

Tutorial Week 2

Topics: metrics, topologies, continuous functions.

2.1. Let X be a set and d the discrete metric on X , that is $d(x_1, x_2) = 1$ for all $x_1 \neq x_2$; see also [Exercise 2.6](#). Prove that the topology defined by d is the discrete topology.

Solution. By the definition of the discrete metric, we have $\mathbf{B}_1(x) = \{x\}$ for every element x of X . If S is a subset of X , then

$$S = \bigcup_{x \in S} \{x\} = \bigcup_{x \in S} \mathbf{B}_1(x),$$

and thus S is open. Therefore, every subset of X is open in (X, d) ; in other words, the topology defined by d is the discrete topology. \square

2.2. Is the word “finite” necessary in the statement of [Proposition 2.12](#)? If no, give a proof of the statement without “finite”. If yes, give an example of an infinite collection of open sets whose intersection is not an open set.

Solution. The word “finite” is necessary. For a counterexample to the more general statement, for each $n \in \mathbf{Z}_{\geq 1}$ take $U_n = (-1/n, 1/n)$ as an open set in \mathbf{R} with the Euclidean metric. I claim that

$$U := \bigcap_{n \in \mathbf{Z}_{\geq 1}} U_n = \{0\}.$$

This can be proved by contradiction: suppose $u \in U$, $u \neq 0$. Let $n \in \mathbf{Z}_{\geq 1}$ be such that $n \geq \frac{1}{|u|}$. Then $|u| \geq \frac{1}{n}$, therefore $u \notin (-1/n, 1/n) = U_n$, contradiction.

Finally, U is not open: for any $r \in \mathbf{R}_{>0}$, $\frac{r}{2} \in \mathbf{B}_r(0)$ but $\frac{r}{2} \notin \{0\} = U$, so $\mathbf{B}_r(0)$ is not a subset of U . \square

2.3. Find all topologies on the set $\{0, 1\}$ and determine which of them is metrisable.

Solution. Let \mathcal{T} be a topology on $\{0, 1\}$. Since \emptyset and $\{0, 1\}$ must belong to \mathcal{T} , there are four possibilities:

- $\mathcal{T} = \mathcal{P}(\{0, 1\})$. This is the discrete topology on $\{0, 1\}$.
- $\mathcal{T} = \{\emptyset, \{1\}, \{0, 1\}\}$. Since \mathcal{T} is finite, it suffices to verify it is closed under binary intersection and binary union. We can prove this by enumeration:

$$\begin{array}{lll} \emptyset \cap \{1\} = \emptyset, & \emptyset \cap \{0, 1\} = \emptyset, & \{1\} \cap \{0, 1\} = \{1\}, \\ \emptyset \cup \{1\} = \{1\}, & \emptyset \cup \{0, 1\} = \{0, 1\}, & \{1\} \cup \{0, 1\} = \{0, 1\}. \end{array}$$

Therefore, \mathcal{T} is a topology.

- $\mathcal{T} = \{\emptyset, \{0\}, \{0, 1\}\}$. This can be proved to be a topology similarly.
- $\mathcal{T} = \{\emptyset, \{0, 1\}\}$. This is the trivial topology on $\{0, 1\}$.

Now suppose d is a metric on $\{0, 1\}$. Since $d(0, 1) > 0$, we can pick a positive real number r smaller than $d(0, 1)$. It follows that

$$\mathbf{B}_r(0) = \{0\} \quad \text{and} \quad \mathbf{B}_r(1) = \{1\},$$

and the metric topology defined by d is thus the discrete topology. Therefore, the only metrisable topology on $\{0, 1\}$ is the discrete topology. \square

2.4. Let X be a set and S a subset of $\mathcal{P}(X)$. Prove that the topology generated by S is the intersection of all topologies \mathcal{T} on X containing S , and is thus the coarsest among such topologies.

Solution. Let \mathcal{T}_S be the topology generated by S and \mathcal{T}'_S the intersection of all topologies \mathcal{T} on X containing S .

We start with proving \mathcal{T}'_S is a topology:

- Both \emptyset and X belong to all topologies containing S , and thus belong to the intersection \mathcal{T}'_S .
- If $\{U_i \in \mathcal{T}'_S : i \in I\}$ is a collection of members of \mathcal{T}'_S , then $U_i \in \mathcal{T}$ for every $i \in I$ and every topology \mathcal{T} containing S . It follows that $\bigcup_{i \in I} U_i \in \mathcal{T}$ for every topology \mathcal{T} containing S , and thus $\bigcup_{i \in I} U_i \in \mathcal{T}'_S$.
- If U_1, \dots, U_n are members of \mathcal{T}'_S , then they belong to every topology \mathcal{T} containing S . It follows that $\bigcap_{i=1}^n U_i \in \mathcal{T}$ for every topology \mathcal{T} containing S , and thus $\bigcap_{i=1}^n U_i \in \mathcal{T}'_S$.

It follows from the definition of \mathcal{T}_S that $S \subseteq \mathcal{T}_S$, so \mathcal{T}_S is finer than \mathcal{T}'_S . However, for \mathcal{T}'_S to be a topology, it has to be closed under arbitrary union and finite intersection, and thus contains all members of \mathcal{T}_S ; in other words, \mathcal{T}'_S has to be finer than \mathcal{T}_S . Hence $\mathcal{T}'_S = \mathcal{T}_S$. \square

2.5. Let X and Y be two topological spaces, where the topology on X is the discrete topology. Prove that every function from X to Y is continuous.

Solution. Consider a function $f: X \rightarrow Y$. Since the topology on X is discrete, it follows that $f^{-1}(U)$ is open for every open subset U of Y , and thus f is continuous. \square

2.6. Let $f: X \rightarrow Y$ be a function between topological spaces. Suppose the topology on Y is generated by a subset S of $\mathcal{P}(Y)$. Prove that the function f is continuous if and only if $f^{-1}(U)$ is open for every element U of S .

Solution. The ‘only if’ part follows directly from the definition of continuity.

Conversely, suppose that the inverse image of every member of S is open. It follows that the final topology \mathcal{T}'_Y induced by f (see [Exercise 2.14](#)) contains S , and is thus finer than \mathcal{T}_Y by [Question 2.4](#). By part (b) of [Exercise 2.14](#), this implies that f is continuous. \square

2.7. Let $f: X \rightarrow Y$ be a function and \mathcal{T}_Y a topology on Y . Define

$$\mathcal{T}_X = \{f^{-1}(U) : U \in \mathcal{T}_Y\}.$$

- (a) Prove that \mathcal{T}_X is the coarsest topology on X such that f is continuous. (This topology is called the *initial topology* induced by f .)
- (b) Let \mathcal{T} be another topology on X . Prove that $f: (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}_Y)$ is continuous if and only if \mathcal{T} is finer than \mathcal{T}_X .
- (c) Use an example to prove that \mathcal{T}_X need not be metrisable even when \mathcal{T}_Y is a metric topology.
- (d) Give an example in which \mathcal{T}_X is metrisable but \mathcal{T}_Y is not.
- (e) Suppose \mathcal{T}_Y is generated by a subset S of $\mathcal{P}(Y)$. Prove that \mathcal{T}_X is generated by the set

$$\{f^{-1}(U) : U \in S\}.$$

[Hint: For (c) and (d), consider using [Question 2.3](#).]

Note: There is a “dual” setting where you start with a topology on X and look for the finest topology on Y such that f is continuous, see [Exercise 2.14](#).

Solution.

(a) We start with proving that \mathcal{T}_X is a topology:

- Since $\emptyset = f^{-1}(\emptyset)$ and $X = f^{-1}(Y)$, it follows that \mathcal{T}_X contains \emptyset and X .
- If $\{f^{-1}(U_i) : i \in I\}$ is a collection of members of \mathcal{T}_X , then

$$\bigcup_{i \in I} f^{-1}(U_i) = f^{-1}\left(\bigcup_{i \in I} U_i\right) \in \mathcal{T}_X.$$

- If $f^{-1}(U_1), \dots, f^{-1}(U_n)$ are members of \mathcal{T}_X , then

$$\bigcap_{i=1}^n f^{-1}(U_i) = f^{-1}\left(\bigcap_{i=1}^n U_i\right) \in \mathcal{T}_X.$$

If \mathcal{T} is a topology on X such that f is continuous, then $f^{-1}(U) \in \mathcal{T}$ for every member U of \mathcal{T}_Y , and thus $\mathcal{T}_X \subseteq \mathcal{T}$. Therefore, \mathcal{T}_X is the coarsest topology such that f is continuous.

(b) The ‘only if’ part has been proven in part (a), so it suffices to prove the ‘if’ part.

Suppose \mathcal{T} is finer than \mathcal{T}_X . If U is a member of \mathcal{T}_Y , then $f^{-1}(U) \in \mathcal{T}_X \subseteq \mathcal{T}$. Hence f is continuous.

(c) Put $X = \{0, 1\}$, $Y = \{1\}$, $\mathcal{T}_Y = \mathcal{P}(Y)$. Let $f: X \rightarrow Y$ be the function sending both 0 and 1 to 1. It follows that $\mathcal{T}_X = \{\emptyset, \{0, 1\}\}$. The topology \mathcal{T}_Y is defined by the discrete metric (see [Question 2.1](#)), but \mathcal{T}_X is not metrisable (see [Question 2.3](#)).

(d) Put $X = \{1\}$, $Y = \{0, 1\}$, $\mathcal{T}_Y = \{\emptyset, \{1\}, \{0, 1\}\}$. Let $f: X \rightarrow Y$ be the inclusion function, which sends 1 to 1. It follows that $\mathcal{T}_X = \mathcal{P}(X)$. The topology \mathcal{T}_X is defined by the discrete metric (see [Question 2.1](#)), but \mathcal{T}_Y is not metrisable (see [Question 2.3](#)).

(e) Let \mathcal{T}'_X be the topology on X generated by the set

$$\{f^{-1}(U) : U \in S\}.$$

Since the topology \mathcal{T}_X contains $f^{-1}(U)$ for every member U of S , it follows from [Question 2.4](#) that $\mathcal{T}'_X \subseteq \mathcal{T}_X$. By [Question 2.6](#), f is continuous when the topology on X is \mathcal{T}'_X , so part (a) implies that $\mathcal{T}_X \subseteq \mathcal{T}'_X$. Hence $\mathcal{T}'_X = \mathcal{T}_X$. \square

2.8. Prove that a function $f: X \rightarrow Y$ between metric spaces is continuous if and only if it satisfies the usual ϵ - δ definition: for every point x of X and every positive real number ϵ , there exists a positive real number δ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$.

Solution. It follows from the definition of open balls that the condition ‘ $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$ ’ means $f(\mathbf{B}_\delta(x)) \subseteq \mathbf{B}_\epsilon(f(x))$. We will use the rephrased statement in this proof.

Suppose $f: X \rightarrow Y$ is continuous. If $x \in X$ and ϵ is a positive real number, then the inverse image of $\mathbf{B}_\epsilon(f(x))$ in X is open, and thus contains $\mathbf{B}_\delta(x)$ for some positive real number δ . It follows that $f(\mathbf{B}_\delta(x)) \subseteq \mathbf{B}_\epsilon(f(x))$.

Conversely, suppose f satisfies the usual ϵ - δ definition and consider an open subset U of Y . If $f(x) \in U$ for some element x of X , then the openness of U implies the existence of positive real number ϵ such that $\mathbf{B}_\epsilon(f(x)) \subseteq U$. Since f satisfies the usual ϵ - δ definition, there exists a positive real number δ such that $f(\mathbf{B}_\delta(x)) \subseteq \mathbf{B}_\epsilon(f(x)) \subseteq U$, which implies $\mathbf{B}_\delta(x) \subseteq f^{-1}(U)$. It follows that $f^{-1}(U)$ is open in X , and thus f is continuous. \square

2.9.

- (a) Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be functions, where X, Y, Z are sets, and let $S \subseteq Z$. Then

$$f^{-1}(g^{-1}(S)) = (g \circ f)^{-1}(S).$$

- (b) Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be continuous functions, where X, Y, Z are topological spaces. Prove that $g \circ f: X \rightarrow Z$ is continuous.

Solution.

- (a) We have $x \in (g \circ f)^{-1}(S)$ iff $(g \circ f)(x) \in S$ iff $g(f(x)) \in S$ iff $f(x) \in g^{-1}(S)$ iff $x \in f^{-1}(g^{-1}(S))$.
- (b) Let $W \subseteq Z$ be open. As $g: Y \rightarrow Z$ is continuous, $g^{-1}(W) \subseteq Y$ is open. As $f: X \rightarrow Y$ is continuous, $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \subseteq X$ is open. So $g \circ f$ is continuous. \square