Tutorial Week 11

Topics: projections, self-adjoint operators, normal operators.

11.1. Consider the function $g\colon \ell^2 \longrightarrow \mathbf{F}$ given by

$$g(x) = \sum_{n=1}^{\infty} \frac{x_n}{n^2}.$$

(a) Find $y \in \ell^2$ such that

$$g(x) = \langle x, y \rangle \qquad \text{for all } x \in \ell^2.$$

(b) Deduce that g is linear and Lipschitz and find its norm $\|g\|$.

[*Hint*: You may use without proof the fact that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.]

Solution. (a) Setting $y = (y_n)$ with

$$y_n = \frac{1}{n^2},$$

we certainly have for all $x = (x_n) \in \ell^2$:

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y}_n = \sum_{n=1}^{\infty} \frac{x_n}{n^2} = g(x).$$

We should check that $y \in \ell^2$:

$$\|y\|_{\ell^2}^2 = \sum_{n=1}^{\infty} y_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(b) From the previous part we know that $g = y^\vee$, so certainly g is linear and Lipschitz. We also have

$$\|g\| = \|y^\vee\| = \|y\|_{\ell^2} = \frac{\pi^2}{3\sqrt{10}},$$

as we have seen in the previous part. □

11.2. Let H be a Hilbert space. Prove that a projection $\pi\colon H \longrightarrow H$ is an orthogonal projection if and only if π is self-adjoint.

Solution. Suppose $\pi\colon H \longrightarrow H$ is an orthogonal projection and suppose x and y are elements of H. Since

$$\pi\big(x - \pi(x)\big) = \pi(x) - \pi^2(x) = \pi(x) - \pi(x) = 0,$$

it follows that $x - \pi(x) \in \ker(\pi) = \big(\mathrm{im}(\pi)\big)^\perp$, and similarly $y - \pi(y) \in \big(\mathrm{im}(\pi)\big)^\perp$. In particular, we have

$$\big\langle \pi(x), y - \pi(y) \big\rangle = \big\langle x - \pi(x), \pi(y) \big\rangle = 0,$$

which implies

$$\begin{aligned}
\big\langle \pi(x), y \big\rangle &= \big\langle \pi(x), \pi(y) \big\rangle + \big\langle \pi(x), y - \pi(y) \big\rangle \\
&= \big\langle \pi(x), \pi(y) \big\rangle \\
&= \big\langle \pi(x), \pi(y) \big\rangle + \big\langle x - \pi(x), \pi(y) \big\rangle \\
&= \big\langle x, \pi(y) \big\rangle.
\end{aligned}$$

Hence π is self-adjoint.

Now suppose π is self-adjoint. If $x \in \ker(\pi)$, then for every element y of H we have

$$\big\langle x, \pi(y)\big\rangle = \big\langle \pi(x), y\big\rangle = \langle 0, y\rangle = 0,$$

so $\ker(\pi) \subseteq \big(\operatorname{im}(\pi)\big)^{\perp}$. If $x \in \big(\operatorname{im}(\pi)\big)^{\perp}$, then for every element y of H we have

$$\big\langle \pi(x), y\big\rangle = \big\langle x, \pi(y)\big\rangle = 0,$$

so $\pi(x) = 0$ because the inner product is positive-definite. It follows that $x \in \ker(\pi)$, and hence $\ker(\pi) = \big(\operatorname{im}(\pi)\big)^{\perp}$. □

11.3. Let H be a Hilbert space. Prove that a projection $\pi\colon H \longrightarrow H$ is orthogonal if and only if $\operatorname{id}_H - \pi$ is an orthogonal projection.

Solution. Since $\pi = \operatorname{id}_H - (\operatorname{id}_H - \pi)$, it suffices to prove π being an orthogonal projection implies $\operatorname{id}_H - \pi$ is an orthogonal projection.

It follows from part (a) of Proposition 3.38 that $\operatorname{id}_H - \pi$ is a projection. Question 11.2 implies π is self-adjoint, so $\operatorname{id}_H - \pi$ is self-adjoint. Hence $\operatorname{id}_H - \pi$ is an orthogonal projection because of Question 11.2. □

11.4. Let $f \in L(H)$ with H a Hilbert space. Suppose that f is invertible with continuous inverse. Then the adjoint f^* is invertible and

$$\big(f^*\big)^{-1} = \big(f^{-1}\big)^*.$$

Solution. We want to prove that

$$\big(f^{-1}\big)^* \circ f^* = \operatorname{id}_H = f^* \circ \big(f^{-1}\big)^*.$$

We have for all $x, y \in H$:

$$\Big\langle x, \big(f^{-1}\big)^*\big(f^*(y)\big)\Big\rangle = \big\langle f^{-1}(x), f^*(y)\big\rangle = \Big\langle f\big(f^{-1}(x)\big), y\Big\rangle = \langle x, y\rangle$$

implying that $\big(f^{-1}\big)^* \circ f^* = \operatorname{id}_H$, and similarly for the other composition. □

11.5. Let H be a Hilbert space and let α be a scalar. Prove that $\alpha \operatorname{id}_H$ is normal (that is, commutes with its adjoint).

Solution. If $x \in H$, then

$$\begin{aligned}\big(\alpha\operatorname{id}_H \circ (\alpha\operatorname{id}_H)^*\big)(x) &= \alpha(\alpha\operatorname{id}_H)^*(x)\\ &= (\alpha\operatorname{id}_H)^*(\alpha x) && \text{(linearity)}\\ &= \big((\alpha\operatorname{id}_H)^* \circ \alpha\operatorname{id}_H\big)(x).\end{aligned}$$

□

Solution. (Alternative): Since id_H is self-adjoint, it follows that $(\alpha \operatorname{id}_H)^* = \overline{\alpha} \operatorname{id}_H$. Hence

$$(\alpha \operatorname{id}_H)^* \circ (\alpha \operatorname{id}_H) = \overline{\alpha}\alpha \operatorname{id}_H = |\alpha|^2 \operatorname{id}_H = \alpha\overline{\alpha} \operatorname{id}_H = (\alpha \operatorname{id}_H) \circ (\alpha \operatorname{id}_H)^*.$$

□

11.6. Let $R\colon \ell^2 \longrightarrow \ell^2$ and $L\colon \ell^2 \longrightarrow \ell^2$ be the operators defined by

$$R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots) \quad \text{and} \quad L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$$

Find the adjoints of R and L and prove that neither R nor L is normal.

Solution. It follows from

$$\langle R(x), y\rangle = \sum_{n=2}^{\infty} x_{n-1}\overline{y_n} = \sum_{n=1}^{\infty} x_n\overline{y_{n+1}} = \langle x, L(y)\rangle$$

that $R^* = L$, and therefore $L^* = (R^*)^* = R$.

Let e_j be the sequence whose j-th entry is 1, and all the others are 0. We have

$$(R^* \circ R)(e_1) = L(e_2) = e_1 \qquad \text{and} \qquad (R \circ R^*)(e_1) = R(0) = 0,$$

so R is not normal. Since $L = R^*$, the equations above also imply that L is not normal. □

11.7. Equip $\mathbf{R}^n$ with the standard Euclidean inner product and let $f\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be a linear transformation with standard matrix representation A.

(a) Prove that the adjoint $f^*\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ has standard matrix representation A^t, the transpose of A.

(b) Prove that f is self-adjoint if and only if A is symmetric.

Solution.

(a) Recall that the standard Euclidean inner product can be written as

$$\langle x, y\rangle = y^t x.$$

It follows that

$$\langle x, f^*(y)\rangle = \langle f(x), y\rangle = y^t A x = (A^t y)^t x.$$

Hence the standard matrix representation of f^* is A^t.

(b) It follows from part (a) that $f = f^*$ if and only if $A = A^t$. □

11.8. Let $f \in L(H)$ with H a Hilbert space. Then the maps

$$p = f^* \circ f \qquad \text{and} \qquad s = f + f^*$$

are self-adjoint.

Solution. Since f is continuous, the adjoint f^* is continuous, so the composition $p = f^* \circ f$ and the sum $s = f + f^*$ are both continuous.

Then

$$p^* = \left(f^* \circ f\right)^* = f^* \circ \left(f^*\right)^* = f^* \circ f = p$$
$$s^* = \left(f + f^*\right)^* = f^* + \left(f^*\right)^* = f^* + f = f + f^* = s.$$ □

11.9. Let H be a real Hilbert space. Prove that self-adjoint continuous linear operators on H form a subspace of $L(H)$.

If H is a complex Hilbert space, does the statement still hold? If yes, give a proof for the statement. If no, find a counterexample, and then find and prove a closest statement that holds.

Solution. If f and g are self-adjoint continuous linear operators on H, then

$$(f+g)^* = f^* + g^* = f + g,$$

so $f+g$ is self-adjoint.

If f is a self-adjoint continuous linear operator on H and α is a real number, then

$$(\alpha f)^* = \alpha f^* = \alpha f,$$

so αf is self-adjoint.

If H is complex, then the statement does not hold. The identity function $\mathrm{id}_H \colon H \longrightarrow H$ is self-adjoint, but it is easy to verify $(i\,\mathrm{id}_H)^* = -i\,\mathrm{id}_H$, so $i\,\mathrm{id}_H$ is not self-adjoint. Hence the set of self-adjoint linear operators on H is not closed under scalar multiplication.

Instead, if we treat $L(H)$ as a real vector space, then the set of self-adjoint linear operators on H is a subspace of $L(H)$. The proof for closure under addition is the same as the case for real Hilbert spaces. If f is a self-adjoint continuous linear operator on H and α is a real number, then

$$(\alpha f)^* = \overline{\alpha} f^* = \alpha f,$$

so αf is self-adjoint. □

11.10. The composition of two self-adjoint maps on a Hilbert space is self-adjoint if and only if the maps commute.

Solution. We have

$$\langle f(g(x)), y\rangle = \langle g(x), f(y)\rangle = \langle x, g(f(y))\rangle$$

by the self-adjointness of f and g.

So $f \circ g$ is self-adjoint if and only if $g \circ f = f \circ g$, as claimed. □