Tutorial Week 3

Topics: Equivalent metrics, local continuity, closure, interior, denseness, product, Hausdorff.

3.1. Let X be a topological space. Prove that a subset U of X is open if and only if it is a neighbourhood of every element of itself.

Solution. If U is open, then it is an open neighbourhood of its elements by definition.

Conversely, suppose U is a neighbourhood of every element of itself. If x is an element of U, then U contains some open neighbourhood V_x of x. Now $U = \bigcup_{x \in U} V_x$, so U is open. □

3.2. Let (X, d) be a metric space.

(a) Prove that the metric topology on (X, d) is generated by open balls of radii smaller than 1.

(b) Define $d' \colon X \times X \longrightarrow \mathbf{R}_{\geqslant 0}$ by

$$d'(x, y) = \min\left\{d(x, y), 1\right\}.$$

Prove that d' is a metric.

(c) Prove that d and d' are equivalent (that is, they give rise to the same topology on X).

Solution. (a) Since the metric topology on (X, d) is generated by open balls of arbitrary radii, it suffices to prove that every open ball is in the topology generated by open balls of radii smaller than 1. Let x be a point in X and let r be an arbitrary positive real number. If $y \in \mathbf{B}_r(x)$, then $d(x, y) < r$, so $\mathbf{B}_{r-d(x,y)}(y) \subseteq \mathbf{B}_r(x)$ by the triangle inequality. It follows that

$$\mathbf{B}_r(x) = \bigcup_{y \in \mathbf{B}_r(x)} \mathbf{B}_{r-d(x,y)}(y) = \bigcup_{y \in \mathbf{B}_r(x)} \mathbf{B}_{r(y)}(y),$$

where $r(y) = \min\{r - d(x, y), 1\}$. Hence $\mathbf{B}_r(x)$ is in the topology generated by open balls of radii smaller than 1.

(b) It is clear that $d'(y, x) = d'(x, y)$ and that $d'(x, y) = 0$ if and only if $d(x, y) = 0$ if and only if $x = y$.

For the triangle inequality: $d'(x, y) \leqslant 1$ so if at least one of $d'(x, t)$, $d'(t, y)$ is 1, the triangle inequality holds. So we may assume that $d'(x, t) = d(x, t)$ and $d'(t, y) = d(t, y)$. Then

$$d'(x, y) \leqslant d(x, y) \leqslant d(x, t) + d(t, y) = d'(x, t) + d'(t, y).$$

(c) It follows from the definition of d that $\mathbf{B}_r^d(x) = \mathbf{B}_r^{d'}(x)$ for every point x in X and every positive real number r smaller than 1. It then follows from part (a) that the metric topologies induced by d and d' are generated by the same collection of open balls, so the two metric topologies are the same. Hence d and d' are equivalent. □

3.3. Let $f \colon X \longrightarrow Y$ be a function between topological spaces. Given $x \in X$, we say that f is *continuous at* x if the inverse image $f^{-1}(N)$ of every neighbourhood N of $f(x)$ is a neighbourhood of x. Prove that f is continuous if and only if it is continuous at every $x \in X$.

Solution. Suppose $f\colon X \longrightarrow Y$ is continuous. If x is a point in X and N is a neighbourhood of $f(x)$, then N contains some open neighbourhood U of $f(x)$, whose inverse image $f^{-1}(U)$ is an open neighbourhood of x because of continuity. Since $f^{-1}(U) \subseteq f^{-1}(N)$, it follows that $f^{-1}(N)$ is a neighbourhood of x.

Conversely, suppose $f\colon X \longrightarrow Y$ is continuous at every point of X. If U be an open subset of Y, then $f^{-1}(U)$ is a neighbourhood of every element of itself. By Question 3.1, this implies $f^{-1}(U)$ is open. Hence f is continuous. $\square$

3.4. Let A and B be subsets of a topological space X.

(a) Suppose $A \subseteq B$. Prove that $\overline{A} \subseteq \overline{B}$ and $A^\circ \subseteq B^\circ$.

(b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(c) Prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Find an example in which $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

(d) Prove that $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$. Find an example in which $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.

[*Hint*: For (c) and (d), think of some subsets of $\mathbf{R}$.]

Solution.

(a) Since $A \subseteq B \subseteq \overline{B}$ and $\overline{A}$ is the smallest closed subset of X containing A, it follows that $\overline{A} \subseteq \overline{B}$.

Since $A^\circ \subseteq A \subseteq B$ and B° is the largest open subset of B, it follows that $A^\circ \subseteq B^\circ$.

(b) Since $A \cup B \subseteq \overline{A} \cup \overline{B}$, it follows that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

For the other inclusion, $A \subseteq A \cup B \subseteq \overline{A \cup B}$ implies $\overline{A} \subseteq \overline{A \cup B}$, and similarly we have $\overline{B} \subseteq \overline{A \cup B}$. Hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Since $A^\circ \cap B^\circ \subseteq A \cap B$, it follows that $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$.

For the other inclusion, $(A \cap B)^\circ \subseteq A \cap B \subseteq A$ implies $(A \cap B)^\circ \subseteq A^\circ$, and similarly we have $(A \cap B)^\circ \subseteq B^\circ$. Hence $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.

(c) Both $\overline{A}$ and $\overline{B}$ contain $A \cap B$, so $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

For the example, let $A = \mathbf{Q}$, $B = \mathbf{R} \smallsetminus \mathbf{Q}$, and $X = \mathbf{R}$. It follows that $\overline{A} = \overline{B} = X$ (see Example 2.26), so $X = \overline{A} \cap \overline{B}$, but $\overline{A \cap B} = A \cap B = \varnothing$.

(d) Since $A \cup B$ contains both A and B, it follows from part (a) that $(A \cup B)^\circ \supseteq A^\circ$ and $(A \cup B)^\circ \supseteq B^\circ$, and hence $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$.

For the example, let $A = \mathbf{Q}$, $B = \mathbf{R} \smallsetminus \mathbf{Q}$, and $X = \mathbf{R}$. It follows that $(A \cup B)^\circ = X^\circ = X$, but $A^\circ \cup B^\circ = \varnothing$ because $A^\circ = B^\circ = \varnothing$. $\square$

3.5. Prove that $\mathbf{Z}$ is a nowhere dense subset of $\mathbf{R}$.

Solution. First we show that $\overline{\mathbf{Z}} = \mathbf{Z}$: letting $U = \mathbf{R} \smallsetminus \mathbf{Z}$, we have

$$U = \bigcup_{n \in \mathbf{Z}} (n-1, n),$$

so U is a union of open subsets, hence open.

Now we note that $\mathbf{Z}^\circ = \varnothing$: if $V \subseteq \mathbf{R}$ is a nonempty open subset, then V contains a nonempty open interval, hence is uncountable, so it cannot be contained in $\mathbf{Z}$. $\square$

3.6. Let X be a topological space.

(a) Prove that any subset of a nowhere dense subset of X is nowhere dense in X.

(b) Prove that a subset $N \subseteq X$ is nowhere dense if and only if $X \setminus \overline{N}$ is dense in X.

(c) Prove that the union of any finite collection of nowhere dense subsets of X is nowhere dense in X.

Solution.

(a) Let $N \subseteq X$ be nowhere dense and let $M \subseteq N$. Then $\overline{M} \subseteq \overline{N}$ by part (a) of Question 3.4, so $\left(\overline{M}\right)^\circ \subseteq \left(\overline{N}\right)^\circ = \varnothing$ by part (a) of Question 3.4.

(b) Suppose N is nowhere dense and let $U \subseteq X$ be nonempty and open. If $U \cap \left(X \setminus \overline{N}\right) = \varnothing$, then $U \subseteq \overline{N}$, so $U \subseteq \left(\overline{N}\right)^\circ = \varnothing$, contradicting the non-emptiness of U. So it must be that U intersects $X \setminus \overline{N}$ nontrivially, hence $X \setminus \overline{N}$ is dense.

Conversely, suppose $X \setminus \overline{N}$ is dense but N is not nowhere dense, that is there exists a nonempty open $U \subseteq \overline{N}$. Then $U \cap \left(X \setminus \overline{N}\right) = \varnothing$, contradicting the denseness of $X \setminus \overline{N}$.

(c) It suffices to prove the case of two nowhere dense sets M and N. Let $L = M \cup N$. Then by part (b) of Question 3.4 we have $\overline{L} = \overline{M} \cup \overline{N}$ so $X \setminus \overline{L} = \left(X \setminus \overline{M}\right) \cap \left(X \setminus \overline{N}\right)$. As $X \setminus \overline{L}$ is the intersection of two dense open subsets, it is dense and open by Exercise 2.24, hence L is nowhere dense. □

3.7. Let X, Y_1, and Y_2 be topological spaces, and $\pi_1 : Y_1 \times Y_2 \longrightarrow Y_1$ and $\pi_2 : Y_1 \times Y_2 \longrightarrow Y_2$ be the projections. Prove that a function $f : X \longrightarrow Y_1 \times Y_2$ is continuous if and only if both $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Solution. Suppose f is continuous. By Proposition 2.19, both π_1 and π_2 are continuous. It then follows from Question 2.9 that $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Conversely, suppose $\pi_1 \circ f$ and $\pi_2 \circ f$ are both continuous. If U_1 and U_2 are open subsets of Y_1 and Y_2 respectively, then $f^{-1}(U_1 \times U_2) = (\pi_1 \circ f)^{-1}(U_1) \cap (\pi_2 \circ f)^{-1}(U_2)$ because

$$\begin{aligned} x \in f^{-1}(U_1 \times U_2) &\iff \big((\pi_1 \circ f)(x), (\pi_2 \circ f)(x)\big) \in U_1 \times U_2 \\ &\iff (\pi_1 \circ f)(x) \in U_1 \text{ and } (\pi_2 \circ f)(x) \in U_2 \\ &\iff x \in (\pi_1 \circ f)^{-1}(U_1) \text{ and } x \in (\pi_2 \circ f)^{-1}(U_2) \\ &\iff x \in (\pi_1 \circ f)^{-1}(U_1) \cap (\pi_2 \circ f)^{-1}(U_2). \end{aligned}$$

Since $\pi_1 \circ f$ and $\pi_2 \circ f$ are both continuous, it follows that $(\pi_1 \circ f)^{-1}(U_1)$ and $(\pi_2 \circ f)^{-1}(U_2)$ are both open in X; hence $f^{-1}(U_1 \times U_2)$ is open. The topology on $Y_1 \times Y_2$ is generated by rectangles, so it follows from Question 2.6 that f is continuous. □

3.8. Let X and Y be topological spaces and let A and B be subsets of X and Y respectively.

(a) Suppose A and B are closed in X and Y respectively. Prove that if A and B are closed, then $A \times B$ is closed.

(b) Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.

Solution. (a) Since A and B are closed in X and Y respectively, their complements $X \setminus A$ and $Y \setminus B$ are open in X and Y respectively, and therefore $(X \setminus A) \times Y$ and $X \times (Y \setminus B)$ are open in $X \times Y$. It follows that

$$(X \times Y) \setminus (A \times B) = \big((X \setminus A) \times Y\big) \cup \big(X \times (Y \setminus B)\big)$$

is closed in $X \times Y$.

(b) By part (a), $\overline{A} \times \overline{B}$ is closed in $X \times Y$. Since $A \times B \subseteq \overline{A} \times \overline{B}$, it follows that $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$. It remains to prove the other inclusion.

Given an element x of A, define $\iota_x \colon Y \longrightarrow X \times Y$ by $\iota_x(y) = (x, y)$. Let $\pi_X \colon X \times Y \longrightarrow X$ and $\pi_Y \colon X \times Y \longrightarrow Y$ be the projections. The composite function $\pi_X \circ \iota_x$ is the constant function sending every element of Y to x, which is continuous by Exercise 2.21; while $\pi_Y \circ \iota_x$ is the identity function of Y, which is continuous by Exercise 2.23. it then follows from Question 3.7 that ι_x is continuous.

Since $\overline{A \times B}$ is closed in $X \times Y$, it follows from Exercise 2.13 that $\iota_x^{-1}(\overline{A \times B})$ is closed. Now $B \subseteq \iota_x^{-1}(\overline{A \times B})$ implies $\overline{B} \subseteq \iota_x^{-1}(\overline{A \times B})$; in other words, $\{x\} \times \overline{B} \subseteq \overline{A \times B}$. Since x is an arbitrary point in A, this implies $A \times \overline{B} \subseteq \overline{A \times B}$.

Following similar reasoning for points in $\overline{B}$, we can show that $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$. □

3.9. Given a set X, define the *diagonal function*

$$\Delta \colon X \longrightarrow X \times X, \qquad x \longmapsto (x, x).$$

(a) Prove that two subsets A and B of X are disjoint if and only if $\Delta(X)$ and $A \times B$ are disjoint.

(b) If X is a topological space, prove that Δ is continuous.

(c) Prove that a topological space X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.

Solution.

(a) This follows from

$$x \in A \cap B \iff x \in A \text{ and } x \in B \iff (x, x) \in \Delta(X) \cap (A \times B).$$

(b) Let $\mathrm{id}_X \colon X \longrightarrow X$ denote the identity function of X, defined by $\mathrm{id}_X(x) = x$. This function is continuous by Exercise 2.23.

Let $\pi_1, \pi_2 \colon X \times X \longrightarrow X$ be the projections. Since $\pi_1 \circ \Delta = \pi_2 \circ \Delta = \mathrm{id}_X$, it follows from Question 3.7 that Δ is continuous.

(c) Suppose X is Hausdorff. We will prove that $(X \times X) \smallsetminus \Delta(X)$ is open. If $(x, y) \in (X \times X) \smallsetminus \Delta(X)$, then $x \neq y$, so there exist disjoint open neighbourhoods U of x and V of y. It follows from part (a) that $U \times V$ does not intersect $\Delta(X)$, and is thus a subset of $(X \times X) \smallsetminus \Delta(X)$. This implies $(X \times X) \smallsetminus \Delta(X)$ is open, and hence $\Delta(X)$ is closed.

Conversely, suppose $\Delta(X)$ is closed; in other words, $(X \times X) \smallsetminus \Delta(X)$ is open. Let x and y be two distinct points in X. Since $(x, y) \in (X \times X) \smallsetminus \Delta(X)$, it follows that there exist open neighbourhoods U of x and V of y such that $U \times V \subseteq (X \times X) \smallsetminus \Delta(X)$, which implies U and V are disjoint by part (a). Hence X is Hausdorff. □