MAST30026 Assignment 1

Due Friday 22 August at 20:00 on Canvas and Gradescope

Some guidelines:

- Your answers to this assignment can be handwritten (on physical paper and scanned, or on a tablet or other device), or typeset using a system that can produce professional-quality mathematical documents (e.g. LaTeX or Typst, but not Microsoft Word).

  If you are writing by hand, make sure that your writing can appear clearly enough on the document you upload to Gradescope. This is usually achieved by writing legibly with a very readable writing implement.

- Please indicate clearly which question you are writing about at the top of each page. (Ideally, start a new question on a new page.)

  When you upload your document to Gradescope, please mark which pages correspond to which questions.

- The quality of the exposition will be assessed alongside the correctness of the approach.

  There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you submit is a complete explanation.

  "Completeness" of the explanation is somewhat subjective, but: results from the lectures, tutorials, exercises can be used (without having to re-prove them). Make sure you say clearly what result(s) you are using, though.

- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).

- As a large language model, I do not have an opinion about your use of generative AI to complete this assignment.

  Actually... I do have an opinion.

  Whatever resource you tap into, use it in a smart way: know its limitations, and do the work of really understanding what it is that you are submitting. This is true of your mate who is smart but tends to make arithmetic mistakes, of your favourite linear algebra or analysis book that uses completely different notation to ours, or of the chatbot that sounds impressive but hallucinates references or gives you a proof that relies on lots of results we have not seen in the subject (and that's the best case scenario). Do your job: be paranoid, double-check everything, take it apart and put it back together until it makes sense to you.

- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.

- It is possible that not all questions will have the same weight in the assessment.

- No Chegg or anything similar. At all. Please.

This assignment consists of 4 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

1.1. Let $\mathrm{SL}_2(\mathbf{Z})$ denote the group of 2×2 matrices with integer entries and determinant 1. Fix a positive integer N.

(a) Show that the subset

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : c \equiv 0 \pmod{N} \right\}$$

is a subgroup of $\mathrm{SL}_2(\mathbf{Z})$.

(b) Two matrices $A, B \in \mathrm{SL}_2(\mathbf{Z})$ are said to be *N-levelled* if the bottom-left entry of AB^{-1} is divisible by N (in other words, if $AB^{-1} \in \Gamma_0(N)$). Prove that N-levelledness is an equivalence relation.

(c) If $N = p$ is prime, how many N-levelled equivalence classes are there?

1.2. Use the Jordan Normal Form to compute the *matrix exponential*

$$e^M := I + \frac{1}{1!}M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots$$

where

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

1.3. Let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be inner products on $\mathbf{C}^n$, with respective norms $\| \cdot \|_1$ and $\| \cdot \|_2$. Suppose that

$$\|v\|_1 \geqslant \|v\|_2 \quad \text{for all } v \in \mathbf{C}^n.$$

Suppose also that there exists a basis $v_1, \dots, v_n$ of $\mathbf{C}^n$ with the property that

$$\|v_i\|_1 = \|v_i\|_2 \quad \text{for } i = 1, 2, \dots, n.$$

Use the Spectral Theorem for $\mathbf{C}^n$ to prove that $\| \cdot \|_1 = \| \cdot \|_2$.

1.4.

(a) Let $f : X \longrightarrow \mathbf{R}$ be a function with domain $X \subseteq \mathbf{R}$. Prove that f **is not** uniformly continuous on X if and only if there exist $\varepsilon > 0$ and sequences (x_n) and (y_n) in X such that

$$\lim_{n \longrightarrow \infty} |x_n - y_n| = 0 \qquad \text{but} \qquad |f(x_n) - f(y_n)| \geqslant \varepsilon \quad \text{for all } n \in \mathbf{N}.$$

(b) Use the criterion from the previous part to prove that the function $f : (0,1) \longrightarrow \mathbf{R}$ given by $f(x) = \sin(1/x)$ is not uniformly continuous on $(0,1)$.