Tutorial Week 6

Topics: Cauchy sequences, completeness, uniform continuity.

6.1. Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be uniformly continuous functions between metric spaces. Prove that $g \circ f\colon X \longrightarrow Z$ is uniformly continuous.

Solution. Let ϵ be a positive real number. The uniform continuity of f and g implies that there exists a positive real number δ such that $d_Y(y_1, y_2) < \delta$ implies $d_Z\big(g(y_1), g(y_2)\big) < \epsilon$, and there exists a positive real number γ such that $d_X(x_1, x_2) < \gamma$ implies $d_Y\big(f(x_1), f(x_2)\big) < \delta$. Hence $d_X(x_1, x_2) < \gamma$ implies $d_Z\big((g \circ f)(x_1), (g \circ f)(x_2)\big) < \epsilon$, and therefore $g \circ f$ is uniformly continuous. □

6.2. Let S be a subset of a metric space (X, d_X) and let d_S be the induced metric on S.

(a) Prove that the inclusion function $\iota_S\colon S \longrightarrow X$ is uniformly continuous.

(b) Prove that a function $f\colon (Y, d_Y) \longrightarrow (S, d_S)$ is uniformly continuous if and only if $\iota_S \circ f$ is uniformly continuous.

Solution.

(a) Let ϵ be a positive real number. Put $\delta = \epsilon$. If elements x_1 and x_2 of S satisfy $d_S(x_1, x_2) < \delta$, then

$$d_X\big(\iota_S(x_1), \iota_S(x_2)\big) = d_S(x_1, x_2) < \epsilon.$$

Hence ι_S is uniformly continous.

(b) If f is uniformly continuous, then $\iota_S \circ f$ is uniformly continuous because of part (a) and Question 6.1.

Conversely, suppose $\iota_S \circ f$ is uniformly continuous. Let ϵ be a positive real number. Pick a positive real number δ such that $d_Y(y_1, y_2) < \delta$ implies $d_X\big((\iota_S \circ f)(y_1), (\iota_S \circ f)(y_2)\big) < \epsilon$. It follows that $d_Y(y_1, y_2) < \delta$ implies

$$d_S\big(f(y_1), f(y_2)\big) = d_X\big((\iota_S \circ f)(y_1), (\iota_S \circ f)(y_2)\big) < \epsilon.$$

Hence f is uniformly continuous. □

6.3. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces and let d be a metric on $Y \times Z$ such that

$$\max\{d_Y(y_1, y_2), d_Z(z_1, z_2)\} \leqslant d\big((y_1, z_1), (y_2, z_2)\big) \leqslant d_Y(y_1, y_2) + d_Z(z_1, z_2)$$

for every pair of points (y_1, z_1) and (y_2, z_2) in $Y \times Z$.

(a) Prove that the projections $\pi_Y\colon Y \times Z \longrightarrow Y$ and $\pi_Z\colon Y \times Z \longrightarrow Z$ are uniformly continuous.

(b) Prove that a function $f\colon X \longrightarrow Y \times Z$ is uniformly continuous if and only if both $\pi_Y \circ f$ and $\pi_Z \circ f$ are.

Solution.

(a) If ϵ is a positive real number, then $d\big((y_1, z_1), (y_2, z_2)\big) < \epsilon$ implies

$$d_Y\big(\pi_Y(y_1, z_1), \pi_Y(y_2, z_2)\big) = d_Y(y_1, y_2) \leqslant d\big((y_1, z_1), (y_2, z_2)\big) < \epsilon$$

and similarly $d\big(\pi_Z(y_1, z_1), \pi_Z(y_2, z_2)\big) < \epsilon$. Hence π_Y and π_Z are uniformly continuous.

(b) If f is uniformly continuous, then it follows from Question 6.1 and part (a) that both $\pi_Y \circ f$ and $\pi_Z \circ f$ are uniformly continuous.

Conversely, suppose both $\pi_Y \circ f$ and $\pi_Z \circ f$ are uniformly continuous. Let ϵ be a positive real number. It follows from the uniform continuity of $\pi_Y \circ f$ and $\pi_Z \circ f$ that there exist positive real numbers δ_Y resp. δ_Z such that $d_X(x_1, x_2) < \delta_Y$, resp. $d_X(x_1, x_2) < \delta_Z$ imply

$$d_Y\big((\pi_Y \circ f)(x_1), (\pi_Y \circ f)(x_2)\big) < \epsilon/2 \quad \text{resp.} \quad d_Z\big((\pi_Z \circ f)(x_1), (\pi_Z \circ f)(x_2)\big) < \epsilon/2.$$

Let $\delta = \min\{\delta_Y, \delta_Z\}$. It follows that $d_X(x_1, x_2) < \delta$ implies

$$d\big(f(x_1), f(x_2)\big) \leqslant d_Y\big((\pi_Y \circ f)(x_1), (\pi_Y \circ f)(x_2)\big) + d_Z\big((\pi_Z \circ f)(x_1), (\pi_Z \circ f)(x_2)\big) < \epsilon,$$

so f is uniformly continuous. □

6.4. Let (X, d_X) and (Y, d_Y) be metric spaces and let d be the sup norm metric on $X \times Y$.

(a) Prove that the sequence $\big((x_n, y_n)\big)$ is Cauchy in $X \times Y$ if and only if (x_n) is Cauchy in X and (y_n) is Cauchy in Y.

(b) Prove that if X and Y are complete then $X \times Y$ is complete. Is the converse true?

Solution.

(a) Suppose $\big((x_n, y_n)\big)$ is a Cauchy sequence in $(X \times Y, d)$. By part (a) of Question 6.3, both projections $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are uniformly continuous. Hence $(x_n) = \big(\pi_X(x_n, y_n)\big)$ and $(y_n) = \big(\pi_Y(x_n, y_n)\big)$ are Cauchy because of Proposition 2.60.

Conversely, suppose (x_n) is Cauchy in X and (y_n) is Cauchy in Y. Fix $\varepsilon > 0$. Let $N_x \in \mathbf{N}$ be such that for all $m, n \geqslant N_x$ we have $d_X(x_m, x_n) < \varepsilon$. Let $N_y \in \mathbf{N}$ be such that for all $m, n \geqslant N_y$ we have $d_Y(y_m, y_n) < \varepsilon$. Let $N = \max\{N_x, N_y\}$, then for all $m, n \geqslant N$ we have

$$d\big((x_m, y_m), (x_n, y_n)\big) = \max\big\{d_X(x_m, x_n), d_Y(y_m, y_n)\big\} < \varepsilon,$$

so $\big((x_n, y_n)\big)$ is Cauchy in $X \times Y$.

(b) Let $\big((x_n, y_n)\big)$ be a Cauchy sequence in $X \times Y$. By part (a), (x_n) is Cauchy in X and (y_n) is Cauchy in Y. Since X and Y are complete, we have $(x_n) \longrightarrow x \in X$ and $(y_n) \longrightarrow y \in Y$. By Question 5.4, $\big((x_n, y_n)\big) \longrightarrow (x, y) \in X \times Y$.

The converse also holds: suppose $X \times Y$ is complete. Let (x_n) be a Cauchy sequence in X, and fix some $y \in Y$. Then by (a) we have that $\big((x_n, y)\big)$ is Cauchy in $X \times Y$, so $\big((x_n, y)\big) \longrightarrow (x, y) \in X \times Y$, which by Question 5.4 implies that $(x_n) \longrightarrow x \in X$. The same proof gives us that Y is complete. □

6.5. Suppose $f : X \longrightarrow Y$ is a *uniform homeomorphism* between metric spaces; that is, a homeomorphism such that both f and its inverse are uniformly continuous.

(a) Prove that a sequence (x_n) is Cauchy in X if and only if $\big(f(x_n)\big)$ is Cauchy in Y.

(b) Prove that X is complete if and only if Y is complete.

(c) Prove that $f : \mathbf{R} \longrightarrow (-\pi/2, \pi/2)$ given by $f(x) = \arctan(x)$ is uniformly continuous and a homeomorphism, but it is not a uniform homeomorphism.

(d) Do you feel strongly that uniformly continuous functions ought to preserve completeness? (After all, they preserve Cauchy sequences, and completeness is defined in terms of Cauchy sequences.)

Prove that the function f defined in part (c) does not preserve completeness though it is uniformly continuous and a homeomorphism.

Solution. Let $g\colon Y \longrightarrow X$ denote the inverse of f.

(a) By Proposition 2.60, (x_n) being Cauchy implies $\big(f(x_n)\big)$ is Cauchy, while $\big(f((x_n)\big)$ being Cauchy implies $(x_n) = \Big(g\big(f(x_n)\big)\Big)$ is Cauchy.

(b) Since X and Y are interchangeable, it suffices to prove one direction. Suppose X is complete and (y_n) is a Cauchy sequence in Y. It follows that $\big(g(y_n)\big)$ is Cauchy in X, and therefore converges to some point x in X By Theorem 2.52, $(y_n) = \Big(f\big(g(y_n)\big)\Big)$ converges to $f(x)$. Hence Y is complete.

(c) Since f has an inverse $\tan\colon (-\pi/2, \pi/2) \longrightarrow \mathbf{R}$, and both f and tan are continuous. Hence f is a homeomorphism.

Given $x_1 < x_2$, apply the Mean Value Theorem to $f(x) = \arctan(x)$ on $[x_1, x_2]$ to get some $\xi \in (x_1, x_2)$ such that

$$|f(x_2) - f(x_1)| = |f'(\xi)|\,|x_2 - x_1| = \frac{1}{1+\xi^2}\,|x_2 - x_1| \leqslant |x_2 - x_1|.$$

So for any $\varepsilon > 0$ we can take $\delta = \varepsilon$ and conclude that f is uniformly continuous.

However, its inverse $\tan\colon (-\pi/2, \pi/2) \longrightarrow \mathbf{R}$ is not uniformly continuous, because $(-\pi/2, \pi/2)$ is totally bounded (since bounded in $\mathbf{R}$), but $\mathbf{R}$ is not totally bounded. (Use Proposition 2.71.)

(d) The codomain $(-\pi/2, \pi/2)$ is not complete because $(\pi/2 - 1/n)$ is Cauchy but does not converge in $(-\pi/2, \pi/2)$. However, the domain $\mathbf{R}$ is complete. □

6.6. Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \longrightarrow Y$ a surjective continuous function. Suppose that X is complete and for all $x_1, x_2 \in X$ we have

$$d_X\big(x_1, x_2\big) \leqslant d_Y\big(f(x_1), f(x_2)\big).$$

Prove that Y is complete.

In particular, distance-preserving maps preserve completeness.

Solution. Let (y_n) be a Cauchy sequence in Y. For each $n \in \mathbf{N}$, let $x_n \in f^{-1}(y_n)$. I claim that (x_n) is a Cauchy sequence in X. Fix $\varepsilon > 0$. Let $N \in \mathbf{N}$ be such that for all $m, n \geqslant N$ we have $d_Y(y_m, y_n) < \varepsilon$. Then for all $m, n \geqslant N$ we have

$$d_X(x_m, x_n) \leqslant d_Y\big(f(x_m), f(x_n)\big) = d_Y(y_m, y_n) < \varepsilon,$$

so (x_n) is indeed Cauchy in X.

Since X is complete, we have $(x_n) \longrightarrow x \in X$, so that by the continuity of f we conclude that $(y_n) = \big(f(x_n)\big) \longrightarrow f(x) \in Y$. □

6.7. Let (X, d) be a metric space.

(a) Fix an arbitrary element $y \in X$ and consider the function $f\colon X \longrightarrow \mathbf{R}$ given by $f(x) = d(x, y)$. Prove that f is uniformly continuous.

(b) Prove that $d\colon X \times X \longrightarrow \mathbf{R}$ is uniformly continuous with respect to the sup metric D on $X \times X$.

Solution.

(a) Let $\varepsilon > 0$. Set $\delta = \varepsilon$. If $x, x' \in X$ satisfy $d(x, x') < \delta = \varepsilon$, then

$$|f(x) - f(x')| = |d(x, y) - d(x', y)| \leqslant d(x, x') < \varepsilon.$$

(b) Let $\varepsilon > 0$. By part (a), there exists positive real numbers δ_1 and δ_2 such that $d(x_1, x_1') < \delta_1$ and $d(x_2, x_2') < \delta_2$ imply

$$d_{\mathbf{R}}\big(d(x_1, x_2), d(x_1', x_2)\big) < \epsilon/2 \quad \text{and} \quad d_{\mathbf{R}}\big(d(x_1', x_2), d(x_1', x_2')\big) < \epsilon/2.$$

Set $\delta = \min\{\delta_1, \delta_2\}$. If $(x_1, x_2), (x_1', x_2') \in X \times X$ satisfy

$$\max\{d(x_1, x_1'), d(x_2, x_2')\} = D\big((x_1, x_2), (x_1', x_2')\big) < \epsilon$$

then

$$d_{\mathbf{R}}\big(d(x_1, x_2), d(x_1', x_2')\big) \leqslant d_{\mathbf{R}}\big(d(x_1, x_2), d(x_1', x_2)\big) + d_{\mathbf{R}}\big(d(x_1', x_2), d(x_1', x_2')\big) < \epsilon.$$

Hence d is uniformly continuous. $\square$