Tutorial Week 10

Topics: continuous linear transformations, series, projections.

10.1. Prove that all linear transformations between finite-dimensional normed vector spaces are continuous.

Solution. Let $v_1, \ldots, v_m$ be a basis of V and let $w_1, \ldots, w_n$ be a basis of W. By Theorem 3.6, the norms on V and on W are equivalent to the norms $\|\cdot\|_V$ and $\|\cdot\|_W$ defined by

$$\|\alpha_1 v_1 + \cdots + \alpha_m v_m\|_V = |\alpha_1| + \cdots + |\alpha_m| \quad \text{and} \quad \|\beta_1 w_1 + \cdots + \beta_m w_n\|_V = |\beta_1| + \cdots + |\beta_n|.$$

Since continuity of f is determined by the topologies on V and W, and since equivalent norms give rise to the same topology (see Exercise 3.2), we can assume without loss of generality that V and W are equipped with the norms $\|\cdot\|_V$ and $\|\cdot\|_W$ respectively.

Put $M = \max\{\|f(v_1)\|_W, \ldots, \|f(v_n)\|_W\}$. If $v = \alpha_1 v_1 + \cdots \alpha_m v_m$, then

$$\begin{aligned}
\|f(v)\|_W &= \|f(\alpha_1 v_1 + \cdots + \alpha_m v_m)\|_W \\
&= \|\alpha_1 f(v_1) + \cdots + \alpha_m f(v_m)\|_W \\
&\leqslant \|\alpha_1 f(v_1)\|_W + \cdots + \|\alpha_m f(v_m)\|_W \\
&= |\alpha_1|\|f(v_1)\|_W + \cdots + |\alpha_m|\|f(v_m)\|_W \\
&\leqslant \alpha_1 M + \cdots + \alpha_m M \leqslant M\|v\|_V.
\end{aligned}$$

Hence f is Lipschitz, and therefore continuous by Proposition 3.25. □

10.2. Let $f_1 \colon V \longrightarrow W_1$ and $f_2 \colon V \longrightarrow W_2$ be two continuous linear transformations between normed vector spaces. Prove that the function $f \colon V \longrightarrow W_1 \times W_2$ defined by $f(v) = \big(f_1(v), f_2(v)\big)$ is a continuous linear transformation.

Solution. We start with proving that f is linear. If v_1 and v_2 are vectors in V, then

$$\begin{aligned}
f(v_1 + v_2) &= \big(f_1(v_1 + v_2), f_2(v_1 + v_2)\big) \\
&= \big(f_1(v_1) + f_1(v_2), f_2(v_1) + f_2(v_2)\big) \\
&= \big(f_1(v_1), f_2(v_1)\big) + \big(f_1(v_2), f_2(v_2)\big) = f(v_1) + f(v_2).
\end{aligned}$$

If α is a scalar and v is a vector in V, then

$$f(\alpha v) = \big(f_1(\alpha v), f_2(\alpha v)\big) = \big(\alpha f_1(v), \alpha f_2(v)\big) = \alpha f(v).$$

Hence f is linear.

It remains to prove that f is continuous. Let $\pi_1 \colon W_1 \times W_2 \longrightarrow V_1$ and $\pi_2 \colon W_1 \times W_2 \longrightarrow W_2$ be the projections. Since $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$, it follows from Question 3.7 that f is continuous. □

10.3. Let c_{00} be the space of sequences with only finitely many nonzero terms (see Example 3.23), which is considered as a subspace of ℓ^∞. Let $f \colon c_{00} \longrightarrow \mathbf{F}^{\mathbf{N}}$ be the function defined by $\big(f(v)\big)_n = n v_n$.

(a) Prove that the image of the function f is contained in ℓ^∞.

(b) Let $g \colon c_{00} \longrightarrow \ell^\infty$ be the function defined by $g(v) = f(v)$. Prove that g is not continuous.

(c) Prove that there exists a discontinuous linear transformation from ℓ^∞ to itself.

In this part, you can use the following fact:

> *Let V and W be $\mathbf{F}$-vector spaces. If S is a subspace of V and if $\phi\colon S \longrightarrow W$ is a linear transformation, then there exists a linear transformation $\tilde{\phi}\colon V \longrightarrow W$ such that $\phi = \tilde{\phi}|_S$.*

Solution.

(a) Let $v \in c_{00}$. By the definition of c_{00}, there exists a positive integer N such that $n > N$ implies $v_n = 0$. It follows that $\big(f(v)\big)_n \leqslant N v_n$ for every positive integer n. Hence $\|f(v)\|_{\ell^\infty} \leqslant N\|v\|_{\ell^\infty} < \infty$, which implies $f(v) \in \ell^\infty$.

(b) For every positive integer n, let $v^{(m)}$ be the vector in ℓ^∞ defined by

$$v_n^{(m)} = \begin{cases} 1 & \text{if } n \leqslant m, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify $\|v^{(m)}\|_{\ell^\infty} = 1$ but $\|g\big(v^{(m)}\big)\|_{\ell^\infty} = m$ for every positive integer m. Hence g is not Lipschitz, which implies g is not continuous by Proposition 3.25.

(c) Apply the fact to the discontinuous linear transformation $g\colon c_{00} \longrightarrow \ell^\infty$ in part (b) to obtain a linear transformation $\tilde{g}\colon \ell^\infty \longrightarrow \ell^\infty$, which cannot be continuous because otherwise its restriction g would be continuous. $\square$

10.4. If $f \in L(V, W)$ with V, W normed spaces, and the series

$$\sum_{n=1}^{\infty} \alpha_n v_n, \qquad \alpha_n \in \mathbf{F}, v_n \in V,$$

converges in V, then the series

$$\sum_{n=1}^{\infty} \alpha_n f(v_n)$$

converges in W to the limit

$$f\left(\sum_{n=1}^{\infty} \alpha_n v_n\right).$$

Solution. Let

$$x_m = \sum_{n=1}^{m} \alpha_n v_n, \qquad x = \sum_{n=1}^{\infty} \alpha_n v_n.$$

We know that $(x_m) \longrightarrow x$ in V.

Since $f \in L(V, W)$ is continuous, we have that $\big(f(x_m)\big) \longrightarrow f(x)$ in W.

But f is also linear, so

$$f(x_m) = \sum_{n=1}^{m} \alpha_n f(v_n).$$

Hence

$$\left(\sum_{n=1}^{m} \alpha_n f(v_n)\right) \longrightarrow f(x),$$

so that the series

$$\sum_{n=1}^{\infty} \alpha_n f(v_n) \quad \text{converges to } f(x).$$

$\square$

10.5.

(a) Prove that the function $f\colon \ell^1 \longrightarrow \mathbf{F}$ defined by

$$f\big((a_n)\big) = \sum_{n=1}^{\infty} a_n.$$

is continuous.

(b) Prove that the following subset is a closed subspace of ℓ^1:

$$S = \Big\{ (a_n) \in \ell^1 : \sum_{n=1}^{\infty} a_n = 0 \Big\}.$$

Solution.

(a) First note that this is a reasonable definition, because the infinite series on the right hand side converges in $\mathbf{F}$:

$$\Big|\sum_{n=1}^{N} a_n\Big| \leqslant \sum_{n=1}^{N} |a_n|,$$

and the latter converges as $N \longrightarrow \infty$ since $(a_n) \in \ell^1$.

The function f is linear. It is also Lipschitz, because as we have just seen:

$$\big|f\big((a_n)\big)\big| = \Big|\sum_{n=1}^{\infty} a_n\Big| \leqslant \sum_{n=1}^{\infty} |a_n| = \|(a_n)\|_{\ell^1}.$$

Hence f is continuous by Proposition 3.25.

(b) As the kernel of the continuous linear functional f, the subset S of ℓ^1 is a closed subspace. □

10.6. Fix $j \in \mathbf{N}$ and consider the map $\pi_j\colon \mathbf{F}^{\mathbf{N}} \longrightarrow \mathbf{F}$ given by

$$\pi_j\big((a_n)\big) = a_j.$$

(a) Show that π_j is linear.

(b) Prove that the restriction of π_j to ℓ^p for $1 \leqslant p \leqslant \infty$ is continuous and surjective.

Solution.

(a) Straightforward.

(b) We have for $a \in \ell^\infty$:

$$|\pi_j(a)| = |a_j| \leqslant \sup_{n\geqslant 1}\{|a_n|\} = \|a\|_{\ell^\infty},$$

so π_j is bounded.

Similarly for $a \in \ell^p$:

$$|\pi_j(a)| = |a_j| = \big(|a_j|^p\big)^{1/p} \leqslant \Big(\sum_{n\geqslant 1} |a_n|^p\Big)^{1/p} = \|a\|_{\ell^p}.$$

For the surjectivity we note that for any $a \in \mathbf{F}$ we have $\pi_j\big((0,\dots,0,a,0,\dots)\big) = a$ and $(0,\dots,0,a,0,\dots) \in \ell^1 \subseteq \ell^p$ for all $1 \leqslant p \leqslant \infty$. □

10.7. Let V be a normed space and φ, ψ be commuting projections: $\varphi \circ \psi = \psi \circ \varphi$. Prove that $\varphi \circ \psi$ is a projection with image $\operatorname{im}\varphi \cap \operatorname{im}\psi$.

Solution. We know that the composition of continuous linear maps is continuous linear, so this is true for $\varphi \circ \psi$. To conclude that it is a projection, we need to compute its square:

$$\big(\varphi \circ \psi\big) \circ \big(\varphi \circ \psi\big) = \big(\varphi \circ \varphi\big) \circ \big(\psi \circ \psi\big) = \varphi \circ \psi,$$

where it was crucial that φ and ψ commute.

For the statement about the image, note that $w \in \operatorname{im}\big(\varphi \circ \psi\big)$ if and only if there exists $v \in V$ such that

$$w = \varphi\big(\psi(v)\big) = \psi\big(\varphi(v)\big),$$

which implies that $w \in \operatorname{im}\varphi \cap \operatorname{im}\psi$. So $\operatorname{im}\big(\varphi \circ \psi\big) \subseteq \operatorname{im}\varphi \cap \operatorname{im}\psi$.

Conversely, suppose $w \in \operatorname{im}\varphi \cap \operatorname{im}\psi$, then there exists $v \in V$ such that $w = \psi(v)$. But $w \in \operatorname{im}\varphi$ and φ is a projection, so that

$$w = \varphi(w) = \varphi\big(\psi(v)\big) \in \operatorname{im}\big(\varphi \circ \psi\big).$$ □

10.8. Let φ be a nonzero orthogonal projection (that is, φ is not the constant function 0) on an inner product space V. Prove that $\|\varphi\| = 1$.

Solution. We know that $\big(\operatorname{im}\varphi\big)^{\perp} = \ker\varphi$. For any x we have

$$\varphi\big(x - \varphi(x)\big) = \varphi(x) - \varphi^2(x) = \varphi(x) - \varphi(x) = 0,$$

so $x - \varphi(x) \in \ker\varphi$. Therefore

$$\langle x, \varphi(x)\rangle - \|\varphi(x)\|^2 = \langle x - \varphi(x), \varphi(x)\rangle = 0,$$

so

$$\|\varphi(x)\|^2 = \langle x, \varphi(x)\rangle \leqslant \|x\|\,\|\varphi(x)\|$$

by the Cauchy–Schwarz Inequality. Hence $\|\varphi(x)\| \leqslant \|x\|$ for all x, hence $\|\varphi\| \leqslant 1$. However for $x \in \operatorname{im}\varphi$ we have $\varphi(x) = x$ so $\|\varphi(x)\| = \|x\|$ and we conclude that $\|\varphi\| = 1$. □