

Tutorial Week 5

Topics: Topological groups, sequences.

5.1. Let d_1 and d_2 be equivalent metrics (they define the same topology) on a set X . Prove that a sequence converges to a point x in (X, d_1) if and only if it converges to x in (X, d_2) .

Solution. Since d_1 and d_2 are interchangeable, it suffices to prove the ‘only if’ part. Let (x_n) be a sequence converging to x in (X, d_1) . By [Proposition 2.21](#), the identity function $\text{id}_X: X \rightarrow X$ defined by $\text{id}_X(x) = x$ is continuous as a function from (X, d_1) to (X, d_2) . The result then follows from [Theorem 2.52](#). \square

5.2. Let (x_n) be a sequence in X , let $\varphi: \mathbf{N} \rightarrow \mathbf{N}$ be an injective function, and consider the sequence $(y_n) = (x_{\varphi(n)})$ in X . Prove that if (x_n) converges to x , then so does (y_n) .

Does the converse hold?

Solution. Suppose $(x_n) \rightarrow x$. Given $\varepsilon > 0$, let $N \in \mathbf{N}$ be such that $x_n \in \mathbf{B}_\varepsilon(x)$ for all $n \geq N$.

Since $\varphi: \mathbf{N} \rightarrow \mathbf{N}$ is injective, the inverse image $\varphi^{-1}(\{1, \dots, N-1\})$ is a finite set, so it has a maximal element M . (If the set is empty, just take $M = 0$.) For all $n \geq M+1$, we have $\varphi(n) \geq N$, so $y_n = x_{\varphi(n)} \in \mathbf{B}_\varepsilon(x)$.

The converse does not hold. For instance, take $(x_n) = (1, 0, 1, 0, 1, 0, \dots)$ and $\varphi(n) = 2n$, then the sequence $(y_n) = (0, 0, 0, \dots)$ converges to 0 but (x_n) does not converge. \square

5.3. Let $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$ and define

$$\mathcal{T} = \mathcal{P}(\mathbf{N}) \cup \{U \in \mathcal{P}(\mathbf{N}^*): \infty \in U \text{ and } \mathbf{N}^* \setminus U \text{ is finite}\}.$$

- (a) Prove that \mathcal{T} is a topology on \mathbf{N}^* .
- (b) Prove that $(\mathbf{N}^*, \mathcal{T})$ is compact.
- (c) Let X be a metric space and $f: (\mathbf{N}^*, \mathcal{T}) \rightarrow X$. Prove that f is continuous if and only if $(f(n))$ converges to $f(\infty)$. (In other words, convergent sequences in X are exactly continuous functions from $(\mathbf{N}^*, \mathcal{T})$ to X .)
- (d) Let X be a metric space and let (x_n) be a sequence in X that converges to a point x in X . Prove that $\{x\} \cup \{x_n: n \in \mathbf{N}\}$ is compact.

Solution.

- (a) It is clear that \emptyset and \mathbf{N}^* belongs to \mathcal{T} .

Suppose $\{U_i: i \in I\}$ is a collection of members of \mathcal{T} . If $\{U_i: i \in I\} \subseteq \mathcal{P}(\mathbf{N})$, then $\bigcup_{i \in I} U_i \in \mathcal{P}(\mathbf{N}) \subseteq \mathcal{T}$. Otherwise, there exists a member V of $\{U_i: i \in I\}$ such that $\infty \in V$. It then follows from

$$\mathbf{N}^* \setminus \left(\bigcup_{i \in I} U_i \right) \subseteq \mathbf{N}^* \setminus V$$

that $\mathbf{N}^* \setminus \left(\bigcup_{i \in I} U_i \right)$ is finite, and therefore $\bigcup_{i \in I} U_i \in \mathcal{T}$.

For closure under intersection, it suffices to prove \mathcal{T} is closed under binary intersections. Let U and V be members of \mathcal{T} . If at most one of U and V contains ∞ , then $U \cap V \in \mathcal{P}(\mathbf{N})$. Otherwise, it then follows from

$$\mathbf{N}^* \setminus (U \cap V) = (\mathbf{N}^* \setminus U) \cup (\mathbf{N}^* \setminus V)$$

that $\mathbf{N}^* \setminus (U \cap V)$ is finite, and therefore $U \cap V \in \mathcal{T}$.

(b) Let $\{U_i: i \in I\}$ be an open cover of \mathbf{N}^* . Pick a member V of the open cover such that $\infty \in V$. Since $V \in \mathcal{T}$, it follows that $\mathbf{N}^* \setminus V$ is finite. For each element x of $\mathbf{N}^* \setminus V$, pick a member V_x of the open cover such that $x \in V_x$. It follows that $\{V\} \cup \{V_x: x \in \mathbf{N}^* \setminus V\}$ is a finite sub-cover of $\{U_i: i \in I\}$. Hence \mathbf{N}^* is compact.

(c) Suppose f is continuous. It follows that for every positive real number ϵ , the inverse image $f^{-1}(\mathbf{B}_\epsilon(f(\infty)))$ is open, and therefore $\mathbf{N}^* \setminus f^{-1}(\mathbf{B}_\epsilon(f(\infty)))$ is finite. Hence there exists a natural number N such that $n \geq N$ implies $f(n) \in \mathbf{B}_\epsilon(f(\infty))$.

Conversely, suppose $(f(n))$ converges to $f(\infty)$. The space \mathbf{N} is discrete as a subspace of \mathbf{N}^* , so $f|_{\mathbf{N}}$ is continuous; this implies f is continuous at every natural number by [Question 3.3](#). To apply [Question 3.3](#), it remains to prove f is continuous at ∞ . Let M be a neighbourhood of ∞ and pick a positive real number ϵ such that $\mathbf{B}_\epsilon(f(\infty)) \subseteq M$. Since $f(n) \rightarrow f(\infty)$ as $n \rightarrow \infty$, there exists a natural number N such that $n \geq N$ implies $f(n) \in \mathbf{B}_\epsilon(f(\infty))$. This implies

$$\mathbf{N}^* \setminus f^{-1}(\mathbf{B}_\epsilon(f(\infty))) \subseteq \{1, \dots, N\},$$

so $f^{-1}(\mathbf{B}_\epsilon(f(\infty)))$ is open. Since $f^{-1}(\mathbf{B}_\epsilon(f(\infty))) \subseteq f^{-1}(M)$, it follows that $f^{-1}(M)$ is a neighbourhood of ∞ , so f is continuous at ∞ . Now apply [Question 3.3](#) to f , we see that f is continuous.

(d) Define a function $f: \mathbf{N}^* \rightarrow X$ by

$$f(n) = \begin{cases} x_n & \text{if } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$

By part (c), f is continuous, so it follows from [Proposition 2.37](#) that $\{x\} \cup \{x_n: n \in \mathbf{N}\} = f(\mathbf{N}^*)$ is compact. \square

5.4. Let (X, d_X) and (Y, d_Y) be metric spaces and let d be the sup norm metric on $X \times Y$:

$$d((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2)).$$

Prove that $((x_n, y_n)) \rightarrow (x, y) \in X \times Y$ if and only if $(x_n) \rightarrow x \in X$ and $(y_n) \rightarrow y \in Y$.

Solution. Suppose $(x_n) \rightarrow x$ and $(y_n) \rightarrow y$. Let $\epsilon > 0$, $N_x \in \mathbf{N}$ such that $x_n \in \mathbf{B}_\epsilon(x)$ for all $n \geq N_x$, and $N_y \in \mathbf{N}$ such that $y_n \in \mathbf{B}_\epsilon(y)$ for all $n \geq N_y$. Set $N = \max\{N_x, N_y\}$, then

$$d((x_n, y_n), (x, y)) = \max\{d_X(x_n, x), d_Y(y_n, y)\} < \epsilon \quad \text{for all } n \geq N.$$

Conversely, suppose $((x_n, y_n)) \rightarrow (x, y)$. Given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $(x_n, y_n) \in \mathbf{B}_\epsilon((x, y))$ for all $n \geq N$, so

$$\max\{d_X(x_n, x), d_Y(y_n, y)\} = d((x_n, y_n), (x, y)) < \epsilon,$$

and hence both $d_X(x_n, x)$ and $d_Y(y_n, y)$ are bounded by ϵ for all $n \geq N$. \square

Solution. (Alternative): Define a function $f: \mathbf{N}^* \rightarrow X \times Y$ by

$$f(n) = \begin{cases} (x_n, y_n) & \text{if } n \in \mathbf{N}, \\ (x, y) & \text{otherwise.} \end{cases}$$

Let $\pi_X: X \times Y \rightarrow X$ and $\pi_Y: X \times Y \rightarrow Y$ be the projections. The result follows from the following:

- f is continuous if and only if π_X and π_Y are both continuous (see [Question 3.7](#)).
- f is continuous if and only if (x_n, y_n) converges to (x, y) (part (c) of [Question 5.3](#)).
- $\pi_X \circ f$ is continuous if and only if x_n converges to x (part (c) of [Question 5.3](#)).
- $\pi_Y \circ f$ is continuous if and only if y_n converges to y (part (c) of [Question 5.3](#)). □

5.5. Let G be a topological group and let H be a subgroup of G .

- (a) Prove that H is closed if it is open. Does the converse hold?
- (b) Prove that H is open if it is closed and has finite index. Does the converse hold?
- (c) Suppose G is compact and H is open. Prove that H has finite index.
- (d) Is the compactness of G necessary in part (c)?

Solution.

- (a) Suppose H is open. If g is an element of G , then gH is open because $gH = L_{g^{-1}}^{-1}(H)$ and $L_{g^{-1}}$ is continuous by [Proposition 2.44](#). Now the result follows from the equation

$$G \setminus H = \bigcup_{g \notin H} gH.$$

The converse does not true. If $G = \mathbf{R}$, which is given the Euclidean topology, and if $H = \{0\}$, then H is a closed subgroup of G but it is not open.

- (b) Suppose H is closed. If g is an element of G , then $L_{g^{-1}}$ is continuous by [Proposition 2.44](#), so $gH = L_{g^{-1}}^{-1}(H)$ is closed because of [Exercise 2.13](#). Since H is of finite index, it has are only finitely many cosets H, g_1H, \dots, g_nH . It follows that

$$G \setminus H = \bigcup_{n=1}^n g_nH = G,$$

which is closed because it is a finite union of closed sets. Hence H is open.

The converse does not hold. Let $G = \mathbf{R}$ but endow it with the discrete topology, and let $H = \mathbf{Z}$. Then H is open in G but it is not of finite index (because if it is, then \mathbf{R} is a finite union of countable sets, and is thus countable by [Exercise 1.2](#)).

- (c) Arguing as in part (a), we have

$$G = \bigcup_{g \in G} gH,$$

so $\{gH : g \in G\}$ is an open cover of G . Since G is compact, this open cover admits a finite sub-cover, which implies that H has finite index.

- (d) Yes. Let G be any infinite group with the discrete topology, and let $H = \{e\}$, then H is open in G but it does not have finite index. □

5.6. Let S and T be subsets of a topological group G . Define

$$ST = \{st : s \in S \text{ and } t \in T\}.$$

- (a) Suppose S and T are open. Prove that ST is open.
- (b) Suppose S and T are connected. Prove that ST is connected.

- (c) Suppose S and T are compact. Prove that ST is compact.
- (d) Suppose S is compact and T is closed. Prove that ST is closed.
 [Hint: Use [Theorem 2.39](#) after checking that

$$ST = \pi_2\left(j^{-1}(m^{-1}(T))\right),$$

where $m: G \times G \rightarrow G$ is the multiplication map of G , j is the inclusion of $S^{-1} \times G$ into $G \times G$, and $\pi_2: S^{-1} \times G \rightarrow G$ is the projection onto the second factor.]

- (e) Assuming without proof the fact that $\mathbf{Z} + \pi\mathbf{Z}$ is dense in \mathbf{R} , convince yourself that ST need not be closed even if both S and T are.

Solution.

- (a) If g is an element of S , then gT is open because $gT = L_{g^{-1}}^{-1}(T)$ and $L_{g^{-1}}$ is continuous by [Proposition 2.44](#). It then follows from

$$ST = \bigcup_{s \in S} sT$$

that ST is open.

- (b) If S or T is empty, then $ST \neq \emptyset$, so it is connected. Otherwise, the product $S \times T$ is connected by [Question 4.4](#), so $ST = m(S \times T)$ is connected by [Proposition 2.30](#).
- (c) The product $S \times T$ is compact by [Theorem 2.39](#), so $ST = m(S \times T)$ is compact by [Proposition 2.37](#).
- (d) Since inversion is a homeomorphism, it follows from [Proposition 2.37](#) that S^{-1} is compact. The inclusion $j: S^{-1} \times G \rightarrow G \times G$ is continuous by [Exercise 2.23](#). Since T is closed, it follows from [Exercise 2.13](#) that $m^{-1}(T) \subseteq G \times G$ is closed and then $j^{-1}(m^{-1}(T)) \subseteq S^{-1} \times G$ is closed.

We now claim that

$$ST = \pi_2\left(j^{-1}(m^{-1}(T))\right);$$

and this implies ST is closed because π_2 is closed by [Theorem 2.39](#) (here we crucially need S^{-1} to be compact). To prove this equation, we start with an element g of ST . Since $g \in ST$, there exists an element s of S and an element t of T such that $g = st$. It follows that $(s^{-1}, g) \in j^{-1}(m^{-1}(T))$, so

$$g \in \pi_2\left(j^{-1}(m^{-1}(T))\right).$$

For the other inclusion, suppose $(s', g) \in j^{-1}(m^{-1}(T))$. It follows that $s'g \in T$, so $g \in s'^{-1}T$, which implies $g \in ST$ because $s' \in S^{-1}$. Hence the equation holds.

- (e) Since $\mathbf{Z} + \pi\mathbf{Z} = \bigcup_{n \in \mathbf{Z}} (n + \pi\mathbf{Z})$, it follows from [Exercise 1.2](#) that $\mathbf{Z} + \pi\mathbf{Z} \neq \mathbf{R}$; but we know it is dense in \mathbf{R} , so it cannot be closed. Hence \mathbf{Z} and $\pi\mathbf{Z}$ are closed in \mathbf{R} , but $\mathbf{Z} + \pi\mathbf{Z}$ is not closed. \square