

Tutorial Week 4

Topics: Connectedness, closed functions, compactness.

4.1. Prove that if a topological space X admits a connected dense subset D , then X is itself connected.

Solution. If the topological space is empty, then the statement follows from [Example 2.28](#). So we can suppose X is nonempty. Since the closure of \emptyset in X is still empty, we see that the dense subset D cannot be empty, so we can pick an element x of D .

Let $f: X \rightarrow \{0, 1\}$ be a continuous function, where $\{0, 1\}$ is given the discrete topology. Since f is continuous, it follows from [Exercise 2.13](#) that $f^{-1}(f(x))$ is closed. By [Proposition 2.29](#), the restriction of f to D is constant, so $D \subseteq f^{-1}(f(x))$, and therefore $X = \overline{D} \subseteq f^{-1}(f(x))$. Hence f is constant, which implies that X is connected. \square

Solution. (Alternative): Suppose X is disconnected, so $X = U \cup V$ with U, V open, non-empty, and disjoint. Then $D \subseteq U \cup V$ with $D \cap U \neq \emptyset$, $D \cap V \neq \emptyset$ (because D is dense), and of course $D \cap U \cap V = \emptyset$, implying that D is a disconnected subset of X by [Proposition 2.27](#). \square

4.2. Let C_1 and C_2 be two connected subsets of a topological space X such that $C_1 \cap C_2 \neq \emptyset$. Prove that $C_1 \cup C_2$ is connected.

Solution. Let $f: C_1 \cup C_2 \rightarrow \{0, 1\}$ be a continuous function, where $\{0, 1\}$ is given the discrete topology. Since $C_1 \cap C_2$ is non-empty, we can pick an element x of $C_1 \cap C_2$. By [Proposition 2.29](#), the restriction of f to C_1 and C_2 are both constant. Hence we have $f(x) = f(y)$ for every element y of $C_1 \cup C_2$; in other words, f is a constant function on $C_1 \cup C_2$. By [Proposition 2.29](#), this implies $C_1 \cup C_2$ is connected. \square

4.3. Let X be a topological space. Suppose A is a connected subset of X and $\{C_i : i \in I\}$ is an arbitrary collection of connected subsets of X such that $A \cap C_i \neq \emptyset$ for all $i \in I$. Then

$$A \cup \bigcup_{i \in I} C_i$$

is a connected subset of X .

Solution. Let $f: A \cup \bigcup_{i \in I} C_i \rightarrow \{0, 1\}$ be a continuous function, where $\{0, 1\}$ is given the discrete topology. Pick an element a of A and consider an arbitrary element x of $A \cup \bigcup_{i \in I} C_i$. If $x \in A$, then the connectedness of A and [Proposition 2.29](#) imply $f(x) = f(a)$. If $x \in C_i$ for some $i \in I$, then it follows from [Question 4.3](#) and [Proposition 2.29](#) that $f(x) = f(a)$. Hence f is constant, which implies $A \cup \bigcup_{i \in I} C_i$ is connected. \square

4.4. Let X and Y be non-empty topological spaces. Prove that $X \times Y$ is connected if and only if both X and Y are connected.

Solution. Suppose $X \times Y$ is connected. Recall from [Proposition 2.19](#) that the projections $\pi_X: X \times Y \rightarrow X$ and $\pi_Y: X \times Y \rightarrow Y$ are continuous. It then follows from [Proposition 2.30](#) that $X = \pi_X(X \times Y)$ and $Y = \pi_Y(X \times Y)$ are connected.

Conversely, suppose that both X and Y are connected. Let $f: X \times Y \rightarrow \{0, 1\}$ be a continuous function, where $\{0, 1\}$ is given the discrete topology. Consider two elements (x_1, y_1) and (x_2, y_2) of $X \times Y$. It follows from [Exercise 2.20](#) that $\{x_1\} \times Y$ is homeomorphic to Y , and is therefore connected. This implies that f is constant when restricted to $\{x_1\} \times Y$. Similarly, f is constant when restricted to $X \times \{y_2\}$ because Y is connected. Hence

$$f(x_1, y_1) = f(x_1, y_2) = f(x_2, y_2),$$

and therefore $X \times Y$ is connected. \square

- 4.5. (a) Prove that the composition of two closed maps is a closed map.
 (b) Prove that a continuous bijection between topological spaces is a homeomorphism if and only if it is closed.

Solution.

- (a) Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be the two closed maps. Suppose F is a closed subset of X . Since f is a closed map, it follows that $f(F)$ is a closed subset. Since g is a closed map, we see that $g(f(F)) = (g \circ f)(F)$ is a closed subset. Hence $g \circ f$ is a closed map.
 (b) Let $f: X \rightarrow Y$ be a continuous bijection between topological spaces and let $g: Y \rightarrow X$ be its inverse.

Suppose f is a homeomorphism. If F is a closed subset of X , then $f(F) = g^{-1}(F)$ is closed by continuity of g . Hence f is a closed map.

Conversely, suppose f is a closed map. If F is a closed subset of X , then $g^{-1}(F) = f(F)$ is closed. It then follows from [Exercise 2.13](#) that g is continuous, and therefore f is a homeomorphism. \square

- 4.6. Prove that every finite topological space is compact.

Solution. Let X be a finite topological space and consider an open cover $\{U_i: i \in I\}$ of X . For every point x in X , pick a member U_x of $\{U_i: i \in I\}$ such that $x \in U_x$. Now $\{U_x: x \in X\}$ is a finite sub-cover of $\{U_i: i \in I\}$. Hence X is compact. \square

- 4.7. Let X be a topological space and let K be a subset of X . We will say that K is a *compact subspace* of X if the subspace topology on $K \subseteq X$ makes K into a compact topological space.

Prove that K is a compact subset of X (as defined at the start of [Section 2.5](#) in the lecture notes) if and only if it is a compact subspace of X (as defined above).

(In other words, compactness is an intrinsic property of topological spaces: it does not depend on the ambient topological space.)

Solution. Suppose K is compact as a topological space with the subspace topology from X . Let $\iota: K \rightarrow X$ be the inclusion function, which is continuous by [Exercise 2.23](#). It then follows from [Proposition 2.37](#) that $\iota(K)$ is a compact subset of X .

Conversely, suppose K is a compact subset of X . Let $\{U_i: i \in I\}$ be an open cover of K in the subspace K . By the definition of the subspace topology, for every U_i there exists an open subset V_i of X such that $U_i = K_i \cap K$. Since $\{U_i: i \in I\}$ is an open cover of K in the subspace K , it follows that $\{V_i: i \in I\}$ is an open cover of K in X . The compactness of K as a subset of X then implies there exists a finite subset J of I such that $K \subseteq \bigcup_{j \in J} V_j$, and therefore

$$K = K \cap \left(\bigcup_{j \in J} V_j \right) = \bigcup_{j \in J} (K \cap V_j) = \bigcup_{j \in J} U_j.$$

Hence $\{U_i: i \in I\}$ has a finite sub-cover, which implies K is compact as a subspace of X . \square

- 4.8. Let K and L be compact subsets of a topological space X . Prove that $K \cup L$ is compact.

Solution. Consider an arbitrary open cover of $K \cup L$:

$$K \cup L \subseteq \bigcup_{i \in I} U_i.$$

This is also an open cover of K , so there is a finite subcover that still covers K :

$$K \subseteq \bigcup_{n=1}^N U_{i_n}, \quad i_n \in I.$$

Similarly, we get a finite subcover that covers L :

$$L \subseteq \bigcup_{m=1}^M U_{j_m}, \quad j_m \in I.$$

Letting $S = \{i_1, \dots, i_N\} \cup \{j_1, \dots, j_M\}$, we get a finite subcover that covers $K \cup L$:

$$K \cup L \subseteq \bigcup_{s \in S} U_s. \quad \square$$

4.9. Let X be a discrete topological space.

- (a) Prove that X is compact if and only if X is finite.
- (b) Prove that X is connected if and only if X is empty or is a singleton.

Solution.

- (a) If X is finite, then X is compact by [Question 4.6](#).

Conversely, suppose X is compact and consider the open cover $\{\{x\} : x \in X\}$. Its only subcover is itself (any proper subcollection will miss some points of X), but by compactness it admits a finite subcover, so the cover itself must have been finite, hence X is finite.

- (b) It follows from [Example 2.28](#) that X is connected if it is empty or is a singleton.

Conversely, if $x_1 \neq x_2$ are elements of X , then $\{x_1\}$ and $X \setminus \{x_1\}$ are two disjoint non-empty open subsets of X such that their union is X , so X is disconnected. \square

4.10. Let X be a compact topological space and let Y be a Hausdorff topological space. Prove that every continuous bijection from X to Y is a homeomorphism.

Solution. Let $f: X \rightarrow Y$ be a continuous bijection. We will prove f is a closed map; it will then follow from part (b) of [Question 4.5](#) that f is a homeomorphism.

If F is a closed subset of X , then it is compact by [Proposition 2.36](#). It follows from [Proposition 2.37](#) that $f(F)$ is compact, which implies it is a closed subset by [Proposition 2.35](#). Hence f is a closed map. \square

Recall that a topological space is *totally disconnected* if its only connected subsets are \emptyset and the singletons.

4.11. A topological space X is called *totally separated* if for every pair (x, y) of distinct points in X there exist disjoint clopen neighbourhoods U and V of x and y respectively. Prove that every totally separated space is totally disconnected.

Solution. Let X be a totally separated space and let S be a subset of X with two distinct points x and y . It follows from total separatedness that there exists disjoint clopen neighbourhoods U and V of x and y respectively. Since U is clopen and does not contain y , it follows that $X \setminus U$ is a clopen neighbourhood of y . Moreover, $S \cap U$ and $S \cap (X \setminus U)$ are two disjoint open sets in S such that their union is S . Hence S is not connected, and therefore the only connected subsets of X are the empty set and the singletons; in other words, X is totally disconnected. \square

4.12. Prove that the following are totally disconnected:

- (a) \mathbb{Q} equipped with the Euclidean topology;

(b) every discrete topological space.

Solution. We will prove all of them are totally separated, which implies totally disconnectedness by [Question 4.11](#).

- (a) Let x and y be two distinct rational number. Without loss of generality, we assume that $x < y$. The denseness of $\mathbf{R} \setminus \mathbf{Q}$ (see [Example 2.26](#)) implies that there exists an irrational number z such that $x < z < y$. The open sets $\mathbf{Q} \cap (-\infty, z)$ and $\mathbf{Q} \cap (z, \infty)$ are open in \mathbf{Q} , and their intersection is empty while their union is \mathbf{Q} , so $\mathbf{Q} \cap (-\infty, z)$ is a clopen neighbourhood of x in \mathbf{Q} and $\mathbf{Q} \cap (z, \infty)$ is a clopen neighbourhood of y . Hence \mathbf{Q} is totally disconnected when equipped with the Euclidean topology.
- (b) Let X be a discrete space and let x and y be two points in X . It follows from the definition of the discrete topology that $\{x\}$ and $\{y\}$ are clopen, so they are disjoint clopen neighbourhoods of x and y respective. Hence X is totally separated. \square