# Tutorial Week 4

**Topics:** Connectedness, closed functions, compactness.

**4.1.** Prove that if a topological space $X$ admits a connected dense subset $D$, then $X$ is itself connected.

*Solution.* If the topological space is empty, then the statement follows from Example 2.28. So we can suppose $X$ is nonempty. Since the closure of $\varnothing$ in $X$ is still empty, we see that the dense subset $D$ cannot be empty, so we can pick an element $x$ of $D$.

Let $f\colon X \longrightarrow \{0,1\}$ be a continuous function, where $\{0,1\}$ is given the discrete topology. Since $f$ is continuous, it follows from Exercise 2.13 that $f^{-1}\big(f(x)\big)$ is closed. By Proposition 2.29, the restriction of $f$ to $D$ is constant, so $D \subseteq f^{-1}\big(f(x)\big)$, and therefore $X = \overline{D} \subseteq f^{-1}\big(f(x)\big)$. Hence $f$ is constant, which implies that $X$ is connected. □

*Solution.* (Alternative): Suppose $X$ is disconnected, so $X = U \cup V$ with $U, V$ open, non-empty, and disjoint. Then $D \subseteq U \cup V$ with $D \cap U \neq \varnothing$, $D \cap V \neq \varnothing$ (because $D$ is dense), and of course $D \cap U \cap V = \varnothing$, implying that $D$ is a disconnected subset of $X$ by Proposition 2.27. □

**4.2.** Let $C_1$ and $C_2$ be two connected subsets of a topological space $X$ such that $C_1 \cap C_2 \neq \varnothing$. Prove that $C_1 \cup C_2$ is connected.

*Solution.* Let $f\colon C_1 \cup C_2 \longrightarrow \{0,1\}$ be a continuous function, where $\{0,1\}$ is given the discrete topology. Since $C_1 \cap C_2$ is non-empty, we can pick an element $x$ of $C_1 \cap C_2$. By Proposition 2.29, the restriction of $f$ to $C_1$ and $C_2$ are both constant. Hence we have $f(x) = f(y)$ for every element $y$ of $C_1 \cup C_2$; in other words, $f$ is a constant function on $C_1 \cup C_2$. By Proposition 2.29, this implies $C_1 \cup C_2$ is connected. □

**4.3.** Let $X$ be a topological space. Suppose $A$ is a connected subset of $X$ and $\{C_i : i \in I\}$ is an arbitrary collection of connected subsets of $X$ such that $A \cap C_i \neq \varnothing$ for all $i \in I$. Then

$$A \cup \bigcup_{i \in I} C_i$$

is a connected subset of $X$.

*Solution.* Let $f\colon A \cup \bigcup_{i \in I} C_i \longrightarrow \{0,1\}$ be a continuous function, where $\{0,1\}$ is given the discrete topology. Pick an element $a$ of $A$ and consider an arbitrary element $x$ of $A \cup \bigcup_{i\in I} C_i$. If $x \in A$, then the connectedness of $A$ and Proposition 2.29 imply $f(x) = f(a)$. If $x \in C_i$ for some $i \in I$, then it follows from Question 4.3 and Proposition 2.29 that $f(x) = f(a)$. Hence $f$ is constant, which implies $A \cup \bigcup_{i\in I} C_i$ is connected. □

**4.4.** Let $X$ and $Y$ be non-empty topological spaces. Prove that $X \times Y$ is connected if and only if both $X$ and $Y$ are connected.

*Solution.* Suppose $X \times Y$ is connected. Recall from Proposition 2.19 that the projections $\pi_X\colon X \times Y \longrightarrow X$ and $\pi_Y\colon X \times Y \longrightarrow Y$ are continuous. It then follows from Proposition 2.30 that $X = \pi_X(X \times Y)$ and $Y = \pi_Y(X \times Y)$ are connected.

Conversely, suppose that both $X$ and $Y$ are connected. Let $f\colon X \times Y \longrightarrow \{0,1\}$ be a continuous function, where $\{0,1\}$ is given the discrete topology. Consider two elements $(x_1, y_1)$ and $(x_2, y_2)$ of $X \times Y$. It follows from Exercise 2.20 that $\{x_1\} \times Y$ is homeomorphic to $Y$, and is therefore connected. This implies that $f$ is constant when restricted to $\{x_1\} \times Y$. Similarly, $f$ is constant when restricted to $X \times \{y_2\}$ because $Y$ is connected. Hence

$$f(x_1, y_1) = f(x_1, y_2) = f(x_2, y_2),$$

and therefore $X \times Y$ is connected. □

**4.5.** (a) Prove that the composition of two closed maps is a closed map.

(b) Prove that a continuous bijection between topological spaces is a homeomorphism if and only if it is closed.

*Solution.*

(a) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be the two closed maps. Suppose $F$ is a closed subset of $X$. Since $f$ is a closed map, it follows that $f(F)$ is a closed subset. Since $g$ is a closed map, we see that $g(f(F)) = (g \circ f)(F)$ is a closed subset. Hence $g \circ f$ is a closed map.

(b) Let $f\colon X \longrightarrow Y$ be a continuous bijection between topological spaces and let $g\colon Y \longrightarrow X$ be its inverse.

Suppose $f$ is a homeomorphism. If $F$ is a closed subset of $X$, then $f(F) = g^{-1}(F)$ is closed by continuity of $g$. Hence $f$ is a closed map.

Conversely, suppose $f$ is a closed map. If $F$ is a closed subset of $X$, then $g^{-1}(F) = f(F)$ is closed. It then follows from Exercise 2.13 that $g$ is continuous, and therefore $f$ is a homeomorphism. □

**4.6.** Prove that every finite topological space is compact.

*Solution.* Let $X$ be a finite topological space and consider an open cover $\{U_i\colon i \in I\}$ of $X$. For every point $x$ in $X$, pick a member $U_x$ of $\{U_i\colon i \in I\}$ such that $x \in U_x$. Now $\{U_x\colon x \in X\}$ is a finite sub-cover of $\{U_i\colon i \in I\}$. Hence $X$ is compact. □

**4.7.** Let $X$ be a topological space and let $K$ be a subset of $X$. We will say that $K$ is a *compact subspace* of $X$ if the subspace topology on $K \subseteq X$ makes $K$ into a compact topological space.

Prove that $K$ is a compact subset of $X$ (as defined at the start of Section 2.5 in the lecture notes) if and only if it is a compact subspace of $X$ (as defined above).

(In other words, compactness is an intrinsic property of topological spaces: it does not depend on the ambient topological space.)

*Solution.* Suppose $K$ is compact as a topological space with the subspace topology from $X$. Let $\iota\colon K \longrightarrow X$ be the inclusion function, which is continuous by Exercise 2.23. It then follows from Proposition 2.37 that $\iota(K)$ is a compact subset of $X$.

Conversely, suppose $K$ is a compact subset of $X$. Let $\{U_i\colon i \in I\}$ be an open cover of $K$ in the subspace $K$. By the definition of the subspace topology, for every $U_i$ there exists an open subset $V_i$ of $X$ such that $U_i = K_i \cap K$. Since $\{U_i\colon i \in I\}$ is an open cover of $K$ in the subspace $K$, it follows that $\{V_i \in i \in I\}$ is an open cover of $K$ in $X$. The compactness of $K$ as a subset of $X$ then implies there exists a finite subset $J$ of $I$ such that $K \subseteq \bigcup_{j\in J} V_j$, and therefore

$$K = K \cap \Big(\bigcup_{j\in J} V_j\Big) = \bigcup_{j\in J}(K \cap V_j) = \bigcup_{j\in J} U_j.$$

Hence $\{U_i\colon i \in I\}$ has a finite sub-cover, which implies $K$ is compact as a subspace of $X$. □

**4.8.** Let $K$ and $L$ be compact subsets of a topological space $X$. Prove that $K \cup L$ is compact.

*Solution.* Consider an arbitrary open cover of $K \cup L$:

$$K \cup L \subseteq \bigcup_{i\in I} U_i.$$

This is also an open cover of $K$, so there is a finite subcover that still covers $K$:

$$K \subseteq \bigcup_{n=1}^{N} U_{i_n}, \qquad i_n \in I.$$

Similarly, we get a finite subcover that covers $L$:

$$L \subseteq \bigcup_{m=1}^{M} U_{j_m}, \qquad j_m \in I.$$

Letting $S = \{i_1, \dots, i_N\} \cup \{j_1, \dots, j_M\}$, we get a finite subcover that covers $K \cup L$:

$$K \cup L \subseteq \bigcup_{s \in S} U_s. \qquad \square$$

**4.9.** Let $X$ be a discrete topological space.

(a) Prove that $X$ is compact if and only if $X$ is finite.

(b) Prove that $X$ is connected if and only if $X$ is empty or is a singleton.

*Solution.*

(a) If $X$ is finite, then $X$ is compact by Question 4.6.

Conversely, suppose $X$ is compact and consider the open cover $\{\{x\} : x \in X\}$. Its only subcover is itself (any proper subcollection will miss some points of $X$), but by compactness it admits a finite subcover, so the cover itself must have been finite, hence $X$ is finite.

(b) It follows from Example 2.28 that $X$ is connected if it is empty or is a singleton.

Conversely, if $x_1 \neq x_2$ are elements of $X$, then $\{x_1\}$ and $X \setminus \{x_1\}$ are two disjoint non-empty open subsets of $X$ such that their union is $X$, so $X$ is disconnected. $\square$

**4.10.** Let $X$ be a compact topological space and let $Y$ be a Hausdorff topological space. Prove that every continuous bijection from $X$ to $Y$ is a homeomorphism.

*Solution.* Let $f \colon X \longrightarrow Y$ be a continuous bijection. We will prove $f$ is a closed map; it will then follow from part (b) of Question 4.5 that $f$ is a homeomorphism.

If $F$ is a closed subset of $X$, then it is compact by Proposition 2.36. It follows from Proposition 2.37 that $f(F)$ is compact, which implies it is a closed subset by Proposition 2.35. Hence $f$ is a closed map. $\square$

Recall that a topological space is *totally disconnected* if its only connected subsets are $\varnothing$ and the singletons.

**4.11. (*)** A topological space $X$ is called *totally separated* if for every pair $(x, y)$ of distinct points in $X$ there exist disjoint clopen neighbourhoods $U$ and $V$ of $x$ and $y$ respectively. Prove that every totally separated space is totally disconnected.

*Solution.* Let $X$ be a totally separated space and let $S$ be a subset of $X$ with two distinct points $x$ and $y$. It follows from total separatedness that there exists disjoint clopen neighbourhoods $U$ and $V$ of $x$ and $y$ respectively. Since $U$ is clopen and does not contain $y$, it follows that $X \setminus U$ is a clopen neighbourhood of $y$. Moreover, $S \cap U$ and $S \cap (X \setminus U)$ are two disjoint open sets in $S$ such that their union is $S$. Hence $S$ is not connected, and therefore the only connected subsets of $X$ are the empty set and the singletons; in other words, $X$ is totally disconnected. $\square$

**4.12. (*)** Prove that the following are totally disconnected:

(a) $\mathbf{Q}$ equipped with the Euclidean topology;

(b) every discrete topological space.

*Solution.* We will prove all of them are totally separated, which implies totally disconnectedness by Question 4.11.

(a) Let $x$ and $y$ be two distinct rational number. Without loss of generality, we assume that $x < y$. The denseness of $\mathbf{R} \setminus \mathbf{Q}$ (see Example 2.26) implies that there exists an irrational number $z$ such that $x < z < y$. The open sets $\mathbf{Q} \cap (-\infty, z)$ and $\mathbf{Q} \cap (z, \infty)$ are open in $\mathbf{Q}$, and their intersection is empty while their union is $\mathbf{Q}$, so $\mathbf{Q} \cap (-\infty, z)$ is an clopen neighbourhood of $x$ in $\mathbf{Q}$ and $\mathbf{Q} \cap (z, \infty)$ is a clopen neighbourhood of $y$. Hence $\mathbf{Q}$ is totally disconnected when equipped with the Euclidean topology.

(b) Let $X$ be a discrete space and let $x$ and $y$ be two points in $X$. It follows from the definition of the discrete topology that $\{x\}$ and $\{y\}$ are clopen, so they are disjoint clopen neighbourhoods of $x$ and $y$ respective. Hence $X$ is totally separated. $\square$