MAST30026 Assignment 2

Due Friday 19 September at 20:00 on Canvas and Gradescope

Some guidelines:

- Your answers to this assignment can be handwritten (on physical paper and scanned, or on a tablet or other device), or typeset using a system that can produce professional-quality mathematical documents (e.g. LaTeX or Typst, but not Microsoft Word).

  If you are writing by hand, make sure that your writing can appear clearly enough on the document you upload to Gradescope. This is usually achieved by writing legibly with a very readable writing implement.

- Please indicate clearly which question you are writing about at the top of each page. (Ideally, start a new question on a new page.)

  When you upload your document to Gradescope, please mark which pages correspond to which questions.

- The quality of the exposition will be assessed alongside the correctness of the approach.

  There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you submit is a complete explanation.

  "Completeness" of the explanation is somewhat subjective, but: results from the lectures, tutorials, exercises can be used (without having to re-prove them). Make sure you say clearly what result(s) you are using, though.

- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).

- As a large language model, I do not have an opinion about your use of generative AI to complete this assignment.

  Actually... I do have an opinion.

  Whatever resource you tap into, use it in a smart way: know its limitations, and do the work of really understanding what it is that you are submitting. This is true of your mate who is smart but tends to make arithmetic mistakes, of your favourite linear algebra or analysis book that uses completely different notation to ours, or of the chatbot that sounds impressive but hallucinates references or gives you a proof that relies on lots of results we have not seen in the subject (and that's the best case scenario). Do your job: be paranoid, double-check everything, take it apart and put it back together until it makes sense to you.

- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.

- It is possible that not all questions will have the same weight in the assessment.

- No Chegg or anything similar. At all. Please.

This assignment consists of 6 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

2.1.

(a) Suppose X is a Hausdorff topological space and give $Y \subset X$ the subspace topology. Prove that Y is Hausdorff.

(b) Suppose X_1 and X_2 are Hausdorff topological spaces. Prove that $X_1 \times X_2$ is Hausdorff.

(c) Let X be a Hausdorff topological space and let K be a compact subset of X. Let $\ell \in X \setminus K$. Prove that there exist open sets U, V in X such that $K \subseteq U$, $\ell \in V$, and $U \cap V = \varnothing$.

(d) Let X be a Hausdorff topological space and let K, L be compact subsets of X such that $K \cap L = \varnothing$. Prove that there exist open sets U, V in X such that $K \subseteq U$, $L \subseteq V$, and $U \cap V = \varnothing$.

[**Hint**: Use (c).]

2.2. Let X, Y be topological spaces.

(a) Let $f\colon X \longrightarrow Y$ be a continuous function and let $S \subseteq X$ be given the subspace topology. Let $g := f|_S\colon S \longrightarrow Y$ be the restriction of f to S. Prove that g is a continuous function.

(b) Let $f\colon X \longrightarrow Y$ be a function. Let $\{U_i\colon i \in I\}$ be an open cover of X. For every $i \in I$, let $f_i := f|_{U_i}\colon U_i \longrightarrow Y$ be the restriction of f to U_i.

Prove that f is continuous if and only if f_i is continuous for every $i \in I$.

(c) Suppose you are given an open cover $\{U_i\colon i \in I\}$ of X and, for each $i \in I$, a continuous function $f_i\colon U_i \longrightarrow Y$, such that for all $i, j \in I$ we have

$$f_i(x) = f_j(x) \qquad \text{for all } x \in U_i \cap U_j.$$

Prove that there exists a unique continuous function $f\colon X \longrightarrow Y$ such that for every $i \in I$, $f|_{U_i} = f_i$.

2.3.

(a) Let X be a disconnected topological space and let U, V be non-empty open subsets of X such that $X = U \cup V$ and $U \cap V = \varnothing$.

Prove that if $C \subseteq X$ is connected, then $C \subseteq U$ or $C \subseteq V$.

(b) Let X be a topological space and $C \subseteq X$ a connected subset. Suppose that B satisfies $C \subseteq B \subseteq \overline{C}$. Prove that B is connected.

[**Hint**: Use (a).]

(c) Let X be an infinite set with the cofinite topology. Prove that X is connected.

(d) Suppose X, Y are connected topological spaces and $A \subsetneq X$, $B \subsetneq Y$. Prove that the set $(X \times Y) \setminus (A \times B)$ is connected.

[**Hint**: Use Exercise 1.44.]

2.4. Let X be a topological space.

(a) Prove that, if X is Hausdorff and K and L are compact subsets of X, then $K \cap L$ is compact.

(b) Give an example of a topological space X and a subset K of X such that K is compact but K is not closed in X.

(c) Let $Y = \{0, 1\}$ with the trivial (aka indiscrete) topology, and let $X = Y \times \mathbf{R}$, where $\mathbf{R}$ is given its usual (Euclidean) topology.

Keep in mind the projection map $\pi\colon X = Y \times \mathbf{R} \longrightarrow \mathbf{R}$ and make good use of it in the following.

i. What do the open sets of X look like?

ii. Prove that the topological space X is not Hausdorff.

iii. Define the following two subsets of X:

$$K = \big(\{0\} \times [0, 2)\big) \cup \big(\{1\} \times [2, 3]\big)$$
$$L = \big(\{0\} \times (1, 3]\big) \cup \big(\{1\} \times [0, 1]\big).$$

Prove that K is compact (and convince yourself that your proof can be adapted to show that L is compact).

iv. Prove that $K \cap L$ is not compact.

2.5. Let (X, d) be a metric space and define $d'\colon X \times X \longrightarrow \mathbf{R}$ by

$$d'(x, y) := \min\{1, d(x, y)\}.$$

Note that $d'(x, y) \leqslant 1$ for all $x, y \in X$.

(a) Prove that d' is a metric on X.

(b) Prove that d and d' are topologically equivalent.

(c) Give an example to show that (X, d) and (X, d') are not necessarily isometric.

(d) Prove that any sequence (x_n) is Cauchy in (X, d) if and only if it is Cauchy in (X, d').

(e) Let $x \in X$. Prove that any sequence (x_n) converges to x with respect to d if and only if it converges to x with respect to d'.

(f) Deduce that (X, d) is complete if and only if (X, d') is complete.

2.6.

(a) Let (X, d) be a metric space such that $d(x, y) \leqslant 1$ for all $x, y \in X$. Prove that if $(\widehat{X}, \widehat{d})$ is a completion of (X, d), then $\widehat{d}(\widehat{x}, \widehat{y}) \leqslant 1$ for all $\widehat{x}, \widehat{y} \in \widehat{X}$.

(b) For each $n \in \mathbf{N}$, let (X_n, d_n) be a metric space such that $d_n(x, y) \leqslant 1$ for all $x, y \in X_n$. Consider the product

$$X := \prod_{n=0}^{\infty} X_n.$$

Define $d\colon X \times X \longrightarrow \mathbf{R}$ by

$$d(x, y) := \sum_{n=0}^{\infty} \frac{d_n(x_n, y_n)}{2^n}.$$

i. Prove that (X, d) is a metric space.

ii. For every n, fix a completion $(\widehat{X}_n, \widehat{d}_n)$ of (X_n, d_n) with isometry $\iota_n\colon X_n \longrightarrow \widehat{X}_n$. Define

$$\widetilde{X} := \prod_{n=0}^{\infty} \widehat{X}_n$$

endowed with a metric $\widetilde{d}$ defined in the same way as above:

$$\widetilde{d}(x, y) := \sum_{n=0}^{\infty} \frac{\widehat{d}_n(x_n, y_n)}{2^n}.$$

Check that the function $\iota\colon X \longrightarrow \widetilde{X}$ given by

$$\iota := \prod_{n=0}^{\infty} \iota_n$$

is an isometry, and prove that $(\widetilde{X}, \widetilde{d})$ is a completion of (X, d).