

## Tutorial Week 7

**Topics:** Contractions, (total) boundedness, uniform convergence.

**7.1.** Find a non-empty metric space  $X$  and a contraction  $f: X \rightarrow X$  such that  $f$  has no fixed points.

*Solution.* Let  $X = (0, \infty)$ , which is given the Euclidean topology, and let  $f: X \rightarrow X$  be the function defined by  $f(x) = x/2$ . This is a contraction because if  $x$  and  $y$  are positive real numbers then

$$d_X(f(x), f(y)) = \left| \frac{x-y}{2} \right| = \frac{1}{2}|x-y| = \frac{1}{2}d_X(x, y).$$

It has no fixed points because  $f(x) = x$  implies  $x = 0$ , but  $0 \notin (0, \infty)$ . □

**7.2.** Find a bounded subset of a metric space that is not totally bounded.

*Solution.* Endow  $\mathbf{N}$  with the discrete topology. The set  $\mathbf{N}$  is bounded because  $\mathbf{N} = \mathbf{B}_2(0)$ . However, if  $n$  is an element of  $\mathbf{N}$ , then  $\mathbf{B}_1(n) = \{n\}$ , so it is impossible to cover  $\mathbf{N}$  by finitely many open balls of radius 1. □

**7.3.**

- (a) Prove that every subspace of a totally bounded space is totally bounded.
- (b) Suppose a metric space  $X$  has a totally bounded dense subset  $D$ . Prove that  $X$  is totally bounded.
- (c) Prove that a metric space  $X$  is totally bounded if and only if it is isometric to a subspace of a compact metric space. [*Hint:* Completion.]

*Solution.*

- (a) Let  $S$  be a subspace of a totally bounded space  $X$ . If  $(x_n)$  be a sequence in  $S$ , then it is also a sequence in  $X$ , so it has a Cauchy subsequence by [Proposition 2.73](#). Now it again follows from [Proposition 2.73](#) that  $S$  is totally bounded.
- (b) Let  $\epsilon$  be a positive real number. Since  $D$  is totally bounded, there exists a natural number  $N$  and elements  $x_1, \dots, x_N$  of  $D$  such that

$$D \subseteq \bigcup_{n=1}^N \mathbf{B}_{\epsilon/2}(x_n).$$

Since  $X$  is the closure of  $D$  in  $X$ , it follows that

$$X \subseteq \bigcup_{n=1}^N \overline{\mathbf{B}_{\epsilon/2}(x_n)} \subseteq \bigcup_{n=1}^N \mathbf{B}_{\epsilon}(x_n).$$

- (c) Suppose a metric space  $X$  is totally bounded and let  $\widehat{X}$  be a completion of  $X$  with distance-preserving function  $\iota: X \rightarrow \widehat{X}$ . By the definition of completion, we know that  $X$  is isometric to  $\iota(X)$ , so  $\iota(X)$  is totally bounded by [Proposition 2.71](#). It follows from part (b) that the completion  $\widehat{X}$  is totally bounded, and is therefore compact by the Heine–Borel theorem ([Theorem 2.74](#)). Hence  $X$  is isometric to the subspace  $\iota(X)$  of the compact metric space  $\widehat{X}$ .

Conversely, suppose  $Y$  is a compact subspace,  $S$  is a subspace of  $Y$ , and  $f: S \rightarrow X$  is an isometry. It follows from the Heine–Borel theorem ([Theorem 2.74](#)) that  $Y$  is totally bounded, and therefore  $S$  is totally bounded by part (a). Hence  $X = f(S)$  is totally bounded by [Proposition 2.71](#). □

7.4. For each  $n \in \mathbf{N}$ , consider the function  $f_n: [0, 1] \rightarrow \mathbf{R}$  given by

$$f_n(x) = \frac{x^2}{1 + nx}.$$

- (a) Prove that  $f_n$  is bounded, for all  $n \in \mathbf{N}$ .
- (b) Find the pointwise limit  $f$  of the sequence  $(f_n)$ .
- (c) For any  $n \in \mathbf{N}$ , compute the uniform distance  $d_\infty(f_n, f)$ .
- (d) Does the sequence  $(f_n)$  converge uniformly to  $f$ ?

*Solution.*

- (a) Fix  $n \in \mathbf{N}$ . If  $x \in [0, 1]$  then  $0 \leq x^2 \leq 1$  and  $1 + n \geq 1 + nx \geq 1$ , so  $1/(1+n) \leq 1/(1+nx) \leq 1$ , so

$$0 \leq \frac{x^2}{1 + nx} \leq 1.$$

Thus  $f_n$  is bounded.

- (b) For  $x = 0$  the sequence  $(f_n(x)) = (f_n(0))$  is the constant sequence 0, so  $f(0) = 0$ .

For  $0 < x \leq 1$  we have

$$\lim_{n \rightarrow \infty} \frac{x^2}{1 + nx} = x^2 \lim_{n \rightarrow \infty} \frac{1}{1 + nx} = 0,$$

so  $f(x) = 0$ .

We conclude that the pointwise limit is the constant function  $f = 0$  on  $[0, 1]$ .

- (c) We have

$$d_\infty(f_n, f) = \sup_{x \in [0, 1]} \frac{x^2}{1 + nx}.$$

Since  $f_n$  is continuous on a compact interval, it attains its extremal values in  $[0, 1]$ ; in particular its global maximum is at  $x = 0$  or at  $x = 1$  or at a stationary point in  $(0, 1)$ .

The derivative is

$$f'_n(x) = \frac{x(2 + nx)}{(1 + nx)^2},$$

so the stationary points are 0 and  $-2/n$ , neither of which lies in  $(0, 1)$ . Moreover  $f_n(0) = 0$  and  $f_n(1) = 1/(1 + n)$ , so we conclude that

$$d_\infty(f_n, f) = \frac{1}{1 + n}.$$

- (d) We have  $(d_\infty(f_n), f) \rightarrow 0$  as  $n \rightarrow \infty$ , so the convergence is uniform. □

7.5. Let  $f_0: \mathbf{R} \rightarrow \mathbf{R}$  be the function defined by

$$f_0(x) = \begin{cases} 1 + x & \text{if } -1 \leq x \leq 0, \\ 1 - x & \text{if } 0 < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each positive integer  $n$ , define  $f_n: \mathbf{R} \rightarrow \mathbf{R}$  by

$$f_n(x) = f_0(x - n).$$

- (a) Prove that  $f_n$  is bounded, for all  $n \in \mathbf{N}$ .
- (b) Find the pointwise limit  $f$  of the sequence  $(f_n)$ .
- (c) For any  $n \in \mathbf{N}$ , compute the uniform distance  $d_\infty(f_n, f)$ .
- (d) Does the sequence  $(f_n)$  converge uniformly to  $f$ ?

*Solution.*

- (a) It is straightforward to see that  $f_n(\mathbf{R}) = [0, 1]$  for every natural number  $n$ . Thus  $f_n$  is bounded.
- (b) Fix a real number  $x$  and let  $N$  be the smallest positive integer such that  $x < N$ . It follows from the definition of  $f_n$  that  $f_n(x) = 0$  if  $n > N$ . Hence  $(f_n(x)) \rightarrow 0$  as  $n \rightarrow \infty$  and therefore  $f$  is the constant function sending every real number to 0.
- (c) We have
 
$$d_\infty(f_n, f) = \sup_{x \in \mathbf{R}} \{d_{\mathbf{R}}(f(x), 0)\} = d_{\mathbf{R}}(f_n(n), 0) = 1.$$
- (d) Since  $d_\infty(f_n, f)$  does not converge to 0, the sequence  $(f_n)$  does not converge to  $f$  uniformly. □

**7.6.**

- (a) Prove that every closed interval on  $\mathbf{R}$  is compact.
- (b) Prove that every closed ball in  $\mathbf{R}^n$  is compact.
- (c) (*The classical Heine–Borel theorem*) Prove that a subset of  $\mathbf{R}^n$  is compact if and only if it is bounded and closed.
- (d) Prove that every bounded subset of  $\mathbf{R}^n$  is totally bounded.

*Solution.*

- (a) Let  $a$  and  $b$  be real numbers and let  $r = \max\{|a|, |b|\}$ . Since  $[a, b] \subseteq \mathbf{B}_r(0)$ , it follows that  $[a, b]$  is bounded, and therefore totally bounded by [Example 2.69](#). As a closed subset of the complete space  $\mathbf{R}$ , the closed interval  $[a, b]$  is also complete. Hence  $[a, b]$  is compact by the Heine–Borel theorem ([Theorem 2.74](#)).

- (b) Let  $r$  be a positive real number and let  $v$  be an element of  $\mathbf{R}^n$ .

We start with proving  $\mathbf{D}_r(0)$  is compact. Since  $[-r, r]$  is compact, it follows from [Theorem 2.39](#) that  $[-r, r]^n$  is compact. Since  $\mathbf{D}_r(0)$  is a closed subset of  $[-r, r]^n$ , it follows from [Proposition 2.36](#) that  $\mathbf{D}_r(0)$  is compact.

Let  $R_v : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be the continuous function defined by  $R_v(w) = v + w$  (see [Proposition 2.44](#)). Since

$$\mathbf{D}_r(v) = R_v(\mathbf{D}_r(0)),$$

it follows from [Proposition 2.37](#) that  $\mathbf{D}_r(v)$  is compact.

- (c) Suppose  $K$  is a compact subset of  $\mathbf{R}^n$ . It follows from [Proposition 2.35](#) that  $K$  is closed and it follows from the Heine–Borel theorem ([Theorem 2.74](#)) that  $K$  is totally bounded, which implies  $K$  is bounded by [Exercise 2.47](#).

Conversely, suppose  $K$  is a bounded closed subset of  $\mathbf{R}^n$ . It follows from [Exercise 2.45](#) that  $K$  is contained in some closed ball  $\mathbf{D}_r(v)$ , which is compact by part (b). Hence  $K$  is compact since it is a closed subset of a compact set (see [Proposition 2.36](#)).

- (d) Let  $S$  be a bounded subset of  $\mathbf{R}^n$  and suppose  $S$  is contained in some closed ball  $\mathbf{D}_r(v)$  (see [Exercise 2.45](#)), which is compact by part (b) and therefore totally bounded by the Heine–Borel theorem ([Theorem 2.74](#)). It now follows from part (c) of [Question 7.3](#) that  $S$  is totally bounded.  $\square$

**7.7.** Let  $A = (a_{ij})$  be an  $n \times n$  real matrix with all  $|a_{ij}| < 1$ . Prove that any real eigenvalue  $\lambda$  of  $A$  satisfies  $|\lambda| < n$ .

[*Hint:* Show that if  $|\lambda| \geq n$  then the function  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  given by  $f(v) = \frac{1}{\lambda}Av$  is a contraction for the sup metric topology on  $\mathbf{R}^n$ ; then use the Banach Fixed Point Theorem.]

*Solution.* Suppose  $|\lambda| \geq n$ .

We start by proving that the function  $f$  from the hint is a contraction. If  $v$  and  $w$  are elements of  $\mathbf{R}^n$ , then

$$\begin{aligned} d(f(v), f(w)) &= \max_{i \in \{1, \dots, n\}} |f(v)_i - f(w)_i| = \max_i \left| \sum_{j=1}^n \frac{1}{\lambda} a_{ij}(v_j - w_j) \right| \\ &= \frac{1}{|\lambda|} \max_i \left| \sum_{j=1}^n a_{ij}(v_j - w_j) \right| \leq \frac{1}{|\lambda|} \max_i \sum_{j=1}^n |a_{ij}| |v_j - w_j| \\ &< \frac{1}{|\lambda|} \max_i \sum_{j=1}^n |v_j - w_j| = \frac{1}{|\lambda|} \sum_{j=1}^n |v_j - w_j| \\ &\leq \frac{n}{|\lambda|} \max_{j \in \{1, \dots, n\}} |v_j - w_j| = \frac{n}{|\lambda|} d(v, w) \\ &\leq d(v, w). \end{aligned}$$

Hence  $f$  is a contraction.

$\mathbf{R}$  is complete, so  $\mathbf{R}^n$  is complete by [Question 6.4](#). Now it follows from the Banach Fixed Point Theorem ([Theorem 2.66](#)) that there is a unique element  $v$  of  $\mathbf{R}^n$  such that  $v = f(v) = \frac{1}{\lambda}Av$ , which is equivalent to  $Av = \lambda v$ . Since the zero vector satisfies this condition, this unique vector has to be the zero vector, so  $\lambda$  cannot be an eigenvalue of  $A$ .  $\square$