Tutorial Week 10

Topics: continuous linear transformations, series, projections.

10.1. Prove that all linear transformations between finite-dimensional normed vector spaces are continuous.

10.2. Let $f_1 \colon V \longrightarrow W_1$ and $f_2 \colon V \longrightarrow W_2$ be two continuous linear transformations between normed vector spaces. Prove that the function $f \colon V \longrightarrow W_1 \times W_2$ defined by $f(v) = \big(f_1(v), f_2(v)\big)$ is a continuous linear transformation.

10.3. Let c_{00} be the space of sequences with only finitely many nonzero terms (see Example 3.23), which is considered as a subspace of ℓ^∞. Let $f \colon c_{00} \longrightarrow \mathbf{F}^{\mathbf{N}}$ be the function defined by $\big(f(v)\big)_n = nv_n$.

(a) Prove that the image of the function f is contained in ℓ^∞.

(b) Let $g \colon c_{00} \longrightarrow \ell^\infty$ be the function defined by $g(v) = f(v)$. Prove that g is not continuous.

(c) Prove that there exists a discontinuous linear transformation from ℓ^∞ to itself.

In this part, you can use the following fact:

Let V and W be $\mathbf{F}$-vector spaces. If S is a subspace of V and if $\phi \colon S \longrightarrow W$ is a linear transformation, then there exists a linear transformation $\tilde{\phi} \colon V \longrightarrow W$ such that $\phi = \tilde{\phi}|_S$.

10.4. If $f \in L(V, W)$ with V, W normed spaces, and the series

$$\sum_{n=1}^{\infty} \alpha_n v_n, \qquad \alpha_n \in \mathbf{F}, v_n \in V,$$

converges in V, then the series

$$\sum_{n=1}^{\infty} \alpha_n f(v_n)$$

converges in W to the limit

$$f\left(\sum_{n=1}^{\infty} \alpha_n v_n\right).$$

10.5.

(a) Prove that the function $f \colon \ell^1 \longrightarrow \mathbf{F}$ defined by

$$f\big((a_n)\big) = \sum_{n=1}^{\infty} a_n.$$

is continuous.

(b) Prove that the following subset is a closed subspace of ℓ^1:

$$S = \left\{ (a_n) \in \ell^1 : \sum_{n=1}^{\infty} a_n = 0 \right\}.$$

10.6. Fix $j \in \mathbf{N}$ and consider the map $\pi_j \colon \mathbf{F}^{\mathbf{N}} \longrightarrow \mathbf{F}$ given by

$$\pi_j\big((a_n)\big) = a_j.$$

(a) Show that π_j is linear.

(b) Prove that the restriction of π_j to ℓ^p for $1 \leqslant p \leqslant \infty$ is continuous and surjective.

10.7. Let V be a normed space and φ, ψ be commuting projections: $\varphi \circ \psi = \psi \circ \varphi$. Prove that $\varphi \circ \psi$ is a projection with image $\operatorname{im} \varphi \cap \operatorname{im} \psi$.

10.8. Let φ be a nonzero orthogonal projection (that is, φ is not the constant function 0) on an inner product space V. Prove that $\|\varphi\| = 1$.