Exercises on metric and Hilbert spaces
An invitation to functional analysis

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Contents

1. Metric and topological spaces

Metrics

Exercise 1.1. Let (X, d) be a metric space. Show that

$$|d(x,y) - d(t,y)| \leqslant d(x,t)$$

for all $x, y, t \in X$.

Exercise 1.2. Let (X, d) be a metric space. Show that

$$|d(x,y) - d(s,t)| \leqslant d(x,s) + d(y,t)$$

for all $x, s, y, t \in X$.

Exercise 1.3. Let $n \in \mathbf{N}$, $X = \mathbf{R}^n$ with the dot product $\cdot$, $\|x\| = \sqrt{x \cdot x}$ for $x \in X$, and $d(x,y) = \|x - y\|$ for $x, y \in X$. Then (X, d) is a metric space. (The function d is called the *Euclidean metric* or *ℓ^2 metric* on $\mathbf{R}^n$.)

[**Hint**: The Cauchy–Schwarz inequality can be useful for checking the triangle inequality.]

Exercise 1.4. Draw the unit open balls in the metric spaces $(\mathbf{R}^2, d_1)$ (Example 2.3), $(\mathbf{R}^2, d_2)$ (Exercise 1.3), and $(\mathbf{R}^2, d_\infty)$ (Example 2.4).

Exercise 1.5. Let X be a nonempty set and define

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that (X, d) is a metric space. (The function d is called the *discrete metric* on X.)

Exercise 1.6. Let $n \in \mathbf{N}$, $X = \mathbf{F}_2^n$, and let $d(x,y)$ be the number of indices $i \in \{1, \ldots, n\}$ such that $x_i \neq y_i$. Prove that (X, d) is a metric space. (The function d is called the *Hamming metric*.)

Exercise 1.7. Let (X, d) be a metric space and define

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

Prove that (X, d') is a metric space.

[**Hint**: Before tackling the triangle inequality, show that if $a, b, c \in \mathbf{R}_{\geqslant 0}$ satisfy $c \leqslant a + b$, then $\frac{c}{1+c} \leqslant \frac{a}{1+a} + \frac{b}{1+b}$.]

Exercise 1.8. Let (X, d) be a metric space. Fix $x \in X$ and let $U = X \setminus \{x\}$; prove that U is an open set.

Exercise 1.9. Let (X, d) be a metric space. Prove that any closed ball in X is a closed set.

Exercise 1.10. (*) Let $X = \mathbf{Q}$ and fix a prime number p. We define a metric d_p on X that, in some sense, measures the distance between rational numbers from the point of view of divisibility by p. The definition proceeds in several stages:

(a) Define the *p-adic valuation* $v_p \colon \mathbf{Z} \longrightarrow \mathbf{Z}_{\geqslant 0} \cup \{\infty\}$ by:

$$v_p(n) = \text{the largest power of } p \text{ that divides } n,$$

with the convention that $v_p(0) = \infty$.

Show that $v_p(mn) = v_p(m) + v_p(n)$ for all $m, n \in \mathbf{Z}$.

(b) Extend to the *p-adic valuation* $v_p \colon \mathbf{Q} \longrightarrow \mathbf{Z} \cup \{\infty\}$ by defining

$$v_p\left(\frac{m}{n}\right) = v_p(m) - v_p(n).$$

Show that for all $x, y \in \mathbf{Q}$ we have

$$v_p(xy) = v_p(x) + v_p(y)$$

and

$$v_p(x+y) \geqslant \min\left\{v_p(x), v_p(y)\right\},$$

with equality holding if $v_p(x) \neq v_p(y)$.

(c) Next define the *p-adic absolute value* $|\cdot|_p \colon \mathbf{Q} \longrightarrow \mathbf{Q}_{\geqslant 0}$ by:

$$|x|_p = p^{-v_p(x)},$$

with the convention that $|0|_p = p^{-\infty} = 0$.

Show that for all $x, y \in \mathbf{Q}$ we have

$$|xy|_p = |x|_p\,|y|_p$$

and

$$|x+y|_p \leqslant \max\left\{|x|_p, |y|_p\right\},$$

with equality if $|x|_p \neq |y|_p$.

(d) Finally define the *p-adic metric* on $\mathbf{Q}$ by

$$d_p(x, y) = |x - y|_p.$$

Show that $(\mathbf{Q}, d_p)$ is indeed a metric space.

Exercise 1.11. (*) Fix a prime p and consider the metric space $(\mathbf{Q}, d_p)$ where d_p is the p-adic metric from Exercise 1.10.

(a) Let $p = 3$ and write down 4 elements of $\mathbf{B}_1(2)$ and 4 elements of $\mathbf{B}_{1/9}(3)$.

(b) Back to general prime p now: show that every triangle is isosceles. In other words, given three points in $\mathbf{Q}$, at least two of the three resulting (p-adic) distances are equal.

(c) Show that every point of an open ball is a centre. In other words, take an open ball $\mathbf{B}_r(c)$ with $r \in \mathbf{R}_{\geqslant 0}$ and $c \in \mathbf{Q}$ and suppose $x \in \mathbf{B}_r(c)$; prove that $\mathbf{B}_r(c) = \mathbf{B}_r(x)$.

(d) Show that given any two open balls, either one of them is contained in the other, or they are completely disjoint.

Exercise 1.12. (*) Show that any p-adic open ball in $\mathbf{Q}$ is both an open set and a closed set.

Topological spaces and continuous functions

Exercise 1.13. Show that the union of any finite collection of closed sets is closed. Show that the intersection of any arbitrary collection of closed sets is closed.

Exercise 1.14. Let X be a topological space and $U \subseteq X$ a subset. Prove that U is open in X if and only if: for all $u \in U$, there exists an open neighbourhood V_u of u such that $V_u \subseteq U$.

Exercise 1.15. Let X be a topological space. Prove that a subset U of X is open if and only if it is a neighbourhood of every element of itself.

Exercise 1.16. Let X and Y be topological spaces, where the topology on Y is the trivial topology. Prove that every function from X to Y is continuous.

Exercise 1.17.

(a) Let $f\colon X \longrightarrow Y$ be a function between two sets X and Y, and let $S \subseteq Y$. Prove that

$$f^{-1}(S) = X \smallsetminus f^{-1}(Y \smallsetminus S).$$

(b) Let $f\colon X \longrightarrow Y$ be a function between topological spaces. Prove that f is continuous if and only if: for any closed subset $C \subseteq Y$, the inverse image $f^{-1}(C) \subseteq X$ is a closed subset.

Exercise 1.18. Let $f\colon X \longrightarrow Y$ be a function between topological spaces. Given $x \in X$, we say that f is *continuous at x* if the inverse image $f^{-1}(N)$ of every neighbourhood N of $f(x)$ is a neighbourhood of x. Prove that f is continuous if and only if it is continuous at every $x \in X$.

Exercise 1.19. Prove Proposition 2.15:
Let X be a set and $\mathcal{T}_1, \mathcal{T}_2$ two topologies on X. The following statements are equivalent:

(a) $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$ (that is, $\mathcal{T}_2 \subseteq \mathcal{T}_1$);

(b) for any $x \in X$ and any $\mathcal{T}_2$-open neighbourhood U_x^2 of x, there exists a $\mathcal{T}_1$-open neighbourhood U_x^1 of x such that $U_x^1 \subseteq U_x^2$;

(c) the function $f\colon (X, \mathcal{T}_1) \longrightarrow (X, \mathcal{T}_2)$ given by $f(x) = x$ is continuous.

Exercise 1.20. Prove that every constant function between topological spaces is continuous.

Exercise 1.21. Let X be a topological space and let S be a subset of X. Prove that the inclusion $\iota\colon S \longrightarrow X$ defined by $\iota(x) = x$ is continuous when S is given the subspace topology induced from X.
TODO: The induced topology on S is the coarsest topology such that the inclusion $\iota\colon S \longrightarrow X$ is continuous.
Conclude that the identity function $\mathrm{id}_X\colon X \longrightarrow X$ is continuous.

Exercise 1.22. Let $f\colon X \longrightarrow Y$ be a function between topological spaces. Suppose the topology on Y is generated by a subset S of $\mathcal{P}(Y)$. Prove that the function f is continuous if and only if $f^{-1}(U)$ is open for every element U of S.

Exercise 1.23. This is a variation on Tutorial Question 2.7.
Let $f\colon X \longrightarrow Y$ be a function and $\mathcal{T}_Y$ a topology on Y. Define

$$\mathcal{T}_X = \left\{ f^{-1}(U) : U \in \mathcal{T}_Y \right\}.$$

(a) Prove that $\mathcal{T}_X$ is the coarsest topology on X such that f is continuous. (This topology is called the *initial topology* induced by f.)

(b) Let $\mathcal{T}$ be another topology on X. Prove that $f\colon (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous if and only if $\mathcal{T}$ is finer than $\mathcal{T}_X$.

(c) Suppose $\mathcal{T}_Y$ is generated by a subset S of $\mathcal{P}(Y)$. Prove that $\mathcal{T}_X$ is generated by the set

$$\{f^{-1}(U) : U \in S\}.$$

Exercise 1.24. Let X be a topological space and let $\{y\}$ be a one-point topological space. Prove that $X \times \{y\}$ (with the product topology) is homeomorphic to X.

Exercise 1.25. A map $f : X \longrightarrow Y$ between topological spaces is said to be *open* if for every open subset $U \subseteq X$, the image $f(U) \subseteq Y$ is an open subset.

(a) Show that an open continuous bijective map $f : X \longrightarrow Y$ is a homeomorphism.

(b) Suppose S generates the topology on X and let S' denote the set of all finite intersections of elements of S. Show that f is open if and only if $f(U) \subseteq Y$ is an open subset for all $U \in S'$.

(c) Show that the projection maps $\pi_1 : X_1 \times X_2 \longrightarrow X_1$ and $\pi_2 : X_1 \times X_2 \longrightarrow X_2$ are open maps.

Exercise 1.26. Generalise Exercise 1.8 to the setting of Hausdorff topological spaces; in other words, prove that if X is a Hausdorff topological space, then any singleton $\{x\} \subseteq X$ is a closed set.

Interior and closure

Exercise 1.27. Give an example of a metric space X and an open ball $\mathbf{B}_\varepsilon(x)$ such that

$$\overline{\mathbf{B}_\varepsilon(x)} \neq \mathbf{D}_\varepsilon(x).$$

Exercise 1.28. Let X and Y be topological spaces and let A and B be subsets of X and Y respectively.

(a) Suppose A and B are closed in X and Y respectively. Prove that if A and B are closed, then $A \times B$ is closed.

(b) Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.

(c) Conclude that if A is dense in X and B is dense in Y then $A \times B$ is dense in $X \times Y$.

Exercise 1.29. (*) Give explicit continuous surjective functions $f : \mathbf{R} \longrightarrow I$, where I is:

(a) $\mathbf{R}$ (b) $(0, \infty)$ (c) $(-\infty, 0)$ (d) $(-\infty, 0]$ (e) $[-1, 1]$
(f) $(0, 1]$ (g) $[0, 1)$ (h) $(-\pi/2, \pi/2)$ (i) $\{0\}$.

[**Hint**: Draw some functions you know from calculus and see what their ranges are.]

Exercise 1.30. Let A be a subset of a topological space X. Prove that

$$X \setminus A^\circ = \overline{X \setminus A}.$$

Exercise 1.31. (*) Prove that $\mathbf{Z}$ is a nowhere dense subset of $\mathbf{R}$.

Exercise 1.32. (*) Let X be a topological space.

(a) Prove that any subset of a nowhere dense subset of X is nowhere dense in X.

(b) Prove that a subset $N \subseteq X$ is nowhere dense if and only if $X \smallsetminus \overline{N}$ is dense in X.

(c) Prove that the union of any finite collection of nowhere dense subsets of X is nowhere dense in X.

Metric topologies

Exercise 1.33. Let $f\colon X \longrightarrow Y$ be a function.

(a) Let $\mathcal{T}_X$ be a topology on X and let $\mathcal{T}_Y$ be the final topology induced by f, see Tutorial Question 2.7.

i. Give an example where $\mathcal{T}_Y$ is metrisable but $\mathcal{T}_X$ is not.

ii. Give an example where $\mathcal{T}_X$ is metrisable but $\mathcal{T}_Y$ is not.

(b) Let $\mathcal{T}_Y$ be a topology on Y and let $\mathcal{T}_X$ be the initial topology induced by f, see Exercise 1.23.

i. Give an example where $\mathcal{T}_X$ is metrisable but $\mathcal{T}_Y$ is not.

ii. Give an example where $\mathcal{T}_Y$ is metrisable but $\mathcal{T}_X$ is not.

[**Hint**: Consider using Tutorial Questions 2.1 and 2.3.]

Exercise 1.34. Show that any isometry $f\colon X \longrightarrow Y$ is continuous.
In particular, any bijective isometry is a homeomorphism.

Exercise 1.35. Let X be a set and let d_1, d_2 be two metrics on X.

(a) Suppose that there exist $m, M \in \mathbf{R}_{>0}$ such that

$$m\, d_1(x,y) \leqslant d_2(x,y) \leqslant M\, d_1(x,y) \qquad \text{for all } x,y \in X. \tag{1.1}$$

Show that d_1 and d_2 are equivalent.

(b) Prove that the converse of (a) does not hold.

In other words, find a set X and two equivalent metrics d_1 and d_2 with the property that there **do not** exist positive real numbers m and M such that Equation (1.1) holds.

Exercise 1.36. If (X, d_X) and (Y, d_Y) are two metric spaces, a metric d on $X \times Y$ is said to be *conserving* if

$$d_\infty\big((x_1,y_1),(x_2,y_2)\big) \leqslant d\big((x_1,y_1),(x_2,y_2)\big) \leqslant d_1\big((x_1,y_1),(x_2,y_2)\big)$$

for all $(x_1,y_1),(x_2,y_2) \in X \times Y$.

(For the definitions of d_1 and d_∞, see Examples 2.3 and 2.4.)

Prove that any conserving metric d defines the product topology on $X \times Y$. (In particular, all conserving metrics on $X \times Y$ are equivalent.)

Exercise 1.37. Let X, Y be metric spaces. Show that for any $z_1, z_2 \in X \times Y$ we have

$$\frac{1}{2} d_1(z_1, z_2) \leqslant d_\infty(z_1, z_2) \leqslant d_1(z_1, z_2) \leqslant 2 d_\infty(z_1, z_2).$$

Conclude that for any conserving metric d on $X \times Y$, any $z \in X \times Y$ and any $\varepsilon > 0$ we have

$$\mathbf{B}^{d_\infty}_{\varepsilon/2}(z) \subseteq \mathbf{B}^{d_1}_{\varepsilon}(z) \subseteq \mathbf{B}^{d}_{\varepsilon}(z) \subseteq \mathbf{B}^{d_\infty}_{\varepsilon}(z) \subseteq \mathbf{B}^{d_1}_{2\varepsilon}(z).$$

Exercise 1.38. Let Y be a subset of a metric space (X, d) and consider the induced metric on Y.

(a) Prove that for any $y \in Y$ and any $r \in \mathbf{R}_{\geqslant 0}$ we have

$$\mathbf{B}^Y_r(y) = \mathbf{B}^X_r(y) \cap Y,$$

where $\mathbf{B}^X_r(y)$ is the open ball of radius r centred at y in X, and $\mathbf{B}^Y_r(y)$ is the open ball of radius r centred at y in Y.

(b) Let $A \subseteq Y$. Prove that A is an open set in Y if and only if there exists an open set U in X such that $A = U \cap Y$.

Exercise 1.39. Let (X, d) be a metric space, where X is a finite set. Prove that d is topologically equivalent to the discrete metric on X.

Connectedness

Exercise 1.40. Prove that a subset D of a topological space X is disconnected if and only if there exist open subsets $U, V \subseteq X$ such that

$$D \subseteq U \cup V, \qquad D \cap U \cap V = \varnothing, \qquad D \cap U \neq \varnothing, \qquad D \cap V \neq \varnothing.$$

Exercise 1.41. Prove that a discrete topological space X is connected if and only if X is a singleton.

Exercise 1.42. Let X be a topological space. Suppose $\{C_n \colon n \in \mathbf{N}\}$ is a countable collection of connected subsets of X such that $C_n \cap C_{n+1} \neq \varnothing$ for all $n \in \mathbf{N}$. Then

$$\bigcup_{n \in \mathbf{N}} C_n$$

is a connected subset of X.

Exercise 1.43. Prove that if a non-empty topological space X admits a connected dense subset D, then X is itself connected.

Exercise 1.44. Let X be a topological space. Suppose A is a connected subset of X and $\{C_i \colon i \in I\}$ is an arbitrary collection of connected subsets of X such that $A \cap C_i \neq \varnothing$ for all $i \in I$. Then

$$A \cup \bigcup_{i \in I} C_i$$

is a connected subset of X.

Exercise 1.45. Let X and Y be non-empty topological spaces. Prove that $X \times Y$ is connected if and only if both X and Y are connected.

Exercise 1.46. (*) A topological space X is called *totally separated* if X has at least two distinct elements, and for every two distinct points x, y in X there exist disjoint clopen neighbourhoods U and V of x and y respectively. Prove that every totally separated space is totally disconnected.

Exercise 1.47. (*) Prove that the following are totally disconnected:

(a) $\mathbf{Q}$ equipped with the Euclidean topology;

(b) every discrete topological space with at least two distinct elements.

Compactness

Exercise 1.48. Let X be a topological space. We say that a collection of closed subsets of X has the *finite intersection property* if every finite subcollection has nonempty intersection.

Prove that X is compact if and only if every collection of closed sets with the finite intersection property has nonempty intersection.

Exercise 1.49. Let $\mathbf{S}^1 = \mathbf{S}_1\big((0,0)\big) = \big\{x, y \in \mathbf{R} \colon x^2 + y^2 = 1\big\}$ be the unit circle in $\mathbf{R}^2$.

Consider the function $f \colon [0,1) \longrightarrow \mathbf{S}^1$ given by the parametrisation

$$f(t) = \big(\cos(2\pi t), \sin(2\pi t)\big).$$

Endow $[0,1)$ with the induced metric from $\mathbf{R}$ and $\mathbf{S}^1$ with the induced metric from $\mathbf{R}^2$.

Prove that f is a bijective continuous function, but not a homeomorphism.

(You may use without proof whatever properties of the functions sin and cos you manage to remember from previous subjects.)

Exercise 1.50. Prove that no two of the following spaces are homeomorphic:

(a) the interval $X = [-1, 1]$ in $\mathbf{R}$;

(b) the open unit disc Y in $\mathbf{R}^2$;

(c) the closed unit disc Z in $\mathbf{R}^2$.

Exercise 1.51. Are the following pairs of spaces homeomorphic or not?

(a) the unit circle in $\mathbf{R}^2$ and the unit interval $[0,1]$ in $\mathbf{R}$;

(b) the intervals $[0,1]$ and $(0,1)$ in $\mathbf{R}$;

(c) the intervals $[0,1]$ and $[0,2]$ in $\mathbf{R}$.

Exercise 1.52. Prove Proposition 2.36: A subset K of a topological space X is compact if and only if, for any collection $\{U_i \colon i \in I\}$ of open subsets of X such that

$$K \subseteq \bigcup_{i \in I} U_i,$$

there exist $n \in \mathbf{N}$ and $i_1, \ldots, i_n \in I$ such that

$$K \subseteq U_{i_1} \cup \cdots \cup U_{i_n}.$$

Sequences

Exercise 1.53. Let (X, d) be a metric spaces. Prove that

$$(x_n) \sim (y_n) \qquad \text{if } \big(d(x_n, y_n)\big) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

defines an equivalence relation on the set of sequences in X.

Exercise 1.54. Any sequence has at most one limit.

Exercise 1.55. (*) Let $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$ and define

$$\mathcal{T} = \mathcal{P}(\mathbf{N}) \cup \{U \in \mathcal{P}(\mathbf{N}^*) \colon \infty \in U \text{ and } \mathbf{N}^* \setminus U \text{ is finite}\}.$$

(a) Prove that $\mathcal{T}$ is a topology on $\mathbf{N}^*$.

(b) Prove that $(\mathbf{N}^*, \mathcal{T})$ is compact.

(c) Let X be a metric space and $f\colon (\mathbf{N}^*, \mathcal{T}) \longrightarrow X$. Prove that f is continuous if and only if $\big(f(n)\big)$ converges to $f(\infty)$. (In other words, convergent sequences in X are exactly continuous functions from $(\mathbf{N}^*, \mathcal{T})$ to X.)

(d) Let X be a metric space and let (x_n) be a sequence in X that converges to a point x in X. Prove that $\{x\} \cup \{x_n \colon n \in \mathbf{N}\}$ is compact.

Exercise 1.56. Let (X, d) be a metric space and (x_n), (y_n) Cauchy sequences in X. Prove that $\big(d(x_n, y_n)\big)$ is a Cauchy sequence in $\mathbf{R}$.

Exercise 1.57. Let (X, d) be a metric space and let $(x_n) \sim (y_n)$. Prove that (x_n) is Cauchy if and only if (y_n) is Cauchy.

Uniform continuity and completeness

Exercise 1.58. We temporarily say a topological space X has **property H** if the following holds:

H: for any topological space Y and any continuous functions $f, g\colon Y \longrightarrow X$ such that f and g agree on some dense subset D of Y, we have $f = g$.

(a) Let $f, g\colon Y \longrightarrow X$ be continuous functions between topological spaces. Suppose that X has property H and that f and g agree on some subset S of Y. Prove that f and g agree on the closure $\overline{S}$ of S in Y.

(b) Prove that every Hausdorff topological space has property H.

(c) Prove that every topological space with property H is Hausdorff.

(d) Prove that if X is Hausdorff then any continuous function $f\colon D \longrightarrow X$ has at most one continuous extension $\widetilde{f}\colon Y \longrightarrow X$.

Exercise 1.59. Let S be a subset of a metric space (X, d_X) and let d_S be the induced metric on S.

(a) Prove that the inclusion function $\iota_S\colon S \longrightarrow X$ is uniformly continuous.

(b) Prove that a function $f\colon (Y, d_Y) \longrightarrow (S, d_S)$ is uniformly continuous if and only if $\iota_S \circ f$ is uniformly continuous.

Exercise 1.60. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces and let d be a metric on $Y \times Z$ such that

$$\max\{d_Y(y_1, y_2), d_Z(z_1, z_2)\} \leqslant d\big((y_1, z_1), (y_2, z_2)\big) \leqslant d_Y(y_1, y_2) + d_Z(z_1, z_2)$$

for every pair of points (y_1, z_1) and (y_2, z_2) in $Y \times Z$.

(a) Prove that the projections $\pi_Y\colon Y \times Z \longrightarrow Y$ and $\pi_Z\colon Y \times Z \longrightarrow Z$ are uniformly continuous.

(b) Prove that a function $f\colon X \longrightarrow Y \times Z$ is uniformly continuous if and only if both $\pi_Y \circ f$ and $\pi_Z \circ f$ are.

Exercise 1.61. Suppose $f\colon X \longrightarrow Y$ is a *uniform homeomorphism* between metric spaces; that is, a homeomorphism such that both f and its inverse are uniformly continuous.

(a) Prove that a sequence (x_n) is Cauchy in X if and only if $\big(f(x_n)\big)$ is Cauchy in Y.

(b) Prove that X is complete if and only if Y is complete.

(c) Prove that $f\colon \mathbf{R} \longrightarrow (-\pi/2, \pi/2)$ given by $f(x) = \arctan(x)$ is uniformly continuous and a homeomorphism, but it is not a uniform homeomorphism.

(d) Do you feel strongly that uniformly continuous functions ought to preserve completeness? (After all, they preserve Cauchy sequences, and completeness is defined in terms of Cauchy sequences.)

Prove that the function f defined in part (c) does not preserve completeness though it is uniformly continuous and a homeomorphism.

Exercise 1.62. Give $\mathbf{Q} \subseteq \mathbf{R}$ the induced metric and consider the sequence (x_n) defined recursively by

$$x_1 = 1, \qquad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \quad \text{for } n \in \mathbf{N}.$$

(a) Prove that $1 \leqslant x_n \leqslant 2$ for all $n \in \mathbf{N}$ and breathe a sigh of relief that the recursive definition does not accidentally divide by 0.

(b) For $n \in \mathbf{N}$, let $y_n = x_{n+1} - x_n$. Prove that

$$y_{n+1} = -\frac{y_n^2}{2x_{n+1}} \qquad \text{for all } n \in \mathbf{N}.$$

(c) Prove that

$$|y_n| \leqslant \frac{1}{2^n} \qquad \text{for all } n \in \mathbf{N}.$$

(d) Show that (x_n) is Cauchy.

(e) Show that (x_n) converges to $\sqrt{2}$ in $\mathbf{R}$, and conclude that $\mathbf{Q}$ is not complete.

Exercise 1.63. Let (X, d_X) and (Y, d_Y) be metric spaces and let d be the sup norm metric on $X \times Y$.

(a) Prove that the sequence $\big((x_n, y_n)\big)$ is Cauchy in $X \times Y$ if and only if (x_n) is Cauchy in X and (y_n) is Cauchy in Y.

(b) Prove that if X and Y are complete then $X \times Y$ is complete. Is the converse true?

Exercise 1.64. Let (X, d) be a metric space.

(a) Fix an arbitrary element $y \in X$ and consider the function $f\colon X \longrightarrow \mathbf{R}$ given by $f(x) = d(x, y)$. Prove that f is uniformly continuous.

(b) Prove that $d\colon X \times X \longrightarrow \mathbf{R}$ is uniformly continuous with respect to the sup metric D on $X \times X$.

Exercise 1.65. In the context of the proof of Theorem 2.55, show that if $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$, then

$$\lim_{n\longrightarrow\infty} d(x'_n, y'_n) = \lim_{n\longrightarrow\infty} d(x_n, y_n).$$

Exercise 1.66. Let $X = \mathbf{R}_{>0}$, $Y = \mathbf{R}$, $f\colon X \longrightarrow Y$ given by $f(x) = \frac{1}{x}$. For $\widehat{X} = \mathbf{R}_{\geqslant 0}$ and $\widehat{Y} = Y = \mathbf{R}$, prove that there is no continuous function $\widehat{f}\colon \widehat{X} \longrightarrow \widehat{Y}$ such that $\widehat{f}\,|_X = f$.

Exercise 1.67. Let $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ be a contraction and define $F\colon \mathbf{R} \longrightarrow \mathbf{R}$ by

$$F(x) = x + f(x).$$

(a) Use the Banach Fixed Point Theorem to show that the equation $x + f(x) = a$ has a unique solution for any $a \in \mathbf{R}$.

(b) Deduce that F is a bijection.

(c) Show that F is continuous.

(d) Show that F^{-1} is continuous (so it is a homeomorphism).

Exercise 1.68. Let X be the interval $(0, 1/3)$ in $\mathbf{R}$ with the Euclidean metric. Show that $f\colon X \longrightarrow X$ defined by $f(x) = x^2$ is a contraction, but does not have a fixed point in X. Why does this not contradict the Banach Fixed Point Theorem?

Exercise 1.69. Let (X, d) be a complete metric space and $f\colon X \longrightarrow X$ be a function. Let $g = f \circ f$, that is, $g(x) = f\big(f(x)\big)$. Suppose that $g\colon X \longrightarrow X$ is a contraction. Prove that f has a unique fixed point.

Boundedness and total boundedness

Exercise 1.70. Show that a subset $S \subseteq X$ is bounded if and only if $S \subseteq \mathbf{D}_r(x)$ for some $r \geqslant 0$ and some $x \in X$.

Exercise 1.71. Prove that any Cauchy sequence (x_n) in a metric space (X, d) is *bounded*, that is there exists $C \geqslant 0$ such that $d(x_n, x_m) \leqslant C$ for all $n, m \in \mathbf{N}$.

Exercise 1.72.

(a) Prove that every closed interval on $\mathbf{R}$ is compact.

(b) Prove that every closed ball in $\mathbf{R}^n$ is compact.

(c) (**The classical Heine–Borel theorem**) Prove that a subset of $\mathbf{R}^n$ is compact if and only if it is bounded and closed.

(d) Prove that every bounded subset of $\mathbf{R}^n$ is totally bounded.

Exercise 1.73. Let (X, d) be a metric space and let A, B be bounded sets. Then $A \cup B$ is bounded.

Exercise 1.74. Prove that a function $f : X \longrightarrow Y$ between metric spaces is bounded if and only if $f(X)$ is a bounded subset of Y.

Exercise 1.75. Let X, Y be metric spaces and $S \subseteq X$, $T \subseteq Y$ totally bounded subsets. Prove that $S \times T$ is a totally bounded subset of $X \times Y$ (where the latter is equipped with a conserving metric d).

Exercise 1.76. Suppose X and Y are metric spaces with the property that every bounded subset of either of them is totally bounded. Prove that the same is true in the product $X \times Y$ (equipped with a conserving metric).

Exercise 1.77. (*) Let K be a sequentially compact subset of a metric space X. Prove that any open cover of K has a Lebesgue number.

Exercise 1.78. Which of the following metric spaces are compact?

(a) The unit circle in $\mathbf{R}^2$.

(b) The unit open disk in $\mathbf{R}^2$.

(c) The closed unit ball in the space ℓ^∞ of bounded real sequences $\big(a_1, a_2, \dots\big)$.

Exercise 1.79. Let C be a nonempty compact subset of a metric space (X, d). Prove that there exist points $a, b \in C$ such that

$$d(a, b) = \sup\big\{d(x, y) : x, y \in C\big\}.$$

In other words, the diameter of C is realised as the distance between two points of C.

Exercise 1.80.

(a) Suppose $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a continuous function and S is a bounded subset of $\mathbf{R}^n$. Prove that $f(S)$ is bounded.

(b) Find a uniformly continuous function $f : X \longrightarrow Y$ between metric spaces and a bounded subset B of X such that $f(B)$ is unbounded.

Exercise 1.81.

(a) Suppose $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a continuous function and S is a totally bounded subset of $\mathbf{R}^n$. Prove that $f(S)$ is totally bounded.

(b) Find a continuous function $f : X \longrightarrow Y$ between metric spaces and a totally bounded subset S of X such that $f(S)$ is not totally bounded.

Function spaces

Exercise 1.82. Let X be a set and Y a metric space, and consider the metric space $B(X,Y)$ of bounded functions $X \longrightarrow Y$, with the uniform metric d_∞ (see Proposition 2.67).

Fix $x \in X$ and consider the "evaluation at x" function $\mathrm{ev}_x \colon B(X,Y) \longrightarrow Y$ given by $\mathrm{ev}_x(f) = f(x)$. Prove that ev_x is uniformly continuous.

Exercise 1.83. For each $n \in \mathbf{N}$ define $f_n \colon [0,1] \longrightarrow \mathbf{R}$ by

$$f_n(x) = \frac{nx^2}{1+nx}.$$

Convince yourself that f_n is continuous.

Find the pointwise limit f of the sequence (f_n) and determine whether the sequence converges uniformly to f.

Exercise 1.84. Let X, Y be metric spaces and let (f_n) be a sequence in $C_0(X,Y)$ that converges uniformly to $f \in C_0(X,Y)$. If $(x_n) \longrightarrow x$ in X, then $\big(f_n(x_n)\big) \longrightarrow f(x)$ in Y.

Exercise 1.85. (*) Let $p_1(x) = 0$ and

$$p_{n+1}(x) = p_n(x) - \frac{p_n(x)^2 - x^2}{2} = p_n(x) - \frac{\big(p_n(x) - |x|\big)\big(p_n(x) + |x|\big)}{2} \qquad \text{for } n \geqslant 1.$$

Prove that, for all $x \in [-1,1]$ and all $n \geqslant 1$:

(a) $0 \leqslant p_n(x) \leqslant |x|$;

(b) $p_n(x) \leqslant p_{n+1}(x)$;

(c) $|x| - p_{n+1}(x) \leqslant |x|\left(1 - \frac{|x|}{2}\right)^n$.

Exercise 1.86. (*) Fix $n \geqslant 1$ and consider the function $f \colon [0,1] \longrightarrow \mathbf{R}$ given by

$$f(t) = t\left(1 - \frac{t}{2}\right)^n.$$

Prove that

$$f(t) < \frac{2}{n+1} \qquad \text{for all } t \in [0,1].$$

Exercise 1.87. Prove that for any $a > 0$, there is a sequence (q_n) in $x\mathbf{R}[x]$ such that $(q_n) \longrightarrow |x|$ uniformly on $[-a,a]$.

[**Hint**: Take the sequence of polynomials given by Lemma 2.73 and use it to construct (q_n).]

Exercise 1.88. (*) Let X be a metric space with at least two points.

Prove that a subalgebra $\mathcal{A}$ of $C_0(X,\mathbf{R})$ separates points of X and is non-vanishing on X if and only if, for every $(x_1,y_1),(x_2,y_2) \in X \times \mathbf{R}$ with $x_1 \neq x_2$, there exists $h \in \mathcal{A}$ such that

$$f(x_1) = y_1 \qquad \text{and} \qquad f(x_2) = y_2.$$

[**Hint**: For the "if" direction, given $(x_1,y_1),(x_2,y_2) \in X \times \mathbf{R}$ with $x_1 \neq x_2$, find elements $k_1, k_2 \in \mathcal{A}$ such that $k_1(x_1) = 0$, $k_1(x_2) \neq 0$, $k_2(x_1) \neq 0$, $k_2(x_2) = 0$.]

Exercise 1.89. If X is a compact metric space and $f\colon X \longrightarrow \mathbf{C}$ is a function, then we write $\overline{f}\colon X \longrightarrow \mathbf{C}$ for the function defined by

$$\overline{f}(x) = \overline{f(x)}.$$

Given a subalgebra $\mathcal{C}$ of $C_0(X, \mathbf{C})$, we say $\mathcal{C}$ is *closed under complex conjugation* if $f \in \mathcal{C}$ implies $\overline{f} \in \mathcal{C}$.

Let X be a compact metric space and let $\mathcal{C}$ be a $\mathbf{C}$-subalgebra of $C_0(X, \mathbf{C})$. Suppose $\mathcal{C}$ is closed under complex conjugation, is non-vanishing, and separates points.

Let $\mathcal{C}_{\mathbf{R}} = \mathcal{C} \cap C_0(X, \mathbf{R})$.

(a) Prove that if $g \in \mathcal{C}$ then $\mathrm{Re}(f), \mathrm{Im}(f) \in \mathcal{C}_{\mathbf{R}}$.

(b) Prove that $\mathcal{C}_{\mathbf{R}}$ is dense in $C_0(X, \mathbf{R})$.

(c) Prove that $\mathcal{C}$ is dense in $C_0(X, \mathbf{C})$.

Exercise 1.90. Let $X = [0,1] \times [0,1]$ be the unit square with the induced topology from $\mathbf{R}^2$.

Find a subalgebra $\mathcal{A}$ of $C_0(X, \mathbf{C})$ that is dense.

[**Hint**: Check out Tutorial Question 8.3 and Exercise 1.89.]

Exercise 1.91. Let $X = \mathbf{S}^1$ be the unit circle with the induced topology from $\mathbf{R}^2$.

Find a subalgebra $\mathcal{A}$ of $C_0(X, \mathbf{C})$ that is dense.

[**Hint**: A sneaky way is to use Exercise 1.90. A nice way is to get complex exponentials involved.]

(*) Compactness in function spaces

Exercise 1.92. (*) We say that a collection $\mathcal{F}$ of functions $X \longrightarrow Y$ between metric spaces is *equicontinuous* if given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{F}$ and all $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$ we have $d_Y(f(x_1), f(x_2)) < \varepsilon$.

(a) Prove that a singleton $\mathcal{F} = \{f\}$ is equicontinuous if and only if f is uniformly continuous.

(b) Prove that the set $\mathcal{F}$ of all contractions $X \longrightarrow Y$ is equicontinuous.

Exercise 1.93. (*) Let X be a totally bounded metric space and Y a complete metric space. Suppose $\mathcal{F} = (f_n)$ is an equicontinuous sequence in $C_0(X, Y)$ such that $\big(f_n(z)\big)$ converges in Y for every z in a dense subset Z of X. Then (f_n) converges uniformly in $C_0(X, Y)$.

Exercise 1.94. (*) Let X be a metric space and let Z be a countable subset of X. Then every bounded sequence (f_n) in $C_0(X, \mathbf{R}^m)$ has a subsequence (f_{n_k}) such that $(f_{n_k}(z))$ converges in $\mathbf{R}^m$ for every $z \in Z$.

[**Hint**: Try to replicate the proof of Proposition 2.65.]

Exercise 1.95. (*) (This is the Arzelà–Ascoli Theorem.)

If X is a totally bounded metric space and $K \subseteq C_0(X, \mathbf{R}^m)$ is a bounded, closed, and equicontinuous subset, then K is compact.

Exercise 1.96. (*) If X and Y are metric spaces with X compact and $K \subseteq C_0(X, Y)$ is compact, then K is bounded, closed, and equicontinuous.

(This is a converse to the Arzelà–Ascoli Theorem, see Exercise 1.95.)

2. Normed and Hilbert spaces

Norms and inner products

Exercise 2.1. Let $(V, \|\cdot\|)$ be a normed vector space. Prove that the norm function $\|\cdot\|: V \longrightarrow \mathbf{R}_{\geqslant 0}$ is uniformly continuous.

Exercise 2.2. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V are equivalent if and only if they define the same topology on V.

Exercise 2.3. Let $(V, \|\cdot\|)$ be a normed space and let S, T be subsets of V and $\alpha \in \mathbf{F}$. Prove that

(a) If S and T are bounded, so are $S+T$ and αS.

(b) If S and T are totally bounded, so are $S+T$ and αS.

(c) If S and T are compact, so are $S+T$ and αS.

Exercise 2.4. Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Prove that the inner product is a continuous function.

Exercise 2.5. If $(V, \langle\cdot,\cdot\rangle)$ is an inner product space then

$$4\langle v, w\rangle = \begin{cases} \|v+w\|^2 - \|v-w\|^2 & \text{if } \mathbf{F} = \mathbf{R} \\ \|v+w\|^2 - \|v-w\|^2 + i\|v+iw\|^2 - i\|v-iw\|^2 & \text{if } \mathbf{F} = \mathbf{C}. \end{cases}$$

[**Hint**: Look at the proof of Proposition 3.10.]

Exercise 2.6. (*) Suppose $(V, \|\cdot\|)$ is a normed space over $\mathbf{F} = \mathbf{C}$ such that

$$\|v+w\|^2 + \|v-w\|^2 = 2\big(\|v\|^2 + \|w\|^2\big) \qquad \text{for all } v, w \in V.$$

Define $[\cdot,\cdot]: V \times V \longrightarrow \mathbf{R}$ by

$$4[v, w] := \|v+w\|^2 - \|v-w\|^2.$$

(a) Prove that, for all $v, w \in V$, we have

$$[w, v] = [v, w] \tag{2.1}$$
$$[iv, iw] = [v, w] \tag{2.2}$$
$$[iv, w] = -[v, iw]. \tag{2.3}$$

(b) Prove that

$$[2u, w] + [2v, w] = 2[u+v, w] \quad \text{for all } u, v, w \in V. \tag{2.4}$$

Conclude that

$$[2v, w] = 2[v, w] \quad \text{for all } v, w \in V, \tag{2.5}$$

and then that

$$[u, w] + [v, w] = [u+v, w] \quad \text{for all } u, v, w \in V. \tag{2.6}$$

(c) Prove that, for all $v, w \in V$, we have

$$[nv, w] = n[v, w] \quad \text{for all } n \in \mathbf{N} \tag{2.7}$$
$$[nv, w] = n[v, w] \quad \text{for all } n \in \mathbf{Z} \tag{2.8}$$
$$[qv, w] = q[v, w] \quad \text{for all } q \in \mathbf{Q} \tag{2.9}$$
$$[xv, w] = x[v, w] \quad \text{for all } x \in \mathbf{R}. \tag{2.10}$$

(d) Prove that

$$[v, v] \geqslant 0 \quad \text{for all } v \in V \quad \text{and } [v, v] = 0 \quad \text{if and only if } v = 0.$$

(e) Show that parts (a)–(d) imply that the function $\langle\cdot,\cdot\rangle\colon V \times V \longrightarrow \mathbf{C}$ given by

$$4\langle v, w\rangle := 4[v, w] + 4i[v, iw] = \|v + w\|^2 - \|v - w\|^2 + i\|v + iw\|^2 - i\|v - iw\|^2$$

is an inner product on V with associated norm $\|\cdot\|$.

Exercise 2.7. Let $(V, \|\cdot\|)$ be a normed space and let $S \subseteq V$ be a subset. Prove that the closure $\overline{\mathrm{Span}(S)}$ of the span of S is the smallest closed subspace of V that contains S.

Exercise 2.8. Let v be a non-zero vector in a normed vector space V. Prove that the one-dimensional subspace $\mathbf{F}v := \mathrm{Span}(v)$ of V is isometric to $\mathbf{F}$.

Exercise 2.9. Let W be a finite-dimensional subspace of a normed vector space V. Prove that W is a closed subset of V.

Exercise 2.10. Prove that equivalence of norms is an equivalence relation.

Exercise 2.11. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on a vector space V.

(a) Prove that the identity function $\mathrm{id}_V\colon (V, \|\cdot\|_1) \longrightarrow (V, \|\cdot\|_2)$ is uniformly continuous.

(b) Prove that $(V, \|\cdot\|_1)$ is Banach if and only if $(V, \|\cdot\|_2)$ is Banach.

Exercise 2.12. Let $(V, \langle\cdot,\cdot\rangle)$ be a complex inner product space and let $T\colon V \longrightarrow V$ be a linear operator. Show that $T = 0$ if and only if $\langle Tv, v\rangle = 0$ for every vector v in V.
Is this true for real inner product spaces?

Exercise 2.13. Give an example of a series that converges but does not converge absolutely.

Bounded linear functions

Exercise 2.14. Let V, W be inner product spaces and let $f \in L(V, W)$. Prove that

$$\|f\| = \sup_{\|v\|_V = \|w\|_W = 1} \left|\langle f(v), w\rangle_W\right|.$$

[**Hint**: Use Tutorial Question 9.1.]

Exercise 2.15. Let $V = \mathbf{R}^2$ viewed as a normed space with the Euclidean norm. Compute the norm of each of the following elements $M \in L(V)$ directly from the description of the operator norm:

$$\|M\| = \sup_{\|v\|=1} \|M(v)\|.$$

(a) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$;

(b) $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

(c) $C = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ for $a, b \in \mathbf{R}$.

Exercise 2.16. Let V, W be normed spaces, with V Banach, and let $f \in L(V, W)$. Suppose that there exists a constant $c > 0$ such that

$$\|f(v)\|_W \geqslant c\,\|v\|_V \qquad \text{for all } v \in V.$$

Then $\operatorname{im}(f)$ is a closed subspace of W.

Exercise 2.17. Prove that all linear transformations between finite-dimensional normed vector spaces are continuous.

Exercise 2.18. Let $f_1 \colon V \longrightarrow W_1$ and $f_2 \colon V \longrightarrow W_2$ be two continuous linear transformations between normed vector spaces. Prove that the function $f \colon V \longrightarrow W_1 \times W_2$ defined by $f(v) = \big(f_1(v), f_2(v)\big)$ is a continuous linear transformation.

Exercise 2.19. If $f \in L(V, W)$ with V, W normed spaces, and the series

$$\sum_{n=1}^{\infty} \alpha_n v_n, \qquad \alpha_n \in \mathbf{F}, v_n \in V,$$

converges in V, then the series

$$\sum_{n=1}^{\infty} \alpha_n f(v_n)$$

converges in W to the limit

$$f\left(\sum_{n=1}^{\infty} \alpha_n v_n\right).$$

Convexity

Exercise 2.20. Any interval $I \subseteq \mathbf{R}$ is convex.

Exercise 2.21. Prove that, if $\big(V, \|\cdot\|\big)$ is a normed space, then $f \colon V \longrightarrow \mathbf{R}$ given by $f(v) = \|v\|$ is a convex function.

Exercise 2.22. (*) Let $I \subseteq \mathbf{R}$ be an interval and let $f \colon I \longrightarrow \mathbf{R}$ be a twice-differentiable function.

The aim of this Exercise is to check the familiar calculus fact: f is convex if and only if $f''(x) \geqslant 0$ for all $x \in I$.

It was heavily inspired by Alexander Nagel's Wisconsin notes [1]:

https://people.math.wisc.edu/~ajnagel/convexity.pdf

(a) For any $s, t \in I$ with $s < t$, define the linear function $L_{s,t} \colon [s, t] \longrightarrow \mathbf{R}$ by

$$L_{s,t}(x) = f(s) + \left(\frac{x - s}{t - s}\right)\big(f(t) - f(s)\big).$$

Convince yourself that this is the equation of the secant line joining $\big(s, f(s)\big)$ to $\big(t, f(t)\big)$.

Prove that f is convex on I if any only if

$$f(x) \leqslant L_{s,t}(x) \qquad \text{for all } s, t \in I \text{ such that } s < t \text{ and all } s \leqslant x \leqslant t.$$

(b) Check that for all $s, t \in I$ such that $s < t$ we have

$$L_{s,t}(x) - f(x) = \frac{x-s}{t-s}\big(f(t) - f(x)\big) - \frac{t-x}{t-s}\big(f(x) - f(s)\big).$$

(c) Use the Mean Value Theorem for f twice to prove that there exist ξ, ζ with $x < \xi < t$ and $s < \zeta < x$ such that

$$L_{s,t}(x) - f(x) = \frac{(t-x)(x-s)}{t-s}\big(f'(\xi) - f'(\zeta)\big).$$

(d) Use the Mean Value Theorem once more to conclude that if $f''(x) \geqslant 0$ for all $x \in I$, then f is convex on I.

(e) Now we prove the converse. From this point on, assume that $f\colon I \longrightarrow \mathbf{R}$ is twice-differentiable and convex, and let $s, t \in I^\circ$.

1. Show that if $s < x < t$ then

$$\frac{f(x) - f(s)}{x-s} \leqslant \frac{f(t) - f(x)}{t-x}.$$

2. Conclude that if $s < x_1 < x_2 < t$ then

$$\frac{f(x_1) - f(s)}{x_1 - s} \leqslant \frac{f(t) - f(x_2)}{t - x_2}.$$

3. Conclude that if $s < t$ then $f'(s) \leqslant f'(t)$, and finally that $f''(x) \geqslant 0$ on I.

Exercise 2.23. (a) Prove that the function $\exp\colon \mathbf{R} \longrightarrow \mathbf{R}$ given by $\exp(x) = e^x$ is convex.

(b) Prove that for any $a, b \geqslant 0$ such that $a + b = 1$ we have $x^a y^b \leqslant ax + by$.

Exercise 2.24. Prove that for any $p \geqslant 1$ and $x, y \geqslant 0$ we have

$$x^p + y^p \leqslant (x+y)^p.$$

Exercise 2.25. Let $p \geqslant 1$, $q > 0$, $x, y \geqslant 0$, and $a, b \geqslant 0$ such that $a + b = 1$. Prove that

$$\begin{aligned}\min\{x, y\} &\leqslant \big(ax^{-q} + by^{-q}\big)^{-1/q} \\ &\leqslant x^a y^b \\ &\leqslant \big(ax^{1/p} + by^{1/p}\big)^p \\ &\leqslant ax + by \\ &\leqslant \big(ax^p + by^p\big)^{1/p} \\ &\leqslant \max\{x, y\}.\end{aligned}$$

Exercise 2.26. Let $\big(V, \|\cdot\|\big)$ be a normed space and take $r, s > 0$, $u, v \in V$, $\alpha \in \mathbf{F}^\times$. Show that

(a) $\mathbf{B}_r(u + v) = \mathbf{B}_r(u) + \{v\}$;

(b) $\alpha\,\mathbf{B}_1(0) = \mathbf{B}_{|\alpha|}(0)$;

(c) $\mathbf{B}_r(v) = r\mathbf{B}_1(0) + \{v\}$;

(d) $r\mathbf{B}_1(0) + s\mathbf{B}_1(0) = (r+s)\mathbf{B}_1(0)$;

(e) $\mathbf{B}_r(u) + \mathbf{B}_s(v) = \mathbf{B}_{r+s}(u+v)$;

(f) $\mathbf{B}_1(0)$ is a convex subset of V;

(g) any open ball in V is convex.

Exercise 2.27. Let $f\colon V \longrightarrow W$ is a linear transformation between vector spaces.

(a) If U is a subspace of V, then its image $f(U)$ is a subspace of W.

(b) If U is a subspace of W, then its preimage $f^{-1}(U)$ is a subspace of V.

(c) If S is a convex subset of V, then its image $f(S)$ is a convex subset of W.

(d) If S is a convex subset of W, then its preimage $f^{-1}(S)$ is a convex subset of V.

Sequence spaces

Exercise 2.28. For any $n \in \mathbf{N}$, give a linear isometry $\mathbf{F}^n \longrightarrow \ell^2$. (Take the Euclidean norm on $\mathbf{F}^n$.)

Exercise 2.29. Consider the map $f\colon \ell^1 \longrightarrow \mathbf{F}^{\mathbf{N}}$ given by

$$f\big((a_n)\big) = \left(\frac{a_n}{n}\right).$$

(a) Prove that f maps to ℓ^1 and $f\colon \ell^1 \longrightarrow \ell^1$ is linear, continuous, and injective.

(b) Prove that the image W of f is not closed in ℓ^1.

Exercise 2.30. Prove that the norms on the sequence spaces ℓ^∞ and ℓ^p for $p \neq 2$ cannot defined by inner products.

Exercise 2.31. Prove directly that any Cauchy sequence in ℓ^∞ converges, so that ℓ^∞ is a Banach space.

Exercise 2.32. Consider the subset $c_0 \subseteq \mathbf{F}^{\mathbf{N}}$ of all sequences with limit 0:

$$c_0 = \big\{(a_n) \in \mathbf{F}^{\mathbf{N}} : (a_n) \longrightarrow 0\big\}.$$

(a) Prove that c_0 is a closed subspace of ℓ^∞.

(b) Conclude that c_0 is a Banach space.

(c) Prove that c_0 is separable.

Exercise 2.33. Consider the space ℓ^∞ of bounded sequences.

(a) Let $S \subseteq \ell^\infty$ be the subset of sequences (a_n) such that $a_n \in \{0,1\}$ for all $n \in \mathbf{N}$. Prove that S is an uncountable set.

[**Hint**: Mimic Cantor's diagonal argument.]

(b) Use S to construct an uncountable set T of disjoint open balls in ℓ^∞.

(c) Conclude that ℓ^∞ is not separable.

Exercise 2.34. Consider the subset c of $\mathbf{F}^{\mathbf{N}}$ consisting of all convergent sequences (with any limit).

(a) Convince yourself that c is a vector subspace of ℓ^∞.

(b) Prove that $\lim\colon c \longrightarrow \mathbf{F}$ given by

$$(a_n) \longmapsto \lim_{n\longrightarrow\infty}(a_n)$$

is a continuous surjective linear map.

(c) Prove that the formula

$$J\big((a_n)\big) = R\big((a_n)\big) - \Big(\lim_{n\longrightarrow\infty} a_n\Big)(1,1,\dots)$$

defines a linear homeomorphism $J\colon c \longrightarrow c_0$. (Here R denotes the right shift map.)

(d) Conclude that c is Banach.

[**Hint**: Exercise 2.32 should come in handy here and in the following part.]

(e) Show that c is separable and find a Schauder basis for c.

Exercise 2.35. Consider the maps $H_{\text{even}}, H_{\text{odd}}\colon \mathbf{F}^{\mathbf{N}} \longrightarrow \mathbf{F}^{\mathbf{N}}$ defined by

$$H_{\text{even}}\big((a_n)\big) = \big(a_{2n}\big), \qquad H_{\text{odd}}\big((a_n)\big) = \big(a_{2n-1}\big)$$

and construct $f\colon \mathbf{F}^{\mathbf{N}} \longrightarrow \mathbf{F}^{\mathbf{N}} \times \mathbf{F}^{\mathbf{N}}$ as

$$f(a) = \big(H_{\text{even}}(a), H_{\text{odd}}(a)\big).$$

(a) Prove that the restriction of H_{even} and H_{odd} to ℓ^p gives continuous linear functions $H_{\text{even}}, H_{\text{odd}}\colon \ell^p \longrightarrow \ell^p$ for all $p \in \mathbf{R}_{\geqslant 1}$ and for $p = \infty$.

(b) Prove that f is an invertible linear map.

In the next two parts, recall that on the product $V \times W$ of two normed spaces we can work with the norm given by

$$\|(v,w)\| := \|v\|_V + \|w\|_W.$$

(c) Take $p = 1$ and show that the restriction $f\colon \ell^1 \longrightarrow \ell^1 \times \ell^1$ is a bijective linear isometry.

(Recall that we can work with the norm on $\ell^1 \times \ell^1$ given by

$$\|(x,y)\| := \|x\|_{\ell^1} + \|y\|_{\ell^1}.)$$

(d) Show that the statement from part (c) does not hold for the space ℓ^∞; prove the strongest statement that you can for ℓ^∞.

Dual spaces

Exercise 2.36. (*) Let U, V, W be normed spaces over $\mathbf{F}$ and let $\beta\colon U \times V \longrightarrow W$ be a bilinear map.

We say that β is *bounded* if there exists $c > 0$ such that

$$\|\beta(u,v)\|_W \leqslant c\,\|u\|_U\,\|v\|_V \qquad \text{for all } u \in U, v \in V.$$

Prove that β is continuous at $(0,0)$ if and only if β is bounded if and only if β is continuous on $U \times V$.

Exercise 2.37. (*) Let U, V, W be nonzero normed spaces over $\mathbf{F}$ and let $\beta\colon U \times V \longrightarrow W$ be a nonzero bilinear map. Then β is **not** uniformly continuous.

Exercise 2.38. (*) Let U, V, W be normed spaces.

Define the norm of a continuous bilinear map $\beta\colon U \times V \longrightarrow W$, and show that it is a norm on the vector space $\mathrm{Bil}(U, V; W)$ of continuous bilinear maps $U \times V \longrightarrow W$.

[**Hint**: Have a look at Exercise 2.36 to remember what it says.]

Exercise 2.39. (*) Let U, V, W be normed spaces over $\mathbf{F}$.

Suppose $\beta\colon U \times V \longrightarrow W$ is a continuous bilinear map.

Consider the linear function $\beta_U\colon U \longrightarrow \mathrm{Hom}(V, W)$ given by $\beta_U(u) = f_u$, where

$$f_u\colon V \longrightarrow W \qquad \text{is defined by } f_u(v) = \beta(u,v).$$

(a) Prove that for any $u \in U$, $f_u \in L(V, W)$, in other words f_u is continuous.

(b) By part (a) we can think of β_U as a function $U \longrightarrow L(V, W)$. Prove that $\beta_U\colon U \longrightarrow L(V, W)$ is continuous.

Exercise 2.40. In Theorem 3.29 we saw that the function

$$\beta\colon \ell^\infty \times \ell^1 \longrightarrow \mathbf{F} \qquad \text{defined by } \beta(u,v) \longmapsto \sum_{n=1}^{\infty} u_n v_n$$

is a continuous bilinear map.

Show that there is a continuous linear function $\ell^\infty \longrightarrow (\ell^1)^\vee$ that is a bijective isometry.

Conclude that ℓ^∞ is a Banach space.

Exercise 2.41. Flip the factors in Exercise 2.40:

In Theorem 3.29 we saw that the function

$$\ell^1 \times \ell^\infty \longrightarrow \mathbf{F} \qquad \text{defined by } (u,v) \longmapsto \sum_{n=1}^{\infty} u_n v_n$$

is a continuous bilinear map.

(a) Show that there is a continuous linear function $\ell^1 \longrightarrow (c_0)^\vee$ that is a bijective isometry. (Recall that $c_0 \subseteq \ell^\infty$ consists of all convergent sequences with limit 0.)

[**Hint**: It may be useful to prove surjectivity first, and then the isometry property.]

(b) Conclude that ℓ^1 is a Banach space.

(c) Where in your proof for (a) did you make use of the fact that you are working with c_0 rather than ℓ^∞?

Orthogonality

Exercise 2.42. Let $\big(V, \langle\cdot,\cdot\rangle\big)$ be an inner product space and let R, S be subsets of V.

(a) Prove that $S \cap S^{\perp} = 0$.

(b) Prove that if $R \subseteq S$ then $S^{\perp} \subseteq R^{\perp}$.

(c) Prove that $S \subseteq \big(S^{\perp}\big)^{\perp}$.

(d) Prove that $S^{\perp} = \overline{\operatorname{Span}(S)}^{\perp}$.

Exercise 2.43. Let V be a normed space and φ, ψ be commuting projections: $\varphi \circ \psi = \psi \circ \varphi$. Prove that $\varphi \circ \psi$ is a projection with image $\operatorname{im}\varphi \cap \operatorname{im}\psi$.

Exercise 2.44. We explore the Hilbert Projection Theorem when V is a Banach space but not a Hilbert space.

(a) Let $V = \mathbf{R}^2$ with the ℓ^1-norm, that is

$$\big\|(x_1, x_2)\big\| = |x_1| + |x_2|.$$

Let $Y = \mathbf{B}_1(0)$, the closed unit ball around 0. Find two distinct closest points in Y to $x = (-1, 1) \in V$.

(b) Can you find a similar example for $V = \mathbf{R}^2$ with the ℓ^{∞}-norm:

$$\big\|(x_1, x_2)\big\| = \max\big\{|x_1|, |x_2|\big\}?$$

(c) Let V be a normed space and Y a convex subset of V. Fix $x \in V$. Let $Z \subseteq Y$ be the set of all closest points in Y to x. Prove that Z is convex.

Orthonormal bases

Exercise 2.45. In this question, we re-examine the Cauchy–Schwarz inequality in retrospect.
Let u be a vector of norm 1 in an inner product space V. Define $\pi_u \colon V \longrightarrow V$ by

$$\pi_u(v) = \langle v, u\rangle u.$$

(a) Prove that π_u is a linear transformation.

(b) Let v be a vector in V. Prove that $\pi_u(v)$ is orthogonal to $(\mathrm{id}_V - \pi_u)(v)$.

(c) Let v be a vector in V. Prove that $\|\pi_u(v)\| = |\langle v, u\rangle|$.

(d) Prove the **Cauchy–Schwarz inequality**: if v and w are vectors in V, then

$$\big|\langle v, w\rangle\big| \leqslant \|v\|\,\|w\|.$$

(e) Prove that π_u is an orthogonal projection with image $\mathbf{F}u$.

Exercise 2.46. In this question, we generalise the results in Exercise 2.45.

Let $\{u_1, \dots, u_n\}$ be an orthonormal system in an inner product space V and let U be the span of the orthonormal system. Write $\pi_1, \dots, \pi_n$ for the projections $\pi_{u_1}, \dots, \pi_{u_n}$ defined in Exercise 2.45 and put

$$\pi = \pi_1 + \cdots + \pi_n.$$

(a) Prove that

$$\pi_i \circ \pi_j = \begin{cases} \pi_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Prove that π is an orthogonal projection with image U.

(c) Let v be a vector in V. Prove that

$$\left\|\pi(v)\right\|^2 = \sum_{i=1}^{n} \left|\langle v, u_n\rangle\right|^2.$$

(d) Use part (c) to prove the following finite version of the **Bessel's inequality**: if v is a vector in V, then

$$\|v\|^2 \geqslant \sum_{i=1}^{n} \left|\langle v, u_i\rangle\right|^2.$$

Exercise 2.47. (*) Every nonzero Hilbert space H has an orthonormal basis.

[**Hint**: Use Zorn's Lemma (Lemma B.1) and mimic the proof of the existence of bases for arbitrary vector spaces (Theorem 1.2).]

Exercise 2.48. (*) Let $(u_i)_{i \in I}$ be an orthonormal basis of an inner product space V (not necessarily separable) and let v be a vector in V.

(a) Given a a positive integer n, define

$$J_n = \left\{ i \in I \;\middle|\; \left|\langle v, u_i\rangle\right| > \frac{1}{n} \right\}.$$

Prove that J_n has at most $n^2\|v\|^2$ elements.

(b) Put

$$I_v = \left\{ i \in I \;\middle|\; |\langle v, u_i\rangle| \neq 0 \right\}.$$

Prove that I_v is countable.

(c) Choose a bijection $o\colon \mathbf{N} \longrightarrow I_v$. Prove that

$$v = \sum_{n=1}^{\infty} \left\langle v, u_{o(n)}\right\rangle u_{o(n)}.$$

(d) Justify the notation

$$\sum_{i \in I} \langle v, u_i\rangle u_i$$

and convince yourself that

$$v = \sum_{i \in I} \langle v, u_i\rangle u_i.$$

Exercise 2.49. Let H be an infinite-dimensional Hilbert space and let $D := \mathbf{D}_1(0)$ be the closed unit ball in H.

(a) Use Gram–Schmidt orthogonalisation to produce a countable orthonormal set $\{e_1, e_2, \dots\}$.

(b) Conclude that D is not totally bounded.

(c) Conclude that D is not compact.

(d) Conclude that H is not isometric to $\mathbf{F}^n$ for any $n \in \mathbf{N}$.

Adjoint maps

Exercise 2.50. Prove that $f \longmapsto f^*$ is conjugate-linear, in other words that

$$\big(\alpha f + \beta g\big)^* = \overline{\alpha} f^* + \overline{\beta} g^* \qquad \text{for all } \alpha, \beta \in \mathbf{F}, f, g \in L(X,Y).$$

Exercise 2.51. Prove that $f \longmapsto f^*$ is an involution, in other words that

$$\big(f^*\big)^* = f \qquad \text{for all } f \in L(X,Y).$$

Exercise 2.52. Let X, Y, Z be Hilbert spaces.

(a) Prove that $\big(f \circ g\big)^* = g^* \circ f^*$ for all $g \in L(X,Y)$, $f \in L(Y,Z)$.

(b) Prove that $\mathrm{id}_X^* = \mathrm{id}_X$.

Exercise 2.53. Let $f \in L(X,Y)$ with X, Y Hilbert spaces.

(a) Prove that $\|f^*\| = \|f\|$, so $f \longmapsto f^*$ is an isometry.

(b) Prove that $\|f^* \circ f\| = \|f\|^2$.

Exercise 2.54. Let $f\colon X \longrightarrow Y$ be a continuous linear map of Hilbert spaces. Prove that

$$\ker\big(f^*\big) = \big(\operatorname{im} f\big)^\perp \qquad \text{and} \qquad \overline{\operatorname{im}\big(f^*\big)} = \big(\ker f\big)^\perp.$$

Exercise 2.55. Let X be a Hilbert space, $f \in L(X)$, and W a closed subspace of X. Then W is f-invariant if and only if $W^\perp$ is $\big(f^*\big)$-invariant.

Exercise 2.56. Let $f \in L(H)$ with H a Hilbert space. Suppose that f is invertible with continuous inverse. Then the adjoint f^* is invertible and

$$\big(f^*\big)^{-1} = \big(f^{-1}\big)^*.$$

Exercise 2.57. Let $a = (a_n) \in \ell^\infty$ and consider $f\colon \ell^2 \longrightarrow \mathbf{F}^{\mathbf{N}}$ given by

$$f(x) = \big(a_1 x_1, a_2 x_2, \dots, a_n x_n, \dots\big).$$

(a) Prove that the image of f is contained in ℓ^2 and that $f\colon \ell^2 \longrightarrow \ell^2$ is linear and continuous.

(b) Find the norm $\|f\|$.

(c) Show that if $a_n \in \mathbf{R}$ for all $n \in \mathbf{N}$ then f is self-adjoint.

Exercise 2.58. Let H be a Hilbert space and let $\pi\colon H \longrightarrow H$ be a projection. Prove that π is an **orthogonal** projection if and only if it is self-adjoint.

Exercise 2.59. Use Exercise 2.58 to give an alternative proof of Tutorial Question 11.3:

Let H be a Hilbert space and let $\pi\colon H \longrightarrow H$ be a projection. Prove that π is an orthogonal projection if and only if $\mathrm{id}_H - \pi$ is an orthogonal projection.

Exercise 2.60. Let H be a Hilbert space and let α be a scalar. Prove that $\alpha\,\mathrm{id}_H$ is normal (that is, commutes with its adjoint).

Exercise 2.61. Let H be a real Hilbert space. Prove that self-adjoint continuous linear operators on H form a subspace of $L(H)$.

If H is a complex Hilbert space, does the statement still hold? If yes, give a proof for the statement. If no, find a counterexample, and then find and prove a closest statement that holds.

Exercise 2.62. Consider the function $g\colon \ell^2 \longrightarrow \mathbf{F}$ given by

$$g(x) = \sum_{n=1}^{\infty} \frac{x_n}{n^2}.$$

(a) Find $y \in \ell^2$ such that

$$g(x) = \langle x, y\rangle \qquad \text{for all } x \in \ell^2.$$

(b) Deduce that g is linear and bounded and find its norm $\|g\|$.

[**Hint**: You may use without proof the fact that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.]

Exercise 2.63. Consider the normed subspace of ℓ^∞ given by the sequences with only finitely many nonzero terms:

$$c_{00} = \left\{(a_n) \in \mathbf{F}^{\mathbf{N}} : \text{there exists } N \in \mathbf{N} \text{ such that } a_n = 0 \text{ for all } n \geqslant N\right\}.$$

Prove that c_{00} is not complete.

Exercise 2.64. Let c_{00} be the space of sequences with only finitely many nonzero terms (see Exercise 2.63), and consider it as a subspace of ℓ^∞. Prove that c_{00} is separable.

Exercise 2.65. Let c_{00} be the space of sequences with only finitely many nonzero terms (see Exercise 2.63), which is considered as a subspace of ℓ^∞. Let $f\colon c_{00} \longrightarrow \mathbf{F}^{\mathbf{N}}$ be the function defined by $\big(f(v)\big)_n = nv_n$.

(a) Prove that the image of the function f is contained in ℓ^∞.

(b) Let $g\colon c_{00} \longrightarrow \ell^\infty$ be the function defined by $g(v) = f(v)$. Prove that g is not continuous.

(c) Prove that there exists a discontinuous linear transformation from ℓ^∞ to itself.

In this part, you can use the following fact:

Let V and W be F-vector spaces. If S is a subspace of V and if $\phi\colon S \longrightarrow W$ is a linear transformation, then there exists a linear transformation $\tilde{\phi}\colon V \longrightarrow W$ such that $\phi = \tilde{\phi}|_S$.

Compact operators

Exercise 2.66. Let V and W be normed spaces. If $f\colon V \longrightarrow W$ is a bounded operator such that $f(V)$ is finite-dimensional, then f is compact.

Exercise 2.67. In the realm of normed spaces, the composition of a bounded operator and a compact operator (in either direction) is compact.

Exercise 2.68. Let X and Y be Hilbert spaces. Let (f_n) be a convergent sequence in $L(X,Y)$ and let $f = \lim(f_n)$. Prove that $f^* = \lim(f_n^*)$.

Exercise 2.69. Try to mimic the approach in Example 3.62 with the sequence of functions $f_n\colon \ell^2 \longrightarrow \ell^2$ given by

$$f_n(x)_j = \begin{cases} x_j & \text{if } j \leqslant n, \\ 0 & \text{if } j > n. \end{cases}$$

What happens?

Exercise 2.70. Recall the right shift operator $R\colon \ell^2 \longrightarrow \ell^2$

$$R(a_1, a_2, \dots) = (0, a_1, a_2, \dots).$$

(a) Prove that R has no complex eigenvalues.

(b) Is R a compact map?

A. Appendix: Prerequisites

Equivalence relations

Exercise A.1. Let A, B be sets and $f\colon A \longrightarrow B$ a function. For $x, y \in A$, define $x \sim y$ if $f(x) = f(y)$. Show that this satisfies the properties of an equivalence relation on A.

Exercise A.2. Let $\sim$ be an equivalence relation on a set A and let $\pi\colon A \longrightarrow A/\sim$ be the quotient map.

Under what circumstances (if any) is π a bijection?

Exercise A.3. Let $A = \mathbf{N} \times \mathbf{N}$ and define $(a, b) \sim (c, d)$ if $a + d = b + c$.

(a) Show that this satisfies the conditions of an equivalence relation on A.

(b) Construct a bijective function $(A/\sim) \longrightarrow \mathbf{Z}$.

(Don't forget to prove that your function is well-defined, and that it is bijective.)

Exercise A.4. Let V be a vector space. An endomorphism (aka linear transformation from V to itself) $n\colon V \longrightarrow V$ is *nilpotent* if there exists $k \in \mathbf{Z}_{\geqslant 1}$ such that n^k is the constant zero map $V \longrightarrow V$.

An endomorphism $u\colon V \longrightarrow V$ is *unipotent* if $u - \mathrm{id}_V$ is nilpotent.

Now fix p prime, and let V be the vector space over $\mathbf{F}_p$ consisting of set maps $\mathbf{N} \longrightarrow \mathbf{F}_p$. Given two endomorphisms $f, g\colon V \longrightarrow V$, write $f \sim g$ if there exists a unipotent endomorphism u such that $f = u \circ g$.

(a) Prove that $\sim$ is reflexive.

(b) Prove that $\sim$ is symmetric.

(c) Give an example to show that $\sim$ is **not** transitive, and thus not an equivalence relation.

Exercise A.5. Let V be a vector space over $\mathbf{F}$, and let $W \subseteq V$ be a subspace. For $v, v' \in V$, write $v \sim v'$ if $v - v' \in W$.

(a) Prove that $\sim$ is an equivalence relation.

(b) Prove that the operations $[v] + [v'] := [v + v']$ and $\lambda[v] := [\lambda v]$ are well-defined. This proves that $V/\sim$ has the structure of a vector space over $\mathbf{F}$. We call $V/\sim$ a *quotient space*, and write it as V/W.

(c) Let U be a vector space and $f\colon V \longrightarrow U$ a linear transformation such that $f(w) = 0$ for all $w \in W$. Prove there exists a unique linear transformation $g\colon V/W \longrightarrow U$ such that $f = g \circ \pi$, where $\pi\colon V \longrightarrow V/W$ is the quotient map.

(Un)countability

Exercise A.6. (*) Fix a set Ω and let X be the set of all subsets of Ω. For any $S, T \in X$, write $S \sim T$ if S has the same cardinality as T.

Show that $\sim$ is an equivalence relation on X.

Exercise A.7. Let $f\colon X \longrightarrow Y$ be a function, with X a countable set. Then $\mathrm{im}(f)$ is finite or countable.

[**Hint**: Reduce to the case $f\colon \mathbf{N} \longrightarrow Y$ is surjective; construct a right inverse $g\colon Y \longrightarrow \mathbf{N}$, which has to be injective, of f.]

Exercise A.8. Show that the union S of any countable collection of countable sets is a countable set.

[**Hint**: Construct a surjective function $\mathbf{N} \times \mathbf{N} \longrightarrow S$.]

Exercise A.9. (*) Let W be a $\mathbf{Q}$-vector space with a countable basis B. Show that W is a countable set.

[**Hint**: Use Exercise A.8.]

Conclude that $\mathbf{R}$ does not have a countable basis as a vector space over $\mathbf{Q}$.

Linear algebra

Exercise A.10. Let V be a vector space over $\mathbf{F}$. Prove that $\mathrm{End}(V) := \mathrm{Hom}(V, V)$ is an associative unital $\mathbf{F}$-algebra under composition of functions.

Exercise A.11. Let V, W be vector spaces over $\mathbf{F}$ and let B be a basis of V. Suppose $g\colon B \longrightarrow W$ is a function, and let $f\colon V \longrightarrow W$ be its extension to V by linearity.

Prove that

(a) f is injective if and only if $g(B)$ is linearly independent in W;

(b) f is surjective if and only if $g(B)$ spans W;

(c) f is bijective if and only if $g(B)$ is a basis for W.

Exercise A.12. If S and T are subspaces of a vector space V with field of scalars $\mathbf{F}$, then so are $S + T$ and αS for any $\alpha \in \mathbf{F}$.

Exercise A.13. A *complex quadratic form* with real coefficients is a map $f\colon \mathbf{C}^n \longrightarrow \mathbf{C}$ given by

$$f(\mathbf{x}) = \sum_{1 \leqslant i,j \leqslant n} a_{ij} x_i x_j, \qquad a_{ij} \in \mathbf{R}.$$

Use the Spectral Theorem for $\mathbf{C}^n$ to prove that there exist linear maps $g_1, \dots, g_n\colon \mathbf{C}^n \longrightarrow \mathbf{C}$ and constants $b_1, \dots, b_n \in \mathbf{R}$ such that

$$f\big(g_1(\mathbf{x}), \dots, g_n(\mathbf{x})\big) = b_1 g_1(\mathbf{x})^2 + \dots + b_n g_n(\mathbf{x})^2.$$

Uniform continuity and uniform convergence

Exercise A.14. Let $f_1, f_2, \dots$ be a sequence of continuous functions $\mathbf{R} \longrightarrow \mathbf{R}$ that are **not** uniformly continuous, and that converge pointwise to $f\colon \mathbf{R} \longrightarrow \mathbf{R}$.

(a) Give an example to show that f **can** be uniformly continuous.

(b) If the convergence $f_n \longrightarrow f$ is uniform, prove that f **cannot** be uniformly continuous.

Exercise A.15. Let $f_1, f_2, \dots$ be a sequence of continuous functions $\mathbf{R} \longrightarrow \mathbf{R}$ that converges pointwise to $f \colon \mathbf{R} \longrightarrow \mathbf{R}$.

(a) Give an example to show that f need not be continuous. TODO: this is somewhere in the notes or exercises or tutorials, we should find it and just refer to it.

(b) Suppose that $f_n \longrightarrow f$ uniformly and that every f_n is uniformly continuous. Prove that f is uniformly continuous.

B. Appendix: Miscellaneous

(*) Zorn's Lemma

Exercise B.1. (*) Fix a set Ω and let X be the set of all subsets of Ω. Check that $\subseteq$ is a partial order on X. It is not a total order if Ω has at least two distinct elements.

Exercise B.2. (*) Let $(X, \leqslant)$ be a nonempty finite poset. (This just means that X is a nonempty finite set with a partial order $\leqslant$.) Prove that X has a maximal element.

[**Hint**: You could, for instance, use induction on the number of elements of X.]

Exercise B.3. (*) Prove Theorem 1.2: any vector space V has a basis.

[**Hint**: Let X be the set of all linearly independent subsets of V, partially ordered by inclusion. Prove that X has a maximal element B, and prove that this must also span V.]

Exercise B.4. Let $f\colon X \longrightarrow Y$ be a surjective map of sets. Let

$$P(f) = \{(s_A, A)\colon A \subseteq Y,\ s_A\colon A \longrightarrow X,\ f \circ s_A = \mathrm{id}_A\}.$$

Write $(A, s_A) \leqslant (B, s_B)$ if and only if $A \subseteq B$ and $s_B\big|_A = s_A$.

(a) Prove that $(P(f), \leqslant)$ is a poset.

(b) Prove every nonempty chain in $P(f)$ has an upper bound in $P(f)$.

(c) Deduce that there exists a map $s\colon Y \longrightarrow X$ such that $f \circ s = \mathrm{id}_Y$.

Linear algebra

Exercise B.5. Let $\mathbf{R}^{\mathbf{N}}$ be the set of arbitrary sequences $(x_1, x_2, \dots)$ of elements of $\mathbf{R}$.

This is a vector space under the naturally-defined addition of sequences and multiplication by a scalar.

Let $e_j \in \mathbf{R}^{\mathbf{N}}$ be the sequence whose j-th entry is 1, and all the others are 0. Describe the subspace $\mathrm{Span}\{e_1, e_2, \dots\}$ of $\mathbf{R}^{\mathbf{N}}$. Is the set $\{e_1, e_2, \dots\}$ a basis of $\mathbf{R}^{\mathbf{N}}$?

Exercise B.6. (*) Let $V = \mathbf{R}$ viewed as a vector space over $\mathbf{Q}$.

Let $\alpha \in \mathbf{R}$. Show that the set $T = \{\alpha^n\colon n \in \mathbf{N}\}$ is $\mathbf{Q}$-linearly independent if and only if α is transcendental.

(Note: An element $\alpha \in \mathbf{R}$ is called *algebraic* if there exists a monic polynomial $f \in \mathbf{Q}[x]$ such that $f(\alpha) = 0$. An element $\alpha \in \mathbf{R}$ is called *transcendental* if it is not algebraic.)

Exercise B.7. Let $V = \mathbf{F}[x]$ be the vector space of polynomials in one variable with coefficients in $\mathbf{F}$. Given a scalar $\alpha \in \mathbf{F}$, consider the function $\mathrm{ev}_\alpha\colon V \longrightarrow \mathbf{F}$ given by evaluation at α:

$$\mathrm{ev}_\alpha(f) = f(\alpha).$$

Prove that $\mathrm{ev}_\alpha \in V^\vee$.

Exercise B.8. TODO: this really does not need $W = V$, should just do the general case with bases B for V and C for W, and dual bases $B^\vee$ and $C^\vee$.

In the setup of Proposition B.4, suppose $W = V$ so that $T\colon V \longrightarrow V$ and $T^\vee\colon V^\vee \longrightarrow V^\vee$.

Let M be the matrix representation of T with respect to an ordered basis B of V, and let $M^\vee$ be the matrix representation of $T^\vee$ with respect to the dual basis $B^\vee$.

Express $M^\vee$ in terms of M.

Exercise B.9. Let $v_1, \dots, v_n \in V$. Define $\Gamma\colon V^\vee \longrightarrow \mathbf{F}^n$ by

$$\Gamma(\varphi) = \begin{bmatrix} \varphi(v_1) \\ \vdots \\ \varphi(v_n) \end{bmatrix}.$$

(a) Prove that Γ is a linear transformation.

(b) Prove that Γ is injective if and only if $\{v_1, \dots, v_n\}$ spans V.

(c) Prove that Γ is surjective if and only if $\{v_1, \dots, v_n\}$ is linearly independent.

Exercise B.10. Let $T\colon V \longrightarrow W$ be a linear transformation of finite-dimensional vector spaces over $\mathbf{F}$, and let $T^\vee\colon W^\vee \longrightarrow V^\vee$ be the dual transformation as defined in Proposition B.4.

(a) Prove that if T is surjective, then $T^\vee$ is injective.

(b) Prove that if T is injective, then $T^\vee$ is surjective.

(c) Give an example to show that (b) does not always hold if we relax the condition that V and W are finite-dimensional.

(*) Topological groups

Exercise B.11. (*)

(a) Show that a topological group G is Hausdorff if and only if $\{e\}$ is a closed subset of G.

(b) Show that if G is a Hausdorff topological group then its centre Z is a closed subgroup.

(c) Show that if $f\colon G \longrightarrow H$ is a continuous group homomorphism and H is Hausdorff, then $\ker(f)$ is a closed normal subgroup of G.

Exercise B.12. (*) Let $f\colon G \longrightarrow H$ be a group homomorphism between topological groups. Prove that the following are equivalent:

(a) f is continuous;

(b) f is continuous at some element of G;

(c) f is continuous at the identity element e_G of G.

Exercise B.13. (*)

(a) Let V be a $\mathbf{Q}$-vector space. Prove that every group homomorphism $f\colon \mathbf{Q} \longrightarrow V$ is $\mathbf{Q}$-linear.

(b) What can you say (and prove) about **continuous** group homomorphisms $\mathbf{R} \longrightarrow \mathbf{R}$?

(c) Suppose that a group homomorphism $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ is continuous at some real number. Prove that f is continuous on $\mathbf{R}$, and conclude that f is $\mathbf{R}$-linear.

(d) Let B be a basis for $\mathbf{R}$ as a $\mathbf{Q}$-vector space. (Recall from Exercise B.6 that B is uncountable.) Use two distinct irrational elements of B to construct a $\mathbf{Q}$-linear transformation $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ that is not $\mathbf{R}$-linear.

If you would (and why wouldn't you?), follow the rabbit:

https://en.wikipedia.org/wiki/Cauchy%27s_functional_equation

Exercise B.14. (*) Let G be a topological group and let H be a subgroup of G.

(a) Prove that H is closed if it is open. Does the converse hold?

(b) Prove that H is open if it is closed and has finite index. Does the converse hold?

(c) Suppose G is compact and H is open. Prove that H has finite index.

(d) Is the compactness of G necessary in part (c)?

Exercise B.15. (*) Let S and T be subsets of a topological group G. Define

$$ST = \{st\colon\ s \in S \text{ and } t \in T\}.$$

(a) Suppose S and T are open. Prove that ST is open.

(b) Suppose S and T are connected. Prove that ST is connected.

(c) Suppose S and T are compact. Prove that ST is compact.

(d) Suppose S is compact and T is closed. Prove that ST is closed.

[**Hint**: Use Theorem 2.41 after checking that

$$ST = \pi_2\Big(j^{-1}\big(m^{-1}(T)\big)\Big),$$

where $m\colon G \times G \longrightarrow G$ is the multiplication map of G, j is the inclusion of $S^{-1} \times G$ into $G \times G$, and $\pi_2\colon S^{-1} \times G \longrightarrow G$ is the projection onto the second factor.]

(e) Assuming without proof the fact that $\mathbf{Z} + \pi\mathbf{Z}$ is dense in $\mathbf{R}$, convince yourself that ST need not be closed even if both S and T are.

Exercise B.16. (*) Let V be a normed vector space. Prove that $(V, +)$ is a topological group.

Bibliography

[1] Alexander Nagel. Convexity and the sign of the second derivative. https://people.math.wisc.edu/~ajnagel/convexity.pdf, University of Wisconsin lecture notes, 2010. [Online; accessed 25 August 2023].

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