

Tutorial Week 9

Topics: sequence spaces, series.

9.1 (Hölder's Inequality for ℓ^1 and ℓ^∞). Prove that if $u = (u_n) \in \ell^\infty$ and $v = (v_n) \in \ell^1$, then

$$\sum_{n=1}^{\infty} |u_n v_n| \leq \|u\|_{\ell^\infty} \|v\|_{\ell^1}.$$

9.2. Prove that the norms on the sequence spaces ℓ^∞ and ℓ^p for $p \neq 2$ cannot be defined by inner products.

9.3. Suppose $1 \leq p \leq q$. Prove that

$$\ell^p \subseteq \ell^q.$$

Show that if $p < q$ then the inclusion is strict: $\ell^p \subsetneq \ell^q$.

9.4. Prove that every finite-dimensional normed vector space is separable.

9.5. Let c_{00} be the space of sequences with only finitely many nonzero terms (see [Example 3.23](#)), and consider it as a subspace of ℓ^∞ . Prove that c_{00} is separable.

9.6. Consider the subset $c_0 \subseteq \mathbf{F}^{\mathbf{N}}$ of all sequences with limit 0:

$$c_0 = \{(a_n) \in \mathbf{F}^{\mathbf{N}} : (a_n) \rightarrow 0\}.$$

- (a) Prove that c_0 is a closed subspace of ℓ^∞ .
- (b) Conclude that c_0 is a Banach space.
- (c) Prove that c_0 is separable.

9.7. Consider the space ℓ^∞ of bounded sequences.

- (a) Let $S \subseteq \ell^\infty$ be the subset of sequences (a_n) such that $a_n \in \{0, 1\}$ for all $n \in \mathbf{N}$. Prove that S is an uncountable set.

[Hint: Mimic Cantor's diagonal argument.]

- (b) Use S to construct an uncountable set T of disjoint open balls in ℓ^∞ .
- (c) Conclude that ℓ^∞ is not separable.

9.8. Give an example of a series that converges but does not converge absolutely.

9.9. If a series $\sum_{n=1}^{\infty} a_n$ in a normed space $(V, \|\cdot\|)$ converges absolutely, then

$$\left\| \sum_{n=1}^{\infty} a_n \right\| \leq \sum_{n=1}^{\infty} \|a_n\|.$$