \in \mathbf{N}$. Prove that $S$ is an uncountable set.

[*Hint*: Mimic Cantor's diagonal argument.]

(b) Use $S$ to construct an uncountable set $T$ of disjoint open balls in $\ell^\infty$.

(c) Conclude that $\ell^\infty$ is not separable.

**9.8.** Give an example of a series that converges but does not converge absolutely.

**9.9.** If a series $\sum_{n=1}^{\infty} a_n$ in a normed space $\big(V, \|\cdot\|\big)$ converges absolutely, then

$$\left\|\sum_{n=1}^{\infty} a_n\right\| \leqslant \sum_{n=1}^{\infty} \|a_n\|.$$