Tutorial Week 5

Topics: Topological groups, sequences.

5.1. Let d_1 and d_2 be equivalent metrics (they define the same topology) on a set X. Prove that a sequence converges to a point x in (X, d_1) if and only if it converges to x in (X, d_2).

Solution. Since d_1 and d_2 are interchangeable, it suffices to prove the 'only if' part. Let (x_n) be a sequence converging to x in (X, d_1). By Proposition 2.21, the identity function $\mathrm{id}_X : X \longrightarrow X$ defined by $\mathrm{id}_X(x) = x$ is continuous as a function from (X, d_1) to (X, d_2). The result then follows from Theorem 2.52. □

5.2. Let (x_n) be a sequence in a metric space X, let $\varphi \colon \mathbf{N} \longrightarrow \mathbf{N}$ be an injective function, and consider the sequence $(y_n) = \big(x_{\varphi(n)}\big)$ in X. Prove that if (x_n) converges to x, then so does (y_n).

Does the converse hold?

Solution. Suppose $(x_n) \longrightarrow x$. Given $\varepsilon > 0$, let $N \in \mathbf{N}$ be such that $x_n \in \mathbf{B}_\varepsilon(x)$ for all $n \geqslant N$.

Since $\varphi \colon \mathbf{N} \longrightarrow \mathbf{N}$ is injective, the inverse image $\varphi^{-1}\big(\{1, \dots, N-1\}\big)$ is a finite set, so it has a maximal element M. (If the set is empty, just take $M = 0$.) For all $n \geqslant M + 1$, we have $\varphi(n) \geqslant N$, so $y_n = x_{\varphi(n)} \in \mathbf{B}_\varepsilon(x)$.

The converse does not hold. For instance, take $(x_n) = (1, 0, 1, 0, 1, 0, \dots)$ and $\varphi(n) = 2n$, then the sequence $(y_n) = (0, 0, 0, \dots)$ converges to 0 but (x_n) does not converge. □

5.3. (*) Let $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$ and define

$$\mathcal{T} = \mathcal{P}(\mathbf{N}) \cup \{U \in \mathcal{P}(\mathbf{N}^*) \colon \infty \in U \text{ and } \mathbf{N}^* \smallsetminus U \text{ is finite}\}.$$

(a) Prove that $\mathcal{T}$ is a topology on $\mathbf{N}^*$.

(b) Prove that $(\mathbf{N}^*, \mathcal{T})$ is compact.

(c) Let X be a metric space and $f \colon (\mathbf{N}^*, \mathcal{T}) \longrightarrow X$. Prove that f is continuous if and only if $\big(f(n)\big)$ converges to $f(\infty)$. (In other words, convergent sequences in X are exactly continuous functions from $(\mathbf{N}^*, \mathcal{T})$ to X.)

(d) Let X be a metric space and let (x_n) be a sequence in X that converges to a point x in X. Prove that $\{x\} \cup \{x_n \colon n \in \mathbf{N}\}$ is compact.

Solution.

(a) It is clear that $\varnothing$ and $\mathbf{N}^*$ belong to $\mathcal{T}$.

Suppose $\{U_i \colon i \in I\}$ is a collection of members of $\mathcal{T}$. If $\{U_i \colon i \in I\} \subseteq \mathcal{P}(\mathbf{N})$, then $\bigcup_{i \in I} U_i \in \mathcal{P}(\mathbf{N}) \subseteq \mathcal{T}$. Otherwise, there exists a member V of $\{U_i \colon i \in I\}$ such that $\infty \in V$. It then follows from

$$\mathbf{N}^* \smallsetminus \Big(\bigcup_{i \in I} U_i\Big) \subseteq \mathbf{N}^* \smallsetminus V$$

that $\mathbf{N}^* \smallsetminus \Big(\bigcup_{i \in I} U_i\Big)$ is finite, and therefore $\bigcup_{i \in I} U_i \in \mathcal{T}$.

For closure under finite intersection, it suffices to prove it for any two members U and V of $\mathcal{T}$. If at most one of U and V contains ∞, then $U \cap V \in \mathcal{P}(\mathbf{N})$. Otherwise, it then follows from

$$\mathbf{N}^* \smallsetminus (U \cap V) = (\mathbf{N}^* \smallsetminus U) \cup (\mathbf{N}^* \smallsetminus V)$$

that $\mathbf{N}^* \smallsetminus (U \cap V)$ is finite, and therefore $U \cap V \in \mathcal{T}$.

(b) Let $\{U_i : i \in I\}$ be an open cover of $\mathbf{N}^*$. Pick a member V of the open cover such that $\infty \in V$. Since $V \in \mathcal{T}$, it follows that $\mathbf{N}^* \smallsetminus V$ is finite. For each element x of $\mathbf{N}^* \smallsetminus V$, pick a member V_x of the open cover such that $x \in V_x$. It follows that $\{V\} \cup \{V_x : x \in \mathbf{N}^* \smallsetminus V\}$ is a finite sub-cover of $\{U_i \in i \in I\}$. Hence $\mathbf{N}^*$ is compact.

(c) Suppose f is continuous. It follows that for every positive real number ϵ, the inverse image $f^{-1}\big(\mathbf{B}_\epsilon(f(\infty))\big)$ is open, and therefore $\mathbf{N}^* \smallsetminus f^{-1}\big(\mathbf{B}_\epsilon(f(\infty))\big)$ is finite. Hence there exists a natural number N such that $n \geqslant N$ implies $f(n) \in \mathbf{B}_\epsilon(f(\infty))$.

Conversely, suppose $\big(f(n)\big)$ converges to $f(\infty)$. The space $\mathbf{N}$ is discrete as a subspace of $\mathbf{N}^*$, so $f|_{\mathbf{N}}$ is continuous; this implies f is continuous at every natural number by Question 3.3. To apply Question 3.3, it remains to prove f is continuous at ∞. Let M be a neighbourhood of ∞ and pick a positive real number ϵ such that $\mathbf{B}_\epsilon(f(\infty)) \subseteq M$. Since $f(n) \longrightarrow f(\infty)$ as $n \longrightarrow \infty$, there exists a natural number N such that $n \geqslant N$ implies $f(n) \in \mathbf{B}_\epsilon(f(\infty))$. This implies

$$\mathbf{N}^* \smallsetminus f^{-1}\big(\mathbf{B}_\epsilon(f(\infty))\big) \subseteq \{1, \dots, N\},$$

so $f^{-1}\big(\mathbf{B}_\epsilon(f(\infty))\big)$ is open. Since $f^{-1}\big(\mathbf{B}_\epsilon(f(\infty))\big) \subseteq f^{-1}(M)$, it follows that $f^{-1}(M)$ is a neighbourhood of ∞, so f is continuous at ∞. Now apply Question 3.3 to f, we see that f is continuous.

(d) Define a function $f : \mathbf{N}^* \longrightarrow X$ by

$$f(n) = \begin{cases} x_n & \text{if } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$

By part (c), f is continuous, so it follows from Proposition 2.37 that

$$\{x\} \cup \{x_n : n \in \mathbf{N}\} = f(\mathbf{N}^*)$$

is compact. □

5.4. Let (X, d_X) and (Y, d_Y) be metric spaces and let d be the sup norm metric on $X \times Y$:

$$d\big((x_1, y_1), (x_2, y_2)\big) = \max\big(d_X(x_1, x_2), d_Y(y_1, y_2)\big).$$

Prove that $\big((x_n, y_n)\big) \longrightarrow (x, y) \in X \times Y$ if and only if $(x_n) \longrightarrow x \in X$ and $(y_n) \longrightarrow y \in Y$.

Solution. Suppose $(x_n) \longrightarrow x$ and $(y_n) \longrightarrow y$. Let $\varepsilon > 0$, $N_x \in \mathbf{N}$ such that $x_n \in \mathbf{B}_\varepsilon(x)$ for all $n \geqslant N_x$, and $N_y \in \mathbf{N}$ such that $y_n \in \mathbf{B}_\varepsilon(y)$ for all $n \geqslant N_y$. Set $N = \max\{N_x, N_y\}$, then

$$d\big((x_n, y_n), (x, y)\big) = \max\{d_X(x_n, x), d_Y(y_n, y)\} < \varepsilon \quad \text{for all } n \geqslant N.$$

Conversely, suppose $\big((x_n, y_n)\big) \longrightarrow (x, y)$. Given $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that $(x_n, y_n) \in \mathbf{B}_\varepsilon\big((x, y)\big)$ for all $n \geqslant N$, so

$$\max\{d_X(x_n, x), d_Y(y_n, y)\} = d\big((x_n, y_n), (x, y)\big) < \varepsilon,$$

and hence both $d_X(x_n, x)$ and $d_Y(y_n, y)$ are bounded by ε for all $n \geqslant N$. □

Solution. (Alternative): Define a function $f : \mathbf{N}^* \longrightarrow X \times Y$ by

$$f(n) = \begin{cases} (x_n, y_n) & \text{if } n \in \mathbf{N}, \\ (x, y) & \text{otherwise.} \end{cases}$$

Let $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ be the projections. The result follows from the following:

- f is continuous if and only if π_X and π_Y are both continuous (see Question 3.7).
- f is continuous if and only if (x_n, y_n) converges to (x, y) (part (c) of Question 5.3).
- $\pi_X \circ f$ is continuous if and only if x_n converges to x (part (c) of Question 5.3).
- $\pi_Y \circ f$ is continuous if and only if y_n converges to y (part (c) of Question 5.3). □

5.5. (*) Let G be a topological group and let H be a subgroup of G.

(a) Prove that H is closed if it is open. Does the converse hold?

(b) Prove that H is open if it is closed and has finite index. Does the converse hold?

(c) Suppose G is compact and H is open. Prove that H has finite index.

(d) Is the compactness of G necessary in part (c)?

Solution.

(a) Suppose H is open. If g is an element of G, then gH is open because $gH = L_{g^{-1}}^{-1}(H)$ and $L_{g^{-1}}$ is continuous by Proposition 2.44. Now the result follows from the equation

$$G \setminus H = \bigcup_{g \notin H} gH.$$

The converse does not hold. If $G = \mathbf{R}$, which is given the Euclidean topology, and if $H = \{0\}$, then H is a closed subgroup of G but it is not open.

(b) Suppose H is closed. If g is an element of G, then $L_{g^{-1}}$ is continuous by Proposition 2.44, so $gH = L_{g^{-1}}^{-1}(H)$ is closed because of Exercise 2.13. Since H is of finite index, it has are only finitely many cosets $H, g_1 H, \ldots, g_n H$. It follows that

$$G \setminus H = \bigcup_{n=1}^{n} gH = G,$$

which is closed because it is a finite union of closed sets. Hence H is open.

The converse does not hold. Let $G = \mathbf{R}$ but endow it with the discrete topology, and let $H = \mathbf{Z}$. Then H is open in G but it is not of finite index (because if it is, then $\mathbf{R}$ is a finite union of countable sets, and is thus countable by Exercise 1.2).

(c) Arguing as in part (a), we have

$$G = \bigcup_{g \in G} gH,$$

so $\{gH \colon g \in G\}$ is an open cover of G. Since G is compact, this open cover admits a finite sub-cover, which implies that H has finite index.

(d) Yes. Let G be any infinite group with the discrete topology, and let $H = \{e\}$, then H is open in G but it does not have finite index. □

5.6. (*) Let S and T be subsets of a topological group G. Define

$$ST = \{st \colon s \in S \text{ and } t \in T\}.$$

(a) Suppose S and T are open. Prove that ST is open.

(b) Suppose S and T are connected. Prove that ST is connected.

(c) Suppose S and T are compact. Prove that ST is compact.

(d) Suppose S is compact and T is closed. Prove that ST is closed.

[*Hint*: Use Theorem 2.39 after checking that

$$ST = \pi_2\Big(j^{-1}\big(m^{-1}(T)\big)\Big),$$

where $m\colon G \times G \longrightarrow G$ is the multiplication map of G, j is the inclusion of $S^{-1} \times G$ into $G \times G$, and $\pi_2\colon S^{-1} \times G \longrightarrow G$ is the projection onto the second factor.]

(e) Assuming without proof the fact that $\mathbf{Z} + \pi\mathbf{Z}$ is dense in $\mathbf{R}$, convince yourself that ST need not be closed even if both S and T are.

Solution.

(a) If g is an element of S, then gT is open because $gT = L_{g^{-1}}^{-1}(T)$ and $L_{g^{-1}}$ is continuous by Proposition 2.44. It then follows from

$$ST = \bigcup_{s \in S} sT$$

that ST is open.

(b) If S or T is empty, then $ST \neq \varnothing$, so it is connected. Otherwise, the product $S \times T$ is connected by Question 4.4, so $ST = m(S \times T)$ is connected by Proposition 2.30.

(c) The product $S \times T$ is compact by Theorem 2.39, so $ST = m(S \times T)$ is compact by Proposition 2.37.

(d) Since inversion is a homeomorphism, it follows from Proposition 2.37 that S^{-1} is compact. The inclusion $j\colon S^{-1} \times G \longrightarrow G \times G$ is continuous by Exercise 2.23. Since T is closed, it follows from Exercise 2.13 that $m^{-1}(T) \subseteq G \times G$ is closed and then $j^{-1}\big(m^{-1}(T)\big) \subseteq S^{-1} \times G$ is closed.

We now claim that

$$ST = \pi_2\Big(j^{-1}\big(m^{-1}(T)\big)\Big);$$

and this implies ST is closed because π_2 is closed by Theorem 2.39 (here we crucially need S^{-1} to be compact). To prove this equation, we start with an element g of ST. Since $g \in ST$, there exists an element s of S and an element t of T such that $g = st$. It follows that $(s^{-1}, g) \in j^{-1}\big(m^{-1}(T)\big)$, so

$$g \in \pi_2\Big(j^{-1}\big(m^{-1}(T)\big)\Big).$$

For the other inclusion, suppose $(s', g) \in j^{-1}\big(m^{-1}(T)\big)$, It follows that $s'g \in T$, so $g \in s'^{-1}T$, which implies $g \in ST$ because $s' \in S^{-1}$. Hence the equation holds.

(e) Since $\mathbf{Z} + \pi\mathbf{Z} = \bigcup_{n \in \mathbf{Z}}(n + \pi\mathbf{Z})$, it follows from Exercise 1.2 that $\mathbf{Z} + \pi\mathbf{Z} \neq \mathbf{R}$; but we know it is dense in $\mathbf{R}$, so it cannot be closed. Hence $\mathbf{Z}$ and $\pi\mathbf{Z}$ are closed in $\mathbf{R}$, but $\mathbf{Z} + \pi\mathbf{Z}$ is not closed. □