Tutorial Week 8

Topics: normed vector spaces, inner product spaces.

8.1. (*) Let V be a normed vector space. Prove that $(V, +)$ is a topological group.

Solution. By Proposition 3.2, the addition $a\colon V \times V \longrightarrow V$ and the scalar multiplication $s\colon \mathbf{F} \times V \longrightarrow V$ are both continuous. It is straightforward to verify that the inversion $i\colon V \longrightarrow V$ defined by $i(v) = -v$ is equal to the composition $s' \circ f$, where $s'\colon \{-1\} \times V \longrightarrow V$ is the restriction of s to $\{-1\} \times V$ and $f\colon V \longrightarrow \{-1\} \times V$ defined by $f(v) = (-1, v)$. The function s' is continuous because it is the restriction of the continuous function s to $\{-1\} \times V$, while f' is continuous by Exercise 2.20, so the inversion i is continuous by Question 2.9. Hence V is a topological group. □

8.2. Let $(V, \|\cdot\|)$ be a normed space and let $S \subseteq V$ be a subset. Prove that the closure $\overline{\operatorname{Span}(S)}$ of the span of S is the smallest closed subspace of V that contains S.

Solution. We know that $\operatorname{Span}(S)$ is a subspace of V, and by Proposition 3.7 that $\overline{\operatorname{Span}(S)}$ is a closed subspace of V.

Let $W \subseteq V$ be some closed subspace of V that contains S. Then $\operatorname{Span}(S) \subseteq W$, and so $\overline{\operatorname{Span}(S)} \subseteq \overline{W} = W$, whence the minimality property. □

8.3. Let v be a non-zero vector in a normed vector space V. Prove that the one-dimensional subspace $\mathbf{F}v := \operatorname{Span}(v)$ of V is isometric to $\mathbf{F}$.

Solution. Let $f\colon \mathbf{F} \longrightarrow \mathbf{F}v$ be the bijection defined by $f(\alpha) = \frac{\alpha}{\|v\|}v$. If α and β are elements of $\mathbf{F}$, then

$$d_{\mathbf{F}}(\alpha, \beta) = |\alpha - \beta| = \frac{|\alpha - \beta|}{\|v\|}v = \left\| \frac{\alpha}{\|v\|}v - \frac{\beta}{\|v\|}v \right\| = d_V\big(f(\alpha), f(\beta)\big).$$

Hence f is an isometry. □

8.4. Let W be a finite-dimensional subspace of a normed vector space V. Prove that W is a closed subset of V.

Solution. If (w_n) is a sequence in W and it converges to v in V, then (w_n) is Cauchy by Proposition 2.53. It follows from Proposition 3.8 that W is complete, so $v \in W$. Hence W is closed by part (c) of Proposition 2.50. □

8.5. Prove that equivalence of norms is an equivalence relation.

Solution. The relation is reflexive because if $\|\cdot\|$ is a norm on a vector space V, then

$$1 \cdot \|v\| \leqslant \|v\| \leqslant 1 \cdot \|v\| \qquad \text{for all } v \in V.$$

The relation is symmetric because if $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on a vector space V, and if there exists positive real numbers m and M such that

$$m\|v\|_1 \leqslant \|v\|_2 \leqslant M\|v\|_1 \qquad \text{for all } v \in V,$$

then

$$\frac{1}{M}\|v\|_2 \leqslant \|v\|_1 \leqslant \frac{1}{m}\|v\|_2 \qquad \text{for all } v \in V.$$

The relation is transitive because if $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_3$ are norms on a vector space V, and if there exists positive real numbers m, k, M, and K such that

$$m\|v\|_1 \leqslant \|v\|_2 \leqslant M\|v\|_1 \quad \text{and} \quad k\|v\|_2 \leqslant \|v\|_3 \leqslant K\|v\|_2 \qquad \text{for all } v \in V,$$

then

$$mk\|v\|_1 \leqslant \|v\|_2 \leqslant MK\|v\|_1 \qquad \text{for all } v \in V.$$ □

8.6. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on a vector space V.

(a) Prove that the identity function $\mathrm{id}_V\colon (V, \|\cdot\|_1) \longrightarrow (V, \|\cdot\|_2)$ is uniformly continuous.

(b) Prove that $(V, \|\cdot\|_1)$ is Banach if and only if $(V, \|\cdot\|_2)$ is Banach.

Solution.

(a) Let m and M be positive real numbers such that

$$m\|v\|_1 \leqslant \|v\|_2 \leqslant M\|v\|_1 \qquad \text{for all } v \in V$$

and let d_1 and d_2 be the metrics defined by $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively (see Proposition 3.1 for the definition). If ϵ is a positive real number, then $d_1(v,w) < \epsilon/M$ implies

$$d_2\big(\mathrm{id}_V(v), \mathrm{id}_V(w)\big) = d_2(v,w) = \|v-w\|_2 \leqslant M\|v-w\|_1 = Md_1(v,w) < \epsilon.$$

Hence $\mathrm{id}_V\colon (V, \|\cdot\|_1) \longrightarrow (V, \|\cdot\|_2)$ is uniformly continuous.

(b) By Question 8.5 and part (a), the equivalence between $\|\cdot\|_1$ and $\|\cdot\|_2$ implies the identity functions $\mathrm{id}_V\colon (V, \|\cdot\|_1) \longrightarrow (V, \|\cdot\|_2)$ and $\mathrm{id}_V\colon (V, \|\cdot\|_2) \longrightarrow (V, \|\cdot\|_1)$ are both uniformly continuous, so $\mathrm{id}_V\colon (V, \|\cdot\|_1) \longrightarrow (V, \|\cdot\|_2)$ is a uniform homeomorphism (see Question 6.5). It then follows from part (b) of Question 6.5 that $(V, \|\cdot\|_1)$ is complete if and only if $(V, \|\cdot\|_2)$ is complete. □

8.7. Prove that the following norms on $\mathbf{R}^n$ are not defined by inner products:

(a) the ℓ^1-norm defined by

$$\big\|(x_1, \dots, x_n)\big\|_1 = \sum_{i=1}^n |x_i|,$$

(b) the ℓ^∞-norm defined by

$$\big\|(x_1, \dots, x_n)\big\|_\infty = \max\big\{|x_1|, \dots, |x_n|\big\}.$$

Solution. We will verify that neither of the two norms satisfy the Parallelogram Law (Proposition 3.12), so they cannot be defined by inner products. Let $e_1 = (1, 0, \dots, 0)$ and $e_2 = (0, 1, 0, \dots, 0)$.

(a) We have $\|e_1\|_1 = \|e_2\|_1 = 1$ and $\|e_1 + e_2\|_1 = \|e_1 - e_2\|_1 = 2$, so

$$\|e_1+e_2\|_1^2 + \|e_1-e_2\|_1^2 = 8 \neq 4 = 2\big(\|e_1\|_1^2 + \|e_2\|_1^2\big).$$

(b) We have $\|e_1\|_\infty = \|e_2\|_\infty = \|e_1 + e_2\|_\infty = \|e_1 - e_2\|_\infty = 1$, so

$$\|e_1+e_2\|_\infty^2 + \|e_1-e_2\|_\infty^2 = 2 \neq 4 = 2\big(\|e_1\|_\infty^2 + \|e_2\|_\infty^2\big).$$ □

8.8. Let $(V, \langle \cdot\,, \cdot \rangle)$ be a complex inner product space and let $T\colon V \longrightarrow V$ be a linear operator. Show that $T = 0$ if and only if $\langle Tv, v\rangle = 0$ for every vector v in V.

Is this true for real inner product spaces?

Solution. If $T = 0$, then $\langle Tv, v\rangle = \langle 0, v\rangle = 0$ for every vector v in V.

Conversely, suppose $\langle Tv, v\rangle = 0$ for every vector v in V. If v and w are two vectors in V, then

$$\begin{aligned} 0 &= \big\langle T(v+w), v+w \big\rangle \\ &= \langle Tv, v\rangle + \langle Tw, v\rangle + \langle Tv, w\rangle + \langle Tw, w\rangle \\ &= \langle Tv, w\rangle + \langle Tw, v\rangle. \end{aligned}$$

Substituting v by iv gives

$$0 = \big\langle T(iv), w\big\rangle + \langle Tw, iv\rangle = i\langle Tv, w\rangle - i\langle Tw, v\rangle,$$

and it follows that

$$0 = \langle Tv, w\rangle - \langle Tw, v\rangle.$$

Hence $\langle Tv, w\rangle = \langle Tw, w\rangle = 0$. Since the inner product is non-degenerate and w is an arbitrary vector in V, it follows that $Tv = 0$ for every vector v in V, and therefore $T = 0$.

This statement does not hold for real vector spaces. Let $V = \mathbf{R}^2$ with the inner product defined by

$$\big\langle (x_1, y_1), (x_2, y_2) \big\rangle = x_1 x_2 + y_1 y_2$$

and let T be the linear operator defined by $T(x, y) = (y, -x)$. For every vector (x, y) in V, we have

$$\big\langle T(x, y), (x, y)\big\rangle = \big\langle (y, -x), (x, y)\big\rangle = 0,$$

but $T \neq 0$ because $T(1, 0) = (0, -1)$. $\square$