Tutorial Week 9

Topics: sequence spaces, series.

9.1 (Hölder's Inequality for ℓ^1 and ℓ^∞)**.** Prove that if $u = (u_n) \in \ell^\infty$ and $v = (v_n) \in \ell^1$, then

$$\sum_{n=1}^{\infty} |u_n v_n| \leqslant \|u\|_{\ell^\infty} \|v\|_{\ell^1}.$$

Solution. Very straightforward.

By the definition of ℓ^∞ and the ℓ^∞-norm, we have $|u_n| \leqslant \|u\|_{\ell^\infty}$ for all $n \in \mathbf{N}$. Therefore for any $m \in \mathbf{N}$ we have

$$\sum_{n=1}^{m} |u_n v_n| \leqslant \|u\|_\infty \sum_{n=1}^{m} |v_n|,$$

but the latter series converges because $v \in \ell^1$, to $\|v\|_{\ell^1}$ and we get

$$\sum_{n=1}^{\infty} |u_n v_n| \leqslant \|u\|_{\ell^\infty} \|v\|_{\ell^1}. \qquad \square$$

9.2. Prove that the norms on the sequence spaces ℓ^∞ and ℓ^p for $p \neq 2$ cannot defined by inner products.

Solution. We will verify that none of the norms in the question satisfy the Parallelogram Law (Proposition 3.12), so they cannot be defined by inner products. Consider $e_1 = (1, 0, 0, 0, 0, \dots)$ and $e_2 = (0, 1, 0, 0, 0, \dots)$. Then

$$\|e_1 + e_2\|_{\ell^\infty}^2 + \|e_1 - e_2\|_{\ell^\infty}^2 = 2 \neq 4 = 2\big(\|e_1\|_{\ell^\infty}^2 + \|e_2\|_{\ell^\infty}^2\big)$$

and

$$\|e_1 + e_2\|_{\ell^p}^2 + \|e_1 - e_2\|_{\ell^p}^2 = 2 \times 2^{2/p} \neq 4 = 2\big(\|e_1\|_{\ell^p}^2 + \|e_2\|_{\ell^p}^2\big). \qquad \square$$

9.3. Suppose $1 \leqslant p \leqslant q$. Prove that

$$\ell^p \subseteq \ell^q.$$

Show that if $p < q$ then the inclusion is strict: $\ell^p \subsetneq \ell^q$.

Solution. We prove that

$$\|x\|_{\ell^q} \leqslant \|x\|_{\ell^p} \qquad \text{for all } x \in \ell^p.$$

If $\|x\|_{\ell^p} = 0$ then $x = 0$ so $\|x\|_{\ell^q} = 0$ and the inequality obviously holds. So suppose $x \neq 0$, then by dividing through by $\|x\|_{\ell^p}$ we can reduce to proving that

$$\|x\|_{\ell^q} \leqslant 1 \qquad \text{for all } x \text{ such that } \|x\|_{\ell^p} = 1.$$

But if $\|x\|_{\ell^p} = 1$ then

$$\sum_{n=1}^{\infty} |x_n|^p = 1,$$

which means that for all $n \in \mathbf{N}$ we have $|x_n|^p \leqslant 1$, so $|x_n| \leqslant 1$. However, $p \leqslant q$ and $|x_n| \leqslant 1$ implies that $|x_n|^q \leqslant |x_n|^p$ for all $n \in \mathbf{N}$, so that

$$\|x\|_{\ell^q}^q = \sum_{n=1}^{\infty} |x_n|^q \leqslant \sum_{n=1}^{\infty} |x_n|^p = 1.$$

If $p < q$ then $\alpha := q/p > 1$. For each $n \in \mathbf{N}$, let

$$x_n = \frac{1}{n^{1/p}},$$

so that

$$|x_n|^p = \frac{1}{n}, \qquad |x_n|^q = \frac{1}{n^\alpha}.$$

We have

$$\|(x_n)\|_{\ell^p} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty, \qquad \|(x_n)\|_{\ell^q} = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty,$$

so $(x_n) \in \ell^q \setminus \ell^p$. □

9.4. Prove that every finite-dimensional normed vector space is separable.

Solution. Let V be a finite-dimensional normed vector space. If V is a complex normed vector space, then we can restrict the scalars to real numbers to make V a real normed vector space. As a real vector space V is still finite-dimensional because $\dim_{\mathbf{C}} V = 2 \dim_{\mathbf{R}} V$. Hence, without loss of generality, we assume V is a real normed vector space.

Let $v_1, \ldots, v_n$ be a basis of V. By Theorem 3.6, the norm on V is equivalent to the norm $\|\cdot\|_1$ defined by

$$\|\alpha_1 v_1 + \cdots + \alpha_n v_n\|_1 = |\alpha_1| + \cdots + |\alpha_n|.$$

Since separability is a property of topological spaces and equivalent norms give rise to the same topology (see Exercise 3.2), we can assume without loss of generality that V is equipped with the norm $\|\cdot\|_1$.

Let W be the $\mathbf{Q}$-vector space spanned by $\{v_1, \ldots, v_n\}$. We claim that $V = \overline{W}$.

If v is a vector in V, then there exists real numbers $\alpha_1, \ldots, \alpha_n$ such that $v = \sum_{i=1}^n \alpha_i v_i$. For each i, there exists a sequence of real numbers (a_{ij}) converging to α_i as $j \longrightarrow \infty$. Define $v_j = \sum_{i=1}^n a_{ij} v_i$. We now prove $v_j \longrightarrow v$ as $j \longrightarrow \infty$. If ϵ is a positive real number, then for every index i there exists a positive integer N_i such that $n > N_i$ implies $|\alpha_i - a_{ij}| < \epsilon/n$. Put $N = \max\{N_1, \ldots, N_n\}$. It follows that $n > N$ implies

$$\|v - v_j\| = \sum_i |\alpha_i - a_{ij}| < n \cdot (\epsilon/n) = \epsilon.$$

The space W is countable because it is the Cartesian product of n copies of $\mathbf{Q}$. Hence W is a countable dense subset of V. □

9.5. Let c_{00} be the space of sequences with only finitely many nonzero terms (see Example 3.23), and consider it as a subspace of ℓ^∞. Prove that c_{00} is separable.

Solution. Let e_j be the sequence whose j-th entry is 1, and all the others are 0. Since every vector in c_{00} has only finitely many non-zero terms, it follows that c_{00} is spanned by $\{e_1, e_2, \ldots\}$. Let V_n be the span of $\{e_1, \ldots, e_n\}$, which is separable by Question 9.4. Pick a countable dense subset S_n of V_n and put $S = \bigcup_{n=1}^\infty S_n$. We claim that S is a countable dense subset of c_{00}.

As a countable union of countable sets, S is countable because of Exercise 1.2. If v is a vector in c_{00}, then v belongs to V_n for some n, so there exists a sequence in S_n converging to v, which is also a sequence in S converging to v. Hence S is dense in c_{00}.

Alternative: Once you see that c_{00} is spanned by the countable set $\{e_1, e_2, \ldots\}$, use Proposition 3.31. □

9.6. Consider the subset $c_0 \subseteq \mathbf{F}^{\mathbf{N}}$ of all sequences with limit 0:

$$c_0 = \{(a_n) \in \mathbf{F}^{\mathbf{N}} : (a_n) \longrightarrow 0\}.$$

(a) Prove that c_0 is a closed subspace of ℓ^∞.

(b) Conclude that c_0 is a Banach space.

(c) Prove that c_0 is separable.

Solution.

(a) It's pretty clear that c_0 is a subspace of $\mathbf{F}^{\mathbf{N}}$, and hence of ℓ^∞. To show that c_0 is closed in ℓ^∞, let $(x_n) \longrightarrow x \in \ell^\infty$ with $x_n \in c_0$ for all $n \in \mathbf{N}$. We want to prove that $x \in c_0$.

Write $x_n = \big(a_{nm}\big) = \big(a_{n1}, a_{n2}, a_{n3}, \dots\big)$ and $x = (a_m) = (a_1, a_2, a_3, \dots)$. Let $\varepsilon > 0$. There exists $N \in \mathbf{N}$ such that for all $n \geqslant N$ we have

$$\sup_m |a_m - a_{nm}| = \|x - x_n\|_{\ell^\infty} < \frac{\varepsilon}{2}.$$

Consider the sequence $x_N = (a_{Nm}) \in c_0$. It converges to 0, so that there exists $M \in \mathbf{N}$ such that for any $m \geqslant M$ we have

$$|a_{Nm}| < \frac{\varepsilon}{2}.$$

Therefore, for $m \geqslant M$, we get

$$|a_m| = |a_m - a_{Nm} + a_{Nm}| \leqslant |a_m - a_{Nm}| + |a_{Nm}| < \varepsilon.$$

Hence $x = (a_m) \longrightarrow 0$.

(b) Since c_0 is closed and ℓ^∞ is Banach, c_0 is Banach.

(c) I claim that c_0 has the same Schauder basis at the one given in Example 3.32 for ℓ^p: $\{e_1, e_2, \dots\}$ where $e_n = (0, \dots, 0, 1, 0 \dots)$ with the 1 in the n-th spot.

Take $v = (v_n) \in c_0$, then $(v_n) \longrightarrow 0$. I claim that the series

$$\sum_{n=1}^{\infty} v_n e_n$$

converges to v with respect to the norm on c_0, which is the ℓ^∞-norm:

$$\left\| v - \sum_{n=1}^{m} v_n e_n \right\|_{\ell^\infty} = \|(0, \dots, 0, v_{m+1}, v_{m+2}, v_{m+3}, \dots)\|_{\ell^\infty} = \sup_{n \geqslant m+1} |v_n|,$$

and the latter converges to 0 as $m \longrightarrow \infty$, since $(v_n) \longrightarrow 0$. The uniqueness of the coefficients follows in precisely the same way as for Example 3.32. $\square$

9.7. Consider the space ℓ^∞ of bounded sequences.

(a) Let $S \subseteq \ell^\infty$ be the subset of sequences (a_n) such that $a_n \in \{0, 1\}$ for all $n \in \mathbf{N}$. Prove that S is an uncountable set.

[*Hint*: Mimic Cantor's diagonal argument.]

(b) Use S to construct an uncountable set T of disjoint open balls in ℓ^∞.

(c) Conclude that ℓ^∞ is not separable.

Solution.

(a) Suppose S is countable and enumerate its elements:

$$\begin{aligned} a_1 &= \big(a_{11}, a_{12}, a_{13}, \dots\big) \\ a_2 &= \big(a_{21}, a_{22}, a_{23}, \dots\big) \\ a_3 &= \big(a_{31}, a_{32}, a_{33}, \dots\big) \\ &\vdots \end{aligned}$$

Go down the diagonal of this infinite grid of 0's and 1's, and define $b_n = 1 - a_{nn}$ for all $n \in \mathbf{N}$. Then $b = (b_n) \in S$, but $b \neq a_m$ for any $m \in \mathbf{N}$, contradiction.

(b) If $a = (a_n), b = (b_n) \in S$ with $a \neq b$ then

$$\|a - b\| = \sup_n |a_n - b_n| = 1,$$

so $\mathbf{B}_{1/2}(a) \cap \mathbf{B}_{1/2}(b) = \varnothing$.

Therefore we can take

$$T = \big\{\mathbf{B}_{1/2}(s) \colon s \in S\big\}.$$

(c) Any dense subset D of ℓ^∞ must contain at least one point (in fact, must be dense) in each open ball in the set T. Since T is uncountable, D must also be uncountable, so ℓ^∞ is not separable. □

9.8. Give an example of a series that converges but does not converge absolutely.

Solution. In $\mathbf{R}$, consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Taking absolute values we get the harmonic series, which does not converge.

The original series has alternating signs, and its terms in absolute value form a decreasing sequence $(1/n)$ that converges to zero, hence the series converges by the alternating series test. □

9.9. If a series $\sum_{n=1}^{\infty} a_n$ in a normed space $\big(V, \|\cdot\|\big)$ converges absolutely, then

$$\Big\|\sum_{n=1}^{\infty} a_n\Big\| \leqslant \sum_{n=1}^{\infty} \|a_n\|.$$

Solution. This follows from the usual triangle inequality.

For any $m \in \mathbf{N}$, we have

$$\|a_1 + \cdots + a_m\| \leqslant \|a_1\| + \cdots + \|a_m\|.$$

Taking limits as $m \longrightarrow \infty$ we get

$$\Big\|\sum_{n=1}^{\infty} a_n\Big\| = \Big\|\lim_{m \longrightarrow \infty} \sum_{n=1}^{m} a_n\Big\| = \lim_{m \longrightarrow \infty} \Big\|\sum_{n=1}^{m} a_n\Big\| \leqslant \lim_{m \longrightarrow \infty} \sum_{n=1}^{m} \|a_n\| = \sum_{n=1}^{\infty} \|a_n\|.$$ □