Tutorial Week 12

Topics: Orthogonal systems, orthogonal bases, the Stone–Weierstrass theorem.

12.1. Let $(u_i)_{i\in I}$ be an orthonormal basis of an inner product space V and let $v \in V$. Prove that $v = 0$ if and only if $\langle v, u_i\rangle = 0$ for every index $i \in I$.

Solution. If $v = 0$, then $\langle v, u_i\rangle = 0$ by linearity on the first variable.

If $\langle v, u_i\rangle = 0$ for every index i, then

$$\{v\}^\perp \supseteq \{\, u_i \mid i \in I \,\}.$$

It then follows from Proposition 3.40 that

$$\{v\}^\perp \supseteq \overline{\operatorname{Span}\{\, u_i \mid i \in I \,\}} = H.$$

Hence v is orthogonal to itself; in other words, $\langle v, v\rangle = 0$. By positive definiteness of inner products, it follows that $v = 0$. □

12.2. In this question, we re-examine the Cauchy–Schwarz inequality in retrospect.

Let u be a vector of norm 1 in an inner product space V. Define $\pi_u \colon V \longrightarrow V$ by

$$\pi_u(v) = \langle v, u\rangle u.$$

(a) Prove that π_u is a linear transformation.

(b) Let v be a vector in V. Prove that $\pi_u(v)$ is orthogonal to $(\mathrm{id}_V - \pi_u)(v)$.

(c) Let v be a vector in V. Prove that $\|\pi_u(v)\| = |\langle v, u\rangle|$.

(d) Prove the *Cauchy–Schwarz inequality*: if v and w are vectors in V, then

$$\big|\langle v, w\rangle\big| \leqslant \|v\|\,\|w\|.$$

(e) Prove that π_u is an orthogonal projection with image $\mathbf{F}u$.

Solution.

(a) If v and w are vectors in V, then

$$\pi_u(v + w) = \langle v + w, u\rangle u = \langle v, u\rangle u + \langle w, u\rangle u = \pi_u(v) + \pi_u(w).$$

If α is a scalar and v is a vector in V, then

$$\pi_u(\alpha v) = \langle \alpha v, u\rangle u = \alpha\langle v, w\rangle u = \alpha\pi_u(v).$$

(b) The vector $\pi_u(v)$ is orthogonal to $(\mathrm{id}_V - \pi_u)(v)$ because

$$\begin{aligned}\big\langle \pi_u(v), (\mathrm{id}_V - \pi_u)(v)\big\rangle &= \big\langle \langle v, u\rangle u, v - \langle v, u\rangle u\big\rangle \\ &= \big\langle \langle v, u\rangle u, v\big\rangle - \big\langle \langle v, u\rangle u, \langle v, u\rangle u\big\rangle \\ &= \langle v, u\rangle\langle u, v\rangle - \langle v, u\rangle\langle u, u\rangle\overline{\langle v, u\rangle} \\ &= \langle v, u\rangle\overline{\langle v, u\rangle} - \langle v, u\rangle\overline{\langle v, u\rangle} = 0.\end{aligned}$$

(c) The equality follows from

$$\left\|\pi_u(v)\right\|^2 = \left\|\langle v,u\rangle u\right\|^2 = \big\langle \langle v,u\rangle u, \langle v,u\rangle u\big\rangle = \langle v,u\rangle\langle u,u\rangle\overline{\langle v,u\rangle} = \left|\langle v,u\rangle\right|^2.$$

(d) If v is zero, then both sides of the inequality equal 0. Otherwise, put $u = v/\|v\|$. It follows from part (b), the Pythagorean theorem, and part (c) that

$$\|w\|^2 = \left\|\pi_u(w)\right\|^2 + \left\|(\mathrm{id}_V - \pi_u)(w)\right\|^2 \geqslant \left\|\pi_u(w)\right\|^2 = \left|\langle w,u\rangle\right|^2 = \left|\left\langle w, \frac{v}{\|v\|}\right\rangle\right|^2 = \frac{\left|\langle w,v\rangle\right|^2}{\|v\|^2},$$

so $|\langle v,w\rangle| \leqslant \|v\|\,\|w\|$.

(e) If $v \in V$, then it follows from part (c) and the Cauchy–Schwarz inequality that

$$\left\|\pi_u(v)\right\| = \left|\langle v,u\rangle\right| \leqslant \|v\|\,\|u\| = \|v\|,$$

so π_v is continuous. If v is a vector in V, then

$$\pi_u\big(\pi_u(v)\big) = \pi_u\big(\langle v,u\rangle u\big) = \langle v,u\rangle\pi_u(u) = \langle v,u\rangle\langle u,u\rangle u = \langle v,u\rangle u = \pi_u(v),$$

so π_u is a projection. If v and w are vectors in V, then

$$\big\langle \pi_u(v), w\big\rangle = \big\langle \langle v,u\rangle u, w\big\rangle = \langle v,u\rangle\langle u,w\rangle = \langle v,u\rangle\overline{\langle w,u\rangle} = \big\langle v, \langle w,u\rangle u\big\rangle = \big\langle v, \pi_u(w)\big\rangle,$$

so π_u is self-adjoint. Hence π_u is an orthogonal projection by Question 11.2.

If v is a vector in V, then $\pi_u(v) = \langle v,u\rangle u \in \mathbf{F}u$, so $\mathrm{im}(\pi_u) \subseteq \mathbf{F}u$. If α is a scalar, then $\pi_u(\alpha u) = \langle \alpha u, u\rangle u = \alpha\langle u,u\rangle u = \alpha u$, so $\mathbf{F}u \subseteq \mathrm{im}(\pi_u)$. Hence $\mathrm{im}(\pi_u) = \mathbf{F}u$. □

12.3. In this question, we generalise the results in Question 12.2.

Let $\{u_1,\dots,u_n\}$ be an orthonormal system in an inner product space V and let U be the span of the orthonormal system. Write $\pi_1,\dots,\pi_n$ for the projections $\pi_{u_1},\dots,\pi_{u_n}$ defined in Question 12.2 and put

$$\pi = \pi_1 + \cdots + \pi_n.$$

(a) Prove that

$$\pi_i \circ \pi_j = \begin{cases} \pi_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Prove that π is an orthogonal projection with image U.

(c) Let v be a vector in V. Prove that

$$\left\|\pi(v)\right\|^2 = \sum_{i=1}^{n}\left|\langle v,u_n\rangle\right|^2.$$

(d) Use part (c) to prove the following finite version of the *Bessel's inequality*: if v is a vector in V, then

$$\|v\|^2 \geqslant \sum_{i=1}^{n}\left|\langle v,u_i\rangle\right|^2.$$

Solution.

(a) If v is a vector in V, then

$$(\pi_i \circ \pi_j)(v) = \begin{cases} \pi_i\big(\langle v, u_i\rangle u_i\big) = \langle v, u_i\rangle \pi_i(u_i) = \langle v, u_i\rangle u_i = \pi_i(v) & \text{if } i = j, \\ \pi_i\big(\langle v, u_j\rangle u_j\big) = \langle v, u_j\rangle \pi_i(u_j) = \langle v, u_j\rangle \langle u_j, u_i\rangle u_i = 0 & \text{otherwise.} \end{cases}$$

(b) As a sum of continuous linear transformations, π is linear and continuous. It follows from part (a) that

$$\pi \circ \pi = \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i \circ \pi_j = \sum_{i=1}^{n} \pi_i = \pi,$$

so π is a projection. Since self-adjoint operators form a real vector space (see Question 11.9), it follows that π is self-adjoint, and therefore an orthogonal projection by Question 11.2.

If v is a vector in V, then

$$\pi(v) = \pi_1(v) + \cdots + \pi_n(v) \in U,$$

so $\operatorname{im}(\pi) \subseteq U$. If u is a vector in U, then $u = \alpha_1 u_1 + \cdots + \alpha_n u_n$ for some scalars $\alpha_1, \ldots, \alpha_n$. It follows that

$$\pi(u) = \sum_{i=1}^{n} \pi_i\Big(\sum_{j=1}^{n} \alpha_j u_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_j \pi_i(u_j) = \sum_{i=1}^{n} \alpha_i u_i = u,$$

so $\operatorname{im}(\pi) = U$.

(c) Since $\{u_1, \ldots, u_n\}$ is an orthogonal system, the vectors $\pi_i(v)$ and $\pi_j(v)$ are orthogonal unless $i = j$. It then follows from the Pythagorean theorem and part (c) of Question 12.2 that

$$\big\|\pi(v)\big\|^2 = \big\|\pi_1(v) + \cdots + \pi_n(v)\big\|^2 = \sum_{i=1}^{n} \big\|\pi_i(v)\big\|^2 = \sum_{i=1}^{n} \big|\langle v, u_n\rangle\big|^2.$$

(d) Since π is an orthogonal projection, the vectors $\pi(v)$ and $(\mathrm{id}_V - \pi)(v)$ are orthogonal by Proposition 3.38 and the definition of orthogonal projections. It then follows from the Pythagorean theorem and part (c) that

$$\|v\|^2 = \big\|\pi(v)\big\|^2 + \big\|(\mathrm{id}_V - \pi)(v)\big\|^2 \geqslant \big\|\pi(v)\big\|^2 = \sum_{i=1}^{n} \big|\langle v, u_i\rangle\big|^2. \qquad \square$$

We say that a subalgebra $\mathcal{C}$ of $C_0(X, \mathbf{F})$ *separates points* if for every pair of points x and y in X there is a function f in $\mathcal{C}$ such that $f(x) \neq f(y)$. We say that a subalgebra $\mathcal{C}$ of $C_0(X, \mathbf{F})$ is *non-vanishing* if for every point x in X there is a function f in $\mathcal{C}$ such that $f(x) \neq 0$.

12.4. (*) Let $\mathcal{C}$ be a non-vanishing subalgebra of $C_0(X, \mathbf{F})$ that separates points.

(a) Given two points x and y in X, find a function h in $\mathcal{C}$ such that $h(x) = 0$ and $h(y) \neq 0$.

(b) Prove that $\mathcal{C}$ interpolates pairs of points.

Solution.

(a) Pick functions f and g in $C_0(X, \mathbf{F})$ such that $f(x) \neq f(y)$ and $g(y) \neq 0$. Put

$$h = fg - f(x)g.$$

The function h is still in $\mathcal{C}$, and

$$h(x) = f(x)g(x) - f(x)g(x) = 0,$$
$$h(y) = f(y)g(y) - f(x)g(y) = \big(f(y) - f(x)\big)g(y) \neq 0.$$

(b) Let a and b be two scalars. By part (a), there are functions ϕ and ψ in $\mathcal{C}$ such that

$$\phi(x) = 0, \quad \phi(y) \neq 0, \quad \psi(x) \neq 0, \quad \text{and} \quad \psi(y) = 0.$$

Put

$$\Phi = \frac{a\phi}{\phi(x)} + \frac{b\psi}{\psi(y)}.$$

The function Φ is still in $\mathcal{C}$, and we have $\Phi(x) = a$ and $\Phi(y) = b$. Hence $\mathcal{C}$ interpolates pairs of points. □

If X is a compact metric space and $f\colon X \longrightarrow \mathbf{C}$ is a function, then we write $\overline{f}\colon X \longrightarrow \mathbf{C}$ for the function defined by

$$\overline{f}(x) = \overline{f(x)}.$$

Given a subalgebra $\mathcal{C}$ of $C_0(X, \mathbf{C})$, we say $\mathcal{C}$ is *closed under complex conjugation* if $f \in \mathcal{C}$ implies $\overline{f} \in \mathcal{C}$.

12.5. Let X be a compact metric space and let $\mathcal{C}$ be a non-vanishing subalgebra of $C_0(X, \mathbf{C})$. Suppose $\mathcal{C}$ is closed under complex conjugation and separates points.

(a) Let $\mathcal{C}_{\mathbf{R}} = \mathcal{C} \cap C_0(X, \mathbf{R})$. Prove that $\mathcal{C}_{\mathbf{R}}$ is dense in $C_0(X, \mathbf{R})$.

(b) Prove that $\mathcal{C}$ is dense in $C_0(X, \mathbf{C})$.

Solution.

(a) If x and y are elements of X and if a and b are real numbers, then by Question 12.4 there exists a function f in $\mathcal{C}$ such that $f(x) = a$ and $f(y) = b$. Put $g = (f + \overline{f})/2$, which is a real-valued function by definition. Since $g(x) = a$ and $g(y) = b$, it follows that $\mathcal{C}_{\mathbf{R}}$ interpolates pairs of points, so $\mathcal{C}_{\mathbf{R}}$ is dense in $C_0(X, \mathbf{R})$ by the Stone–Weierstrass theorem.

(b) Given a function f in $C_0(X, \mathbf{C})$, we know that $f = g + ih$, where

$$g = \frac{f + \overline{f}}{2} \quad \text{and} \quad h = \frac{f - \overline{f}}{2i}.$$

Since g and h both belong to $C_0(X, \mathbf{R})$, it follows that g and h are both in $\overline{\mathcal{C}}$. The subalgebra $\mathcal{C}$ is a vector subspace of $C_0(X, \mathbf{C})$, so $f \in \overline{\mathcal{C}}$ by Proposition 3.7. Hence $C_0(X, \mathbf{C}) = \overline{\mathcal{C}}$. □

12.6. (*) Let $(u_i)_{i \in I}$ be an orthonormal basis of an inner product space V (not necessarily separable) and let v be a vector in V.

(a) Given a a positive integer n, define

$$J_n = \left\{ i \in I \;\middle|\; |\langle v, u_i \rangle| > \frac{1}{n} \right\}.$$

Prove that J_n has at most $n^2 \|v\|^2$ elements.

(b) Put

$$I_v = \left\{ i \in I \;\middle|\; |\langle v, u_i \rangle| \neq 0 \right\}.$$

Prove that I_v is countable.

(c) Choose a bijection $o\colon \mathbf{N} \longrightarrow I_v$. Prove that

$$v = \sum_{n=1}^{\infty} \langle v, u_{o(n)} \rangle u_{o(n)}.$$

(d) Justify the notation

$$\sum_{i \in I} \langle v, u_i \rangle u_i$$

and convince yourself that

$$v = \sum_{i \in I} \langle v, u_i \rangle u_i.$$

Solution.

(a) If J_n has more than $n^2\|v\|^2$ elements, then choose a finite subset S with more than $n^2\|v\|^2$ elements. It follows from Bessel's inequality (part (a) of Theorem 3.52 or part (d) of Question 12.3) that

$$\|v\|^2 \geqslant \sum_{s \in S} \left|\langle v, u_s \rangle\right|^2 > \frac{n^2\|v\|^2}{n^2} = \|v\|^2,$$

a contradiction. Hence J_n has at most $n^2\|v\|^2$ elements.

(b) The set I_v is countable because

$$I_v = \bigcup_{n \in \mathbf{N}} J_n.$$

(c) Bessel's inequality implies

$$\sum_{n=1}^{N} \left|\langle v, u_{o(n)} \rangle\right|^2 \leqslant \|v\|^2$$

for every positive integer N, so the sequence

$$\left(\sum_{n=1}^{N} \left|\langle v, u_{o(n)} \rangle\right|^2\right)_{N=1}^{\infty}$$

is a non-decreasing sequence with an upper bound $\|v\|^2$. Hence

$$\left(\langle v, u_{o(n)} \rangle\right)_{n=1}^{\infty} \in \ell^2.$$

It then follows from Corollary 3.53 that $\sum_{n=1}^{\infty} \langle v, u_{o(n)} \rangle u_{o(n)}$ converges to some vector v' in V.

It remains to verify $v' = v$. If m is a natural number, then

$$\begin{aligned} \langle v - v', u_{o(m)} \rangle &= \left\langle v - \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} \langle v, u_{o(n)} \rangle u_{o(n)}, u_{o(m)} \right\rangle \\ &= \langle v, u_{o(m)} \rangle - \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} \langle v, u_{o(n)} \rangle \langle u_{o(n)}, u_{o(m)} \rangle \\ &= \langle v, u_{o(m)} \rangle - \langle v, u_{o(m)} \rangle = 0, \end{aligned}$$

where $\lim_{N \longrightarrow \infty} \sum_{n=1}^{N} \langle v, u_{o(n)} \rangle \langle u_{o(n)}, u_{o(m)} \rangle = \langle v, u_{o(m)} \rangle$ because

$$\sum_{n=1}^{N} \langle v, u_{o(n)} \rangle \langle u_{o(n)}, u_{o(m)} \rangle = \begin{cases} 0 & \text{if } n < m, \\ \langle v, u_{o(m)} \rangle & \text{otherwise.} \end{cases}$$

If $i \in I \setminus I_v$, then

$$\begin{aligned}\langle v - v', u_i \rangle &= \Big\langle v - \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} \big\langle v, u_{o(n)} \big\rangle u_{o(n)}, u_i \Big\rangle \\ &= \big\langle v, u_i \big\rangle - \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} \big\langle v, u_{o(n)} \big\rangle \big\langle u_{o(n)}, u_i \big\rangle = 0.\end{aligned}$$

Hence $v - v'$ is orthogonal to v_i for every v_i, which implies $v = v'$ by Question 12.1.

(d) In part (b), we proved that there are only countably many non-zero summands. Hence we can order the non-zero summands and define the sum to be the sum of the series. In part (c), we shown that the sum does not depend on the ordering of non-zero summands, so the sum is well defined. In addition, part (c) implies the identity

$$v = \sum_{i \in I} \langle v, u_i \rangle u_i.$$

□