Exercises on metric and Hilbert spaces
An invitation to functional analysis

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Contents

1. Metric and topological spaces

Metrics

Exercise 1.1. Let (X, d) be a metric space. Show that

$$|d(x,y) - d(t,y)| \leqslant d(x,t)$$

for all $x, y, t \in X$.

Exercise 1.2. Let (X, d) be a metric space. Show that

$$|d(x,y) - d(s,t)| \leqslant d(x,s) + d(y,t)$$

for all $x, s, y, t \in X$.

Exercise 1.3. Let $n \in \mathbf{N}$, $X = \mathbf{R}^n$ with the dot product $\cdot$, $\|x\| = \sqrt{x \cdot x}$ for $x \in X$, and $d(x,y) = \|x - y\|$ for $x, y \in X$. Then (X, d) is a metric space. (The function d is called the *Euclidean metric* or *ℓ^2 metric* on $\mathbf{R}^n$.)

[**Hint**: The Cauchy–Schwarz inequality can be useful for checking the triangle inequality.]

Exercise 1.4. Draw the unit open balls in the metric spaces $(\mathbf{R}^2, d_1)$ (Example 2.3), $(\mathbf{R}^2, d_2)$ (Exercise 1.3), and $(\mathbf{R}^2, d_\infty)$ (Example 2.4).

Exercise 1.5. Let X be a nonempty set and define

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that (X, d) is a metric space. (The function d is called the *discrete metric* on X.)

Exercise 1.6. Let $n \in \mathbf{N}$, $X = \mathbf{F}_2^n$, and let $d(x,y)$ be the number of indices $i \in \{1, \dots, n\}$ such that $x_i \neq y_i$. Prove that (X, d) is a metric space. (The function d is called the *Hamming metric*.)

Exercise 1.7. Let (X, d) be a metric space and define

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

Prove that (X, d') is a metric space.

[**Hint**: Before tackling the triangle inequality, show that if $a, b, c \in \mathbf{R}_{\geqslant 0}$ satisfy $c \leqslant a + b$, then $\frac{c}{1+c} \leqslant \frac{a}{1+a} + \frac{b}{1+b}$.]

Exercise 1.8. Let (X, d) be a metric space. Fix $x \in X$ and let $U = X \setminus \{x\}$; prove that U is an open set.

Exercise 1.9. Let (X, d) be a metric space. Prove that any closed ball in X is a closed set.

Exercise 1.10. Let $X = \mathbf{Q}$ and fix a prime number p. We define a metric d_p on X that, in some sense, measures the distance between rational numbers from the point of view of divisibility by p. The definition proceeds in several stages:

(a) Define the *p-adic valuation* $v_p\colon \mathbf{Z} \longrightarrow \mathbf{Z}_{\geqslant 0} \cup \{\infty\}$ by:

$$v_p(n) = \text{the largest power of } p \text{ that divides } n,$$

with the convention that $v_p(0) = \infty$.

Show that $v_p(mn) = v_p(m) + v_p(n)$ for all $m, n \in \mathbf{Z}$.

(b) Extend to the *p-adic valuation* $v_p\colon \mathbf{Q} \longrightarrow \mathbf{Z} \cup \{\infty\}$ by defining

$$v_p\left(\frac{m}{n}\right) = v_p(m) - v_p(n).$$

Show that for all $x, y \in \mathbf{Q}$ we have

$$v_p(xy) = v_p(x) + v_p(y)$$

and

$$v_p(x+y) \geqslant \min\left\{v_p(x), v_p(y)\right\},$$

with equality holding if $v_p(x) \neq v_p(y)$.

(c) Next define the *p-adic absolute value* $|\cdot|_p\colon \mathbf{Q} \longrightarrow \mathbf{Q}_{\geqslant 0}$ by:

$$|x|_p = p^{-v_p(x)},$$

with the convention that $|0|_p = p^{-\infty} = 0$.

Show that for all $x, y \in \mathbf{Q}$ we have

$$|xy|_p = |x|_p\,|y|_p$$

and

$$|x+y|_p \leqslant \max\left\{|x|_p, |y|_p\right\},$$

with equality if $|x|_p \neq |y|_p$.

(d) Finally define the *p-adic metric* on $\mathbf{Q}$ by

$$d_p(x, y) = |x - y|_p.$$

Show that $(\mathbf{Q}, d_p)$ is indeed a metric space.

Exercise 1.11. Fix a prime p and consider the metric space $(\mathbf{Q}, d_p)$ where d_p is the p-adic metric from Exercise 1.10.

(a) Let $p = 3$ and write down 4 elements of $\mathbf{B}_1(2)$ and 4 elements of $\mathbf{B}_{1/9}(3)$.

(b) Back to general prime p now: show that every triangle is isosceles. In other words, given three points in $\mathbf{Q}$, at least two of the three resulting (p-adic) distances are equal.

(c) Show that every point of an open ball is a centre. In other words, take an open ball $\mathbf{B}_r(c)$ with $r \in \mathbf{R}_{\geqslant 0}$ and $c \in \mathbf{Q}$ and suppose $x \in \mathbf{B}_r(c)$; prove that $\mathbf{B}_r(c) = \mathbf{B}_r(x)$.

(d) Show that given any two open balls, either one of them is contained in the other, or they are completely disjoint.

Exercise 1.12. Show that any p-adic open ball in $\mathbf{Q}$ is both an open set and a closed set.

Topological spaces and continuous functions

Exercise 1.13. Show that the union of any finite collection of closed sets is closed. Show that the intersection of any arbitrary collection of closed sets is closed.

Exercise 1.14. Let X be a topological space and $U \subseteq X$ a subset. Prove that U is open in X if and only if: for all $u \in U$, there exists an open neighbourhood V_u of u such that $V_u \subseteq U$.

Exercise 1.15. Let X be a topological space. Prove that a subset U of X is open if and only if it is a neighbourhood of every element of itself.

Exercise 1.16. Let X and Y be topological spaces, where the topology on Y is the trivial topology. Prove that every function from X to Y is continuous.

Exercise 1.17.

(a) Let $f\colon X \longrightarrow Y$ be a function between two sets X and Y, and let $S \subseteq Y$. Prove that

$$f^{-1}(S) = X \setminus f^{-1}(Y \setminus S).$$

(b) Let $f\colon X \longrightarrow Y$ be a function between topological spaces. Prove that f is continuous if and only if: for any closed subset $C \subseteq Y$, the inverse image $f^{-1}(C) \subseteq X$ is a closed subset.

Exercise 1.18. Let $f\colon X \longrightarrow Y$ be a function between topological spaces. Given $x \in X$, we say that f is *continuous at x* if the inverse image $f^{-1}(N)$ of every neighbourhood N of $f(x)$ is a neighbourhood of x. Prove that f is continuous if and only if it is continuous at every $x \in X$.

Exercise 1.19. Prove Proposition 2.15:
Let X be a set and $\mathcal{T}_1, \mathcal{T}_2$ two topologies on X. The following statements are equivalent:

(a) $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$ (that is, $\mathcal{T}_2 \subseteq \mathcal{T}_1$);

(b) for any $x \in X$ and any $\mathcal{T}_2$-open neighbourhood U_x^2 of x, there exists a $\mathcal{T}_1$-open neighbourhood U_x^1 of x such that $U_x^1 \subseteq U_x^2$;

(c) the function $f\colon (X, \mathcal{T}_1) \longrightarrow (X, \mathcal{T}_2)$ given by $f(x) = x$ is continuous.

Exercise 1.20. Prove that every constant function between topological spaces is continuous.

Exercise 1.21. Let X be a topological space and let S be a subset of X. Prove that the inclusion $\iota\colon S \longrightarrow X$ defined by $\iota(x) = x$ is continuous when S is given the subspace topology induced from X.
TODO: The induced topology on S is the coarsest topology such that the inclusion $\iota\colon S \longrightarrow X$ is continuous.
Conclude that the identity function $\mathrm{id}_X\colon X \longrightarrow X$ is continuous.

Exercise 1.22. Let $f\colon X \longrightarrow Y$ be a function between topological spaces. Suppose the topology on Y is generated by a subset S of $\mathcal{P}(Y)$. Prove that the function f is continuous if and only if $f^{-1}(U)$ is open for every element U of S.

Exercise 1.23. This is a variation on Tutorial Question 2.7.
Let $f\colon X \longrightarrow Y$ be a function and $\mathcal{T}_Y$ a topology on Y. Define

$$\mathcal{T}_X = \left\{ f^{-1}(U) : U \in \mathcal{T}_Y \right\}.$$

(a) Prove that $\mathcal{T}_X$ is the coarsest topology on X such that f is continuous. (This topology is called the *initial topology* induced by f.)

(b) Let $\mathcal{T}$ be another topology on X. Prove that $f: (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous if and only if $\mathcal{T}$ is finer than $\mathcal{T}_X$.

(c) Suppose $\mathcal{T}_Y$ is generated by a subset S of $\mathcal{P}(Y)$. Prove that $\mathcal{T}_X$ is generated by the set

$$\{f^{-1}(U) : U \in S\}.$$

Exercise 1.24. Let X be a topological space and let $\{y\}$ be a one-point topological space. Prove that $X \times \{y\}$ (with the product topology) is homeomorphic to X.

Exercise 1.25. Let X and Y be topological spaces and let A and B be subsets of X and Y respectively.

(a) Suppose A and B are closed in X and Y respectively. Prove that if A and B are closed, then $A \times B$ is closed.

(b) Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.

Exercise 1.26. A map $f: X \longrightarrow Y$ between topological spaces is said to be *open* if for every open subset $U \subseteq X$, the image $f(U) \subseteq Y$ is an open subset.

(a) Show that an open continuous bijective map $f: X \longrightarrow Y$ is a homeomorphism.

(b) Suppose S generates the topology on X and let S' denote the set of all finite intersections of elements of S. Show that f is open if and only if $f(U) \subseteq Y$ is an open subset for all $U \in S'$.

(c) Show that the projection maps $\pi_1: X_1 \times X_2 \longrightarrow X_1$ and $\pi_2: X_1 \times X_2 \longrightarrow X_2$ are open maps.

Exercise 1.27. Generalise Exercise 1.8 to the setting of Hausdorff topological spaces; in other words, prove that if X is a Hausdorff topological space, then any singleton $\{x\} \subseteq X$ is a closed set.

Interior and closure

Exercise 1.28. Give an example of a metric space X and an open ball $\mathbf{B}_\varepsilon(x)$ such that

$$\overline{\mathbf{B}_\varepsilon(x)} \neq \mathbf{D}_\varepsilon(x).$$

Exercise 1.29. Let X and Y be topological spaces and let A and B be subsets of X and Y respectively.

(a) Suppose A and B are closed in X and Y respectively. Prove that if A and B are closed, then $A \times B$ is closed.

(b) Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.

Exercise 1.30. Give explicit continuous surjective functions $f: \mathbf{R} \longrightarrow I$, where I is:

(a) $\mathbf{R}$ (b) $(0, \infty)$ (c) $(-\infty, 0)$ (d) $(-\infty, 0]$ (e) $[-1, 1]$
(f) $(0, 1]$ (g) $[0, 1)$ (h) $(-\pi/2, \pi/2)$ (i) $\{0\}$.

[**Hint**: Draw some functions you know from calculus and see what their ranges are.]

Exercise 1.31. Let A be a subset of a topological space X. Prove that

$$X \setminus A^\circ = \overline{X \setminus A}.$$

Exercise 1.32. Prove that $\mathbf{Z}$ is a nowhere dense subset of $\mathbf{R}$.

Exercise 1.33. Let X be a topological space.

(a) Prove that any subset of a nowhere dense subset of X is nowhere dense in X.

(b) Prove that a subset $N \subseteq X$ is nowhere dense if and only if $X \setminus \overline{N}$ is dense in X.

(c) Prove that the union of any finite collection of nowhere dense subsets of X is nowhere dense in X.

Metric topologies

Exercise 1.34. Let $f: X \longrightarrow Y$ be a function.

(a) Let $\mathcal{T}_X$ be a topology on X and let $\mathcal{T}_Y$ be the final topology induced by f, see Tutorial Question 2.7.

i. Give an example where $\mathcal{T}_Y$ is metrisable but $\mathcal{T}_X$ is not.

ii. Give an example where $\mathcal{T}_X$ is metrisable but $\mathcal{T}_Y$ is not.

(b) Let $\mathcal{T}_Y$ be a topology on Y and let $\mathcal{T}_X$ be the initial topology induced by f, see Exercise 1.23.

i. Give an example where $\mathcal{T}_X$ is metrisable but $\mathcal{T}_Y$ is not.

ii. Give an example where $\mathcal{T}_Y$ is metrisable but $\mathcal{T}_X$ is not.

[**Hint**: Consider using Tutorial Questions 2.1 and 2.3.]

Exercise 1.35. Show that any distance-preserving function $f: X \longrightarrow Y$ is continuous.
In particular, any isometry is a homeomorphism.

Exercise 1.36. Let X be a set and let d_1, d_2 be two metrics on X.

(a) Suppose that there exist $m, M \in \mathbf{R}_{>0}$ such that

$$m\, d_1(x,y) \leqslant d_2(x,y) \leqslant M\, d_1(x,y) \qquad \text{for all } x, y \in X. \tag{1.1}$$

Show that d_1 and d_2 are equivalent.

(b) Prove that the converse of (a) does not hold.

In other words, find a set X and two equivalent metrics d_1 and d_2 with the property that there **do not** exist positive real numbers m and M such that Equation (1.1) holds.

Exercise 1.37. Let X, Y be metric spaces. Show that for any $z_1, z_2 \in X \times Y$ we have

$$\frac{1}{2} d_1(z_1, z_2) \leqslant d_\infty(z_1, z_2) \leqslant d_1(z_1, z_2) \leqslant 2 d_\infty(z_1, z_2).$$

Conclude that for any conserving metric d on $X \times Y$, any $z \in X \times Y$ and any $\varepsilon > 0$ we have

$$\mathbf{B}^{d_\infty}_{\varepsilon/2}(z) \subseteq \mathbf{B}^{d_1}_{\varepsilon}(z) \subseteq \mathbf{B}^{d}_{\varepsilon}(z) \subseteq \mathbf{B}^{d_\infty}_{\varepsilon}(z) \subseteq \mathbf{B}^{d_1}_{2\varepsilon}(z).$$

Exercise 1.38. Let Y be a subset of a metric space (X, d) and consider the induced metric on Y.

(a) Prove that for any $y \in Y$ and any $r \in \mathbf{R}_{\geqslant 0}$ we have

$$\mathbf{B}_r^Y(y) = \mathbf{B}_r^X(y) \cap Y,$$

where $\mathbf{B}_r^X(y)$ is the open ball of radius r centred at y in X, and $\mathbf{B}_r^Y(y)$ is the open ball of radius r centred at y in Y.

(b) Let $A \subseteq Y$. Prove that A is an open set in Y if and only if there exists an open set U in X such that $A = U \cap Y$.

Exercise 1.39. If (X, d_X) and (Y, d_Y) are two metric spaces, a metric d on $X \times Y$ is said to be *conserving* if

$$d_\infty\big((x_1, y_1), (x_2, y_2)\big) \leqslant d\big((x_1, y_1), (x_2, y_2)\big) \leqslant d_1\big((x_1, y_1), (x_2, y_2)\big)$$

for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.

(For the definitions of d_1 and d_∞, see Examples 2.3 and 2.4.)

Prove that any conserving metric d defines the product topology on $X \times Y$. (In particular, all conserving metrics on $X \times Y$ are equivalent.)

2. Normed and Hilbert spaces

A. Appendix: Prerequisites

Equivalence relations

Exercise A.1. Let A, B be sets and $f\colon A \longrightarrow B$ a function. For $x, y \in A$, define $x \sim y$ if $f(x) = f(y)$. Show that this satisfies the properties of an equivalence relation on A.

Exercise A.2. Let $\sim$ be an equivalence relation on a set A and let $\pi\colon A \longrightarrow A/\sim$ be the quotient map.

Under what circumstances (if any) is π a bijection?

Exercise A.3. Let $A = \mathbf{N} \times \mathbf{N}$ and define $(a, b) \sim (c, d)$ if $a + d = b + c$.

(a) Show that this satisfies the conditions of an equivalence relation on A.

(b) Construct a bijective function $(A/\sim) \longrightarrow \mathbf{Z}$.

(Don't forget to prove that your function is well-defined, and that it is bijective.)

Exercise A.4. Let V be a vector space. An endomorphism (aka linear transformation from V to itself) $n\colon V \longrightarrow V$ is *nilpotent* if there exists $k \in \mathbf{Z}_{\geqslant 1}$ such that n^k is the constant zero map $V \longrightarrow V$.

An endomorphism $u\colon V \longrightarrow V$ is *unipotent* if $u - \mathrm{id}_V$ is nilpotent.

Now fix p prime, and let V be the vector space over $\mathbf{F}_p$ consisting of set maps $\mathbf{N} \longrightarrow \mathbf{F}_p$. Given two endomorphisms $f, g\colon V \longrightarrow V$, write $f \sim g$ if there exists a unipotent endomorphism u such that $f = u \circ g$.

(a) Prove that $\sim$ is reflexive.

(b) Prove that $\sim$ is symmetric.

(c) Give an example to show that $\sim$ is **not** transitive, and thus not an equivalence relation.

Exercise A.5. Let V be a vector space over $\mathbf{F}$, and let $W \subseteq V$ be a subspace. For $v, v' \in V$, write $v \sim v'$ if $v - v' \in W$.

(a) Prove that $\sim$ is an equivalence relation.

(b) Prove that the operations $[v] + [v'] := [v + v']$ and $\lambda[v] := [\lambda v]$ are well-defined. This proves that $V/\sim$ has the structure of a vector space over $\mathbf{F}$. We call $V/\sim$ a *quotient space*, and write it as V/W.

(c) Let U be a vector space and $f\colon V \longrightarrow U$ a linear transformation such that $f(w) = 0$ for all $w \in W$. Prove there exists a unique linear transformation $g\colon V/W \longrightarrow U$ such that $f = g \circ \pi$, where $\pi\colon V \longrightarrow V/W$ is the quotient map.

(Un)countability

Exercise A.6. Fix a set Ω and let X be the set of all subsets of Ω. For any $S, T \in X$, write $S \sim T$ if S has the same cardinality as T.

Show that $\sim$ is an equivalence relation on X.

Exercise A.7. Let $f\colon X \longrightarrow Y$ be a function, with X a countable set. Then $\mathrm{im}(f)$ is finite or countable.

[**Hint**: Reduce to the case $f\colon \mathbf{N} \longrightarrow Y$ is surjective; construct a right inverse $g\colon Y \longrightarrow \mathbf{N}$, which has to be injective, of f.]

Exercise A.8. Show that the union S of any countable collection of countable sets is a countable set.

[**Hint**: Construct a surjective function $\mathbf{N} \times \mathbf{N} \longrightarrow S$.]

Exercise A.9. Let W be a $\mathbf{Q}$-vector space with a countable basis B. Show that W is a countable set.

[**Hint**: Use Exercise A.8.]

Conclude that $\mathbf{R}$ does not have a countable basis as a vector space over $\mathbf{Q}$.

Linear algebra

Exercise A.10. Let V be a vector space over $\mathbf{F}$. Prove that $\mathrm{End}(V) := \mathrm{Hom}(V, V)$ is an associative unital $\mathbf{F}$-algebra under composition of functions.

Exercise A.11. Let V, W be vector spaces over $\mathbf{F}$ and let B be a basis of V. Suppose $g\colon B \longrightarrow W$ is a function, and let $f\colon V \longrightarrow W$ be its extension to V by linearity.

Prove that

(a) f is injective if and only if $g(B)$ is linearly independent in W;

(b) f is surjective if and only if $g(B)$ spans W;

(c) f is bijective if and only if $g(B)$ is a basis for W.

Exercise A.12. If S and T are subspaces of a vector space V with field of scalars $\mathbf{F}$, then so are $S + T$ and αS for any $\alpha \in \mathbf{F}$.

Exercise A.13. A *complex quadratic form* with real coefficients is a map $f\colon \mathbf{C}^n \longrightarrow \mathbf{C}$ given by

$$f(\mathbf{x}) = \sum_{1 \leqslant i,j \leqslant n} a_{ij} x_i x_j, \qquad a_{ij} \in \mathbf{R}.$$

Use the Spectral Theorem for $\mathbf{C}^n$ to prove that there exist linear maps $g_1, \dots, g_n\colon \mathbf{C}^n \longrightarrow \mathbf{C}$ and constants $b_1, \dots, b_n \in \mathbf{R}$ such that

$$f\big(g_1(\mathbf{x}), \dots, g_n(\mathbf{x})\big) = b_1 g_1(\mathbf{x})^2 + \cdots + b_n g_n(\mathbf{x})^2.$$

Uniform continuity and uniform convergence

Exercise A.14. Let $f_1, f_2, \dots$ be a sequence of continuous functions $\mathbf{R} \longrightarrow \mathbf{R}$ that are **not** uniformly continuous, and that converge pointwise to $f\colon \mathbf{R} \longrightarrow \mathbf{R}$.

(a) Give an example to show that f **can** be uniformly continuous.

(b) If the convergence $f_n \longrightarrow f$ is uniform, prove that f **cannot** be uniformly continuous.

Exercise A.15. Let $f_1, f_2, \ldots$ be a sequence of continuous functions $\mathbf{R} \longrightarrow \mathbf{R}$ that converges pointwise to $f\colon \mathbf{R} \longrightarrow \mathbf{R}$.

(a) Give an example to show that f need not be continuous. TODO: this is somewhere in the notes or exercises or tutorials, we should find it and just refer to it.

(b) Suppose that $f_n \longrightarrow f$ uniformly and that every f_n is uniformly continuous. Prove that f is uniformly continuous.

B. Appendix: Miscellaneous

Zorn's Lemma

Exercise B.1. Fix a set Ω and let X be the set of all subsets of Ω. Check that $\subseteq$ is a partial order on X. It is not a total order if Ω has at least two distinct elements.

Exercise B.2. Let $(X, \leqslant)$ be a nonempty finite poset. (This just means that X is a nonempty finite set with a partial order $\leqslant$.) Prove that X has a maximal element.

[**Hint**: You could, for instance, use induction on the number of elements of X.]

Exercise B.3. Prove Theorem 1.2: any vector space V has a basis.

[**Hint**: Let X be the set of all linearly independent subsets of V, partially ordered by inclusion. Prove that X has a maximal element B, and prove that this must also span V.]

Exercise B.4. Let $f\colon X \longrightarrow Y$ be a surjective map of sets. Let

$$P(f) = \{(s_A, A)\colon A \subseteq Y,\ s_A\colon A \longrightarrow X,\ f \circ s_A = \mathrm{id}_A\}.$$

Write $(A, s_A) \leqslant (B, s_B)$ if and only if $A \subseteq B$ and $s_B\big|_A = s_A$.

(a) Prove that $(P(f), \leqslant)$ is a poset.

(b) Prove every nonempty chain in $P(f)$ has an upper bound in $P(f)$.

(c) Deduce that there exists a map $s\colon Y \longrightarrow X$ such that $f \circ s = \mathrm{id}_Y$.

Linear algebra

Exercise B.5. Let $\mathbf{R}^{\mathbf{N}}$ be the set of arbitrary sequences $(x_1, x_2, \dots)$ of elements of $\mathbf{R}$.

This is a vector space under the naturally-defined addition of sequences and multiplication by a scalar.

Let $e_j \in \mathbf{R}^{\mathbf{N}}$ be the sequence whose j-th entry is 1, and all the others are 0. Describe the subspace $\operatorname{Span}\{e_1, e_2, \dots\}$ of $\mathbf{R}^{\mathbf{N}}$. Is the set $\{e_1, e_2, \dots\}$ a basis of $\mathbf{R}^{\mathbf{N}}$?

Exercise B.6. Let $V = \mathbf{R}$ viewed as a vector space over $\mathbf{Q}$.

Let $\alpha \in \mathbf{R}$. Show that the set $T = \{\alpha^n\colon n \in \mathbf{N}\}$ is $\mathbf{Q}$-linearly independent if and only if α is transcendental.

(Note: An element $\alpha \in \mathbf{R}$ is called *algebraic* if there exists a monic polynomial $f \in \mathbf{Q}[x]$ such that $f(\alpha) = 0$. An element $\alpha \in \mathbf{R}$ is called *transcendental* if it is not algebraic.)

Exercise B.7. Let $V = \mathbf{F}[x]$ be the vector space of polynomials in one variable with coefficients in $\mathbf{F}$. Given a scalar $\alpha \in \mathbf{F}$, consider the function $\mathrm{ev}_\alpha\colon V \longrightarrow \mathbf{F}$ given by evaluation at α:

$$\mathrm{ev}_\alpha(f) = f(\alpha).$$

Prove that $\mathrm{ev}_\alpha \in V^\vee$.

Exercise B.8. TODO: this really does not need $W = V$, should just do the general case with bases B for V and C for W, and dual bases $B^\vee$ and $C^\vee$.

In the setup of Proposition B.4, suppose $W = V$ so that $T: V \longrightarrow V$ and $T^\vee: V^\vee \longrightarrow V^\vee$.

Let M be the matrix representation of T with respect to an ordered basis B of V, and let $M^\vee$ be the matrix representation of $T^\vee$ with respect to the dual basis $B^\vee$.

Express $M^\vee$ in terms of M.

Exercise B.9. Let $v_1, \ldots, v_n \in V$. Define $\Gamma: V^\vee \longrightarrow \mathbf{F}^n$ by

$$\Gamma(\varphi) = \begin{bmatrix} \varphi(v_1) \\ \vdots \\ \varphi(v_n) \end{bmatrix}.$$

(a) Prove that Γ is a linear transformation.

(b) Prove that Γ is injective if and only if $\{v_1, \ldots, v_n\}$ spans V.

(c) Prove that Γ is surjective if and only if $\{v_1, \ldots, v_n\}$ is linearly independent.

Exercise B.10. Let $T: V \longrightarrow W$ be a linear transformation of finite-dimensional vector spaces over $\mathbf{F}$, and let $T^\vee: W^\vee \longrightarrow V^\vee$ be the dual transformation as defined in Proposition B.4.

(a) Prove that if T is surjective, then $T^\vee$ is injective.

(b) Prove that if T is injective, then $T^\vee$ is surjective.

(c) Give an example to show that (b) does not always hold if we relax the condition that V and W are finite-dimensional.

Bibliography

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