Tutorial Week 11

Topics: projections, self-adjoint operators, normal operators.

11.1. Consider the function $g\colon \ell^2 \longrightarrow \mathbf{F}$ given by

$$g(x) = \sum_{n=1}^{\infty} \frac{x_n}{n^2}.$$

(a) Find $y \in \ell^2$ such that

$$g(x) = \langle x, y \rangle \qquad \text{for all } x \in \ell^2.$$

(b) Deduce that g is linear and Lipschitz and find its norm $\|g\|$.

[*Hint*: You may use without proof the fact that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.]

11.2. Let H be a Hilbert space. Prove that a projection $\pi\colon H \longrightarrow H$ is an orthogonal projection if and only if π is self-adjoint.

11.3. Let H be a Hilbert space. Prove that a projection $\pi\colon H \longrightarrow H$ is orthogonal if and only if $\mathrm{id}_H - \pi$ is an orthogonal projection.

11.4. Let $f \in L(H)$ with H a Hilbert space. Suppose that f is invertible with continuous inverse. Then the adjoint f^* is invertible and

$$\left(f^*\right)^{-1} = \left(f^{-1}\right)^*.$$

11.5. Let H be a Hilbert space and let α be a scalar. Prove that $\alpha\,\mathrm{id}_H$ is normal (that is, commutes with its adjoint).

11.6. Let $R\colon \ell^2 \longrightarrow \ell^2$ and $L\colon \ell^2 \longrightarrow \ell^2$ be the operators defined by

$$R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots) \quad \text{and} \quad L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$$

Find the adjoints of R and L and prove that neither R nor L is normal.

11.7. Equip $\mathbf{R}^n$ with the standard Euclidean inner product and let $f\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be a linear transformation with standard matrix representation A.

(a) Prove that the adjoint $f^*\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ has standard matrix representation A^t, the transpose of A.

(b) Prove that f is self-adjoint if and only if A is symmetric.

11.8. Let $f \in L(H)$ with H a Hilbert space. Then the maps

$$p = f^* \circ f \qquad \text{and} \qquad s = f + f^*$$

are self-adjoint.

11.9. Let H be a real Hilbert space. Prove that self-adjoint continuous linear operators on H form a subspace of $L(H)$.

If H is a complex Hilbert space, does the statement still hold? If yes, give a proof for the statement. If no, find a counterexample, and then find and prove a closest statement that holds.

11.10. The composition of two self-adjoint maps on a Hilbert space is self-adjoint if and only if the maps commute.