Tutorial Week 7

Topics: Contractions, (total) boundedness, uniform convergence.

7.1. Find a non-empty metric space X and a contraction $f\colon X \longrightarrow X$ such that f has no fixed points.

Solution. Let $X = (0, \infty)$, which is given the Euclidean topology, and let $f\colon X \longrightarrow X$ be the function defined by $f(x) = x/2$. This is a contraction because if x and y are positive real numbers then

$$d_X\big(f(x), f(y)\big) = \left|\frac{x-y}{2}\right| = \frac{1}{2}|x-y| = \frac{1}{2}d_X(x,y).$$

It has no fixed points because $f(x) = x$ implies $x = 0$, but $0 \notin (0, \infty)$. □

7.2. Find a bounded subset of a metric space that is not totally bounded.

Solution. Endow $\mathbf{N}$ with the discrete topology. The set $\mathbf{N}$ is bounded because $\mathbf{N} = \mathbf{B}_2(0)$. However, if n is an element of $\mathbf{N}$, then $\mathbf{B}_1(n) = \{n\}$, so it is impossible to cover $\mathbf{N}$ by finitely many open balls of radius 1. □

7.3.

(a) Prove that every subspace of a totally bounded space is totally bounded.

(b) Suppose a metric space X has a totally bounded dense subset D. Prove that X is totally bounded.

(c) Prove that a metric space X is totally bounded if and only if it is isometric to a subspace of a compact metric space. [*Hint*: Completion.]

Solution.

(a) Let S be a subspace of a totally bounded space X. If (x_n) be a sequence in S, then it is also a sequence in X, so it has a Cauchy subsequence by Proposition 2.73. Now it again follows from Proposition 2.73 that S is totally bounded.

(b) Let ϵ be a positive real number. Since D is totally bounded, there exists a natural number N and elements $x_1, \ldots, x_N$ of D such that

$$D \subseteq \bigcup_{n=1}^{N} \mathbf{B}_{\epsilon/2}(x_n).$$

Since X is the closure of D in X, it follows that

$$X \subseteq \bigcup_{n=1}^{N} \overline{\mathbf{B}_{\epsilon/2}(x_n)} \subseteq \bigcup_{n=1}^{N} \mathbf{B}_{\epsilon}(x_n).$$

(c) Suppose a metric space X is totally bounded and let $\widehat{X}$ be a completion of X with distance-preserving function $\iota\colon X \longrightarrow \widehat{X}$. By the definition of completion, we know that X is isometric to $\iota(X)$, so $\iota(X)$ is totally bounded by Proposition 2.71. It follows from part (b) that the completion $\widehat{X}$ is totally bounded, and is therefore compact by the Heine–Borel theorem (Theorem 2.74). Hence X is isometric to the subspace $\iota(X)$ of the compact metric space $\widehat{X}$.

Conversely, suppose Y is a compact subspace, S is a subspace of Y, and $f\colon S \longrightarrow X$ is an isometry. It follows from the Heine–Borel theorem (Theorem 2.74) that Y is totally bounded, and therefore S is totally bounded by part (a). Hence $X = f(S)$ is totally bounded by Proposition 2.71. □

7.4. For each $n \in \mathbf{N}$, consider the function $f_n\colon [0,1] \longrightarrow \mathbf{R}$ given by

$$f_n(x) = \frac{x^2}{1+nx}.$$

(a) Prove that f_n is bounded, for all $n \in \mathbf{N}$.

(b) Find the pointwise limit f of the sequence (f_n).

(c) For any $n \in \mathbf{N}$, compute the uniform distance $d_\infty(f_n, f)$.

(d) Does the sequence (f_n) converge uniformly to f?

Solution.

(a) Fix $n \in \mathbf{N}$. If $x \in [0,1]$ then $0 \leqslant x^2 \leqslant 1$ and $1+n \geqslant 1+nx \geqslant 1$, so $1/(1+n) \leqslant 1/(1+nx) \leqslant 1$, so

$$0 \leqslant \frac{x^2}{1+nx} \leqslant 1.$$

Thus f_n is bounded.

(b) For $x = 0$ the sequence $(f_n(x)) = (f_n(0))$ is the constant sequence 0, so $f(0) = 0$.

For $0 < x \leqslant 1$ we have

$$\lim_{n\longrightarrow\infty} \frac{x^2}{1+nx} = x^2 \lim_{n\longrightarrow\infty} \frac{1}{1+nx} = 0,$$

so $f(x) = 0$.

We conclude that the pointwise limit is the constant function $f = 0$ on $[0,1]$.

(c) We have

$$d_\infty(f_n, f) = \sup_{x\in[0,1]} \frac{x^2}{1+nx}.$$

Since f_n is continuous on a compact interval, it attains its extremal values in $[0,1]$; in particular its global maximum is at $x = 0$ or at $x = 1$ or at a stationary point in $(0,1)$.

The derivative is

$$f_n'(x) = \frac{x(2+nx)}{(1+nx)^2},$$

so the stationary points are 0 and $-2/n$, neither of which lies in $(0,1)$. Moreover $f_n(0) = 0$ and $f_n(1) = 1/(1+n)$, so we conclude that

$$d_\infty(f_n, f) = \frac{1}{1+n}.$$

(d) We have $(d_\infty(f_n), f) \longrightarrow 0$ as $n \longrightarrow \infty$, so the convergence is uniform. □

7.5. Let $f_0\colon \mathbf{R} \longrightarrow \mathbf{R}$ be the function defined by

$$f_0(x) = \begin{cases} 1+x & \text{if } -1 \leqslant x \leqslant 0, \\ 1-x & \text{if } 0 < x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each positive integer n, define $f_n\colon \mathbf{R} \longrightarrow \mathbf{R}$ by

$$f_n(x) = f_0(x-n).$$

(a) Prove that f_n is bounded, for all $n \in \mathbf{N}$.

(b) Find the pointwise limit f of the sequence (f_n).

(c) For any $n \in \mathbf{N}$, compute the uniform distance $d_\infty(f_n, f)$.

(d) Does the sequence (f_n) converge uniformly to f?

Solution.

(a) It is straightforward to see that $f_n(\mathbf{R}) = [0,1]$ for every natural number n. Thus f_n is bounded.

(b) Fix a real number x and let N be the smallest positive integer such that $x < N$. It follows from the definition of f_n that $f_n(x) = 0$ if $n > N$. Hence $\big(f(x)\big) \longrightarrow 0$ as $n \longrightarrow \infty$ and therefore f is the constant function sending every real number to 0.

(c) We have

$$d_\infty(f_n, f) = \sup_{x \in \mathbf{R}} \big\{ d_{\mathbf{R}}\big(f(x), 0\big) \big\} = d_{\mathbf{R}}\big(f_n(n), 0\big) = 1.$$

(d) Since $d_\infty(f_n, f)$ does not converge to 0, the sequence (f_n) does not converge to f uniformly. □

7.6.

(a) Prove that every closed interval on $\mathbf{R}$ is compact.

(b) Prove that every closed ball in $\mathbf{R}^n$ is compact.

(c) (*The classical Heine–Borel theorem*) Prove that a subset of $\mathbf{R}^n$ is compact if and only if it is bounded and closed.

(d) Prove that every bounded subset of $\mathbf{R}^n$ is totally bounded.

Solution.

(a) Let a and b be real numbers and let $r = \max\{|a|, |b|\}$. Since $[a,b] \subseteq \mathbf{B}_r(0)$, it follows that $[a,b]$ is bounded, and therefore totally bounded by Example 2.69. As a closed subset of the complete space $\mathbf{R}$, the closed interval $[a,b]$ is also complete. Hence $[a,b]$ is compact by the Heine–Borel theorem (Theorem 2.74).

(b) Let r be a positive real number and let v be an element of $\mathbf{R}^n$.

We start with proving $\mathbf{D}_r(0)$ is compact. Since $[-r,r]$ is compact, it follows from Theorem 2.39 that $[-r,r]^n$ is compact. Since $\mathbf{D}_r(0)$ is a closed subset of $[-r,r]^n$, it follows from Proposition 2.36 that $\mathbf{D}_r(0)$ is compact.

Let $R_v \colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be the continuous function defined by $R_v(w) = v + w$ (see Proposition 2.44). Since

$$\mathbf{D}_r(v) = R_v\big(\mathbf{D}_r(0)\big),$$

it follows from Proposition 2.37 that $\mathbf{D}_r(v)$ is compact.

(c) Suppose K is a compact subset of $\mathbf{R}^n$. It follows from Proposition 2.35 that K is closed and it follows from the Heine–Borel theorem (Theorem 2.74) that K is totally bounded, which implies K is bounded by Exercise 2.47.

Conversely, suppose K is a bounded closed subset of $\mathbf{R}^n$. It follows from Exercise 2.45 that K is contained in some closed ball $\mathbf{D}_r(v)$, which is compact by part (b). Hence K is compact since it is a closed subset of a compact set (see Proposition 2.36).

(d) Let S be a bounded subset of $\mathbf{R}^n$ and suppose S is contained in some closed ball $\mathbf{D}_r(v)$ (see Exercise 2.45), which is compact by part (b) and therefore totally bounded by the Heine–Borel theorem (Theorem 2.74). It now follows from part (c) of Question 7.3 that S is totally bounded. □

7.7. (*) Let $A = (a_{ij})$ be an $n \times n$ real matrix with all $|a_{ij}| < 1$. Prove that any real eigenvalue λ of A satisfies $|\lambda| < n$.

[*Hint*: Show that if $|\lambda| \geqslant n$ then the function $f\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ given by $f(v) = \frac{1}{\lambda} Av$ is a contraction for the sup metric topology on $\mathbf{R}^n$; then use the Banach Fixed Point Theorem.]

Solution. Suppose $|\lambda| \geqslant n$.

We start by proving that the function f from the hint is a contraction. If v and w are elements of $\mathbf{R}^n$, then

$$\begin{aligned} d\big(f(v), f(w)\big) &= \max_{i \in \{1,\dots,n\}} |f(v)_i - f(w)_i| = \max_i \left| \sum_{j=1}^n \frac{1}{\lambda} a_{ij}(v_j - w_j) \right| \\ &= \frac{1}{|\lambda|} \max_i \left| \sum_{j=1}^n a_{ij}(v_j - w_j) \right| \leqslant \frac{1}{|\lambda|} \max_i \sum_{j=1}^n |a_{ij}|\,|v_j - w_j| \\ &< \frac{1}{|\lambda|} \max_i \sum_{j=1}^n |v_j - w_j| = \frac{1}{|\lambda|} \sum_{j=1}^n |v_j - w_j| \\ &\leqslant \frac{n}{|\lambda|} \max_{j \in \{1,\dots,n\}} |v_j - w_j| = \frac{n}{|\lambda|} d(v, w) \\ &\leqslant d(v, w). \end{aligned}$$

Hence f is a contraction.

$\mathbf{R}$ is complete, so $\mathbf{R}^n$ is complete by Question 6.4. Now it follows from the Banach Fixed Point Theorem (Theorem 2.66) that there is a unique element v of $\mathbf{R}^n$ such that $v = f(v) = \frac{1}{\lambda} Av$, which is equivalent to $Av = \lambda v$. Since the zero vector satisfies this condition, this unique vector has to be the zero vector, so λ cannot be an eigenvalue of A. □