

MAST30026 Assignment 3

Due Friday 17 October at 20:00 on Canvas and Gradescope

Some guidelines:

- Your answers to this assignment can be handwritten (on physical paper and scanned, or on a tablet or other device), or typeset using a system that can produce professional-quality mathematical documents (e.g. $\text{\LaTeX}$ or Typst, but not Microsoft Word).

If you are writing by hand, make sure that your writing can appear clearly enough on the document you upload to Gradescope. This is usually achieved by writing legibly with a very readable writing implement.

- Please indicate clearly which question you are writing about at the top of each page. (Ideally, start a new question on a new page.)

When you upload your document to Gradescope, please mark which pages correspond to which questions.

- The quality of the exposition will be assessed alongside the correctness of the approach. There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you submit is a complete explanation.

“Completeness” of the explanation is somewhat subjective, but: results from the lectures, tutorials, exercises can be used (without having to re-prove them). Make sure you say clearly what result(s) you are using, though.

- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).
- As a large language model, I do not have an opinion about your use of generative AI to complete this assignment.

Actually... I do have an opinion.

Whatever resource you tap into, use it in a smart way: know its limitations, and do the work of really understanding what it is that you are submitting. This is true of your mate who is smart but tends to make arithmetic mistakes, of your favourite linear algebra or analysis book that uses completely different notation to ours, or of the chatbot that sounds impressive but hallucinates references or gives you a proof that relies on lots of results we have not seen in the subject (and that’s the best case scenario). Do your job: be paranoid, double-check everything, take it apart and put it back together until it makes sense to you.

- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.
- It is possible that not all questions will have the same weight in the assessment.
- No Chegg or anything similar. At all. Please.

This assignment consists of 6 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

3.1. Let $D = \mathbf{D}_1(0) \subseteq \mathbf{C}$ be the closed unit disc and $S = \partial D \subseteq \mathbf{C}$ be the unit circle, both equipped with the subspace topology.

- (a) Prove that if two distinct points are removed from D , it is still connected.
- (b) Prove that S is not homeomorphic to D .
- (c) A *simple closed loop* in a topological space X is the image of an injective continuous map $S \rightarrow X$. Prove that there exists no simple closed loop that is dense in D .

3.2. Let V be a Banach space and W a normed space.

- (a) Let $v_0 \in V$ and $r > 0$ and consider the open ball $\mathbf{B}_r(v_0)$. Prove that there exists $\delta > 0$ such that for any $v \in V$ there exists $v' \in \mathbf{B}_r(v_0)$ such that

$$v = \delta \|v\| (v' - v_0).$$

(Note: V does not need to be Banach for this part, just normed.)

- (b) Let $\{f_i : i \in I\} \subseteq L(V, W)$ be a collection of bounded linear operators from V to W . (Careful: I is an arbitrary nonempty set here, it could for instance be uncountable.)

For each $v \in V$, consider the set

$$S_v := \{\|f_i(v)\|_W : i \in I\} \subseteq \mathbf{R}.$$

Consider also the set

$$T := \{\|f_i\| : i \in I\} \subseteq \mathbf{R}.$$

Prove that if S_v is a bounded set for every $v \in V$, then T is a bounded set.

[**Hint:** Apply the Baire Category Theorem to the sequence of subsets C_n of V given by $C_n := \{v \in V : \sup S_v \leq n\}$, then use part (a).]

- (c) Let (g_n) be a sequence of bounded linear operators from V to W . Suppose that for each $v \in V$, the sequence $(g_n(v))$ converges in W .

Prove that there exists a bounded linear operator $g : V \rightarrow W$ such that the sequence (g_n) converges pointwise to g .

3.3. Let V be a normed space over $\mathbf{F}$.

- (a) Show that the intersection of an arbitrary collection of convex subsets of V is convex.
- (b) Show that if C is convex then the closure $\overline{C}$ is convex.
- (c) Prove that if C is convex, then for any $n \in \mathbf{N}$, any $v_1, \dots, v_n \in C$, and any real numbers $\alpha_1, \dots, \alpha_n \geq 0$ such that $\alpha_1 + \dots + \alpha_n = 1$, we have

$$\alpha_1 v_1 + \dots + \alpha_n v_n \in C.$$

3.4. (This is a follow-up to [Assignment Question 3.3.](#))

Let V be a normed space over $\mathbf{F}$. Given a subset A of V , define the *convex hull* of A to be the intersection of all the convex subsets of V that contain A :

$$\text{Conv}(A) := \bigcap_{C \text{ convex}, A \subseteq C} C.$$

- (a) Prove that

$$\text{Conv}(A) = \{\alpha_1 v_1 + \cdots + \alpha_n v_n : n \in \mathbf{N}, v_i \in A, \alpha_i \geq 0, \alpha_1 + \cdots + \alpha_n = 1\}.$$

- (b) Suppose A is an open subset of V . Prove that $\text{Conv}(A)$ is open.

- (c) Let $V = \mathbf{R}^2$ and let $A = (\mathbf{R} \times \{0\}) \cup \{(0, 1)\}$. Show that A is closed but its convex hull $\text{Conv}(A)$ is not closed.

3.5. Let $u = (u_n) \in \ell^a$ and $v = (v_n) \in \ell^b$ where $a, b \geq 2$. Let $c = ab/(a+b)$.

- (a) Let uv be the sequence $(u_n v_n)$. Prove that $uv \in \ell^c$.

[**Hint:** If u^c is the sequence (u_n^c) , note that $u^c \in \ell^p$, where $p = a/c$.]

- (b) Find $u \in \ell^a$ and $v \in \ell^b$ such that $uv \notin \ell^r$ for all $1 \leq r < c$.

3.6.

- (a) Prove that there exists a unique sequence of real numbers $(x_1, x_2, \dots)$ with the property that

$$3x_n = \sin(x_n) + \cos(x_{n+1}) \quad \text{for all } n \in \mathbf{N}.$$

- (b) Give the first five terms in the sequence to three decimals.