Tutorial Week 12

Topics: Orthogonal systems, orthogonal bases, the Stone–Weierstrass theorem.

12.1. Let $(u_i)_{i\in I}$ be an orthonormal basis of an inner product space V and let $v \in V$. Prove that $v = 0$ if and only if $\langle v, u_i\rangle = 0$ for every index $i \in I$.

12.2. In this question, we re-examine the Cauchy–Schwarz inequality in retrospect.

Let u be a vector of norm 1 in an inner product space V. Define $\pi_u \colon V \longrightarrow V$ by

$$\pi_u(v) = \langle v, u\rangle u.$$

(a) Prove that π_u is a linear transformation.

(b) Let v be a vector in V. Prove that $\pi_u(v)$ is orthogonal to $(\mathrm{id}_V - \pi_u)(v)$.

(c) Let v be a vector in V. Prove that $\|\pi_u(v)\| = |\langle v, u\rangle|$.

(d) Prove the *Cauchy–Schwarz inequality*: if v and w are vectors in V, then

$$\left|\langle v, w\rangle\right| \leqslant \|v\|\,\|w\|.$$

(e) Prove that π_u is an orthogonal projection with image $\mathbf{F}u$.

12.3. In this question, we generalise the results in Question 12.2.

Let $\{u_1, \ldots, u_n\}$ be an orthonormal system in an inner product space V and let U be the span of the orthonormal system. Write $\pi_1, \ldots, \pi_n$ for the projections $\pi_{u_1}, \ldots, \pi_{u_n}$ defined in Question 12.2 and put

$$\pi = \pi_1 + \cdots + \pi_n.$$

(a) Prove that

$$\pi_i \circ \pi_j = \begin{cases} \pi_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Prove that π is an orthogonal projection with image U.

(c) Let v be a vector in V. Prove that

$$\left\|\pi(v)\right\|^2 = \sum_{i=1}^{n} \left|\langle v, u_n\rangle\right|^2.$$

(d) Use part (c) to prove the following finite version of the *Bessel's inequality*: if v is a vector in V, then

$$\|v\|^2 \geqslant \sum_{i=1}^{n} \left|\langle v, u_i\rangle\right|^2.$$

We say that a subalgebra $\mathcal{C}$ of $C_0(X, \mathbf{F})$ *separates points* if for every pair of points x and y in X there is a function f in $\mathcal{C}$ such that $f(x) \neq f(y)$. We say that a subalgebra $\mathcal{C}$ of $C_0(X, \mathbf{F})$ is *non-vanishing* if for every point x in X there is a function f in $\mathcal{C}$ such that $f(x) \neq 0$.

12.4. **(*)** Let $\mathcal{C}$ be a non-vanishing subalgebra of $C_0(X, \mathbf{F})$ that separates points.

(a) Given two points x and y in X, find a function h in $\mathcal{C}$ such that $h(x) = 0$ and $h(y) \neq 0$.

(b) Prove that $\mathcal{C}$ interpolates pairs of points.

If X is a compact metric space and $f\colon X \longrightarrow \mathbf{C}$ is a function, then we write $\overline{f}\colon X \longrightarrow \mathbf{C}$ for the function defined by

$$\overline{f}(x) = \overline{f(x)}.$$

Given a subalgebra $\mathcal{C}$ of $C_0(X, \mathbf{C})$, we say $\mathcal{C}$ is *closed under complex conjugation* if $f \in \mathcal{C}$ implies $\overline{f} \in \mathcal{C}$.

12.5. Let X be a compact metric space and let $\mathcal{C}$ be a non-vanishing subalgebra of $C_0(X, \mathbf{C})$. Suppose $\mathcal{C}$ is closed under complex conjugation and separates points.

(a) Let $\mathcal{C}_{\mathbf{R}} = \mathcal{C} \cap C_0(X, \mathbf{R})$. Prove that $\mathcal{C}_{\mathbf{R}}$ is dense in $C_0(X, \mathbf{R})$.

(b) Prove that $\mathcal{C}$ is dense in $C_0(X, \mathbf{C})$.

12.6. (*) Let $(u_i)_{i \in I}$ be an orthonormal basis of an inner product space V (not necessarily separable) and let v be a vector in V.

(a) Given a a positive integer n, define

$$J_n = \left\{ i \in I \,\middle|\, \left|\langle v, u_i \rangle\right| > \frac{1}{n} \right\}.$$

Prove that J_n has at most $n^2 \|v\|^2$ elements.

(b) Put

$$I_v = \left\{ i \in I \,\middle|\, |\langle v, u_i \rangle| \neq 0 \right\}.$$

Prove that I_v is countable.

(c) Choose a bijection $o\colon \mathbf{N} \longrightarrow I_v$. Prove that

$$v = \sum_{n=1}^{\infty} \left\langle v, u_{o(n)} \right\rangle u_{o(n)}.$$

(d) Justify the notation

$$\sum_{i \in I} \langle v, u_i \rangle u_i$$

and convince yourself that

$$v = \sum_{i \in I} \langle v, u_i \rangle u_i.$$