MAST30026 Assignment 1

Due Friday 22 August at 20:00 on Canvas and Gradescope

Some guidelines:

- Your answers to this assignment can be handwritten (on physical paper and scanned, or on a tablet or other device), or typeset using a system that can produce professional-quality mathematical documents (e.g. LaTeX or Typst, but not Microsoft Word).

  If you are writing by hand, make sure that your writing can appear clearly enough on the document you upload to Gradescope. This is usually achieved by writing legibly with a very readable writing implement.

- Please indicate clearly which question you are writing about at the top of each page. (Ideally, start a new question on a new page.)

  When you upload your document to Gradescope, please mark which pages correspond to which questions.

- The quality of the exposition will be assessed alongside the correctness of the approach.

  There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you submit is a complete explanation.

  "Completeness" of the explanation is somewhat subjective, but: results from the lectures, tutorials, exercises can be used (without having to re-prove them). Make sure you say clearly what result(s) you are using, though.

- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).

- As a large language model, I do not have an opinion about your use of generative AI to complete this assignment.

  Actually... I do have an opinion.

  Whatever resource you tap into, use it in a smart way: know its limitations, and do the work of really understanding what it is that you are submitting. This is true of your mate who is smart but tends to make arithmetic mistakes, of your favourite linear algebra or analysis book that uses completely different notation to ours, or of the chatbot that sounds impressive but hallucinates references or gives you a proof that relies on lots of results we have not seen in the subject (and that's the best case scenario). Do your job: be paranoid, double-check everything, take it apart and put it back together until it makes sense to you.

- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.

- It is possible that not all questions will have the same weight in the assessment.

- No Chegg or anything similar. At all. Please.

This assignment consists of 4 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

1.1. Let $\mathrm{SL}_2(\mathbf{Z})$ denote the group of 2×2 matrices with integer entries and determinant 1. Fix a positive integer N.

(a) Show that the subset

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : c \equiv 0 \pmod{N} \right\}$$

is a subgroup of $\mathrm{SL}_2(\mathbf{Z})$.

(b) Two matrices $A, B \in \mathrm{SL}_2(\mathbf{Z})$ are said to be *N-levelled* if the bottom-left entry of AB^{-1} is divisible by N (in other words, if $AB^{-1} \in \Gamma_0(N)$). Prove that N-levelledness is an equivalence relation.

(c) If $N = p$ is prime, how many N-levelled equivalence classes are there?

Solution.

(a) Straightforward:

- The identity matrix is clearly in $\Gamma_0(N)$.
- Let $A, B \in \Gamma_0(N)$, so that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

with $c, y \equiv 0 \pmod{N}$. We have

$$AB^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z & -x \\ -y & w \end{pmatrix} = \begin{pmatrix} az - by & -ax + bw \\ cz - dy & -cx + dw \end{pmatrix}.$$

Now since both c and y are divisible by N, so is $cz - dy$, so $AB^{-1} \in \Gamma_0(N)$.

(b)

- reflexivity: for any $A \in \mathrm{SL}_2(\mathbf{Z})$ we have $AA^{-1} = I \in \Gamma_0(N)$.
- symmetry: if $A \sim B$ then (since $\Gamma_0(N)$ is a subgroup of $\mathrm{SL}_2(\mathbf{Z})$) we have

$$AB^{-1} \in \Gamma_0(N) \Longrightarrow BA^{-1} = \left(AB^{-1}\right)^{-1} \in \Gamma_0(N).$$

- transitivity: if $A \sim B$ and $B \sim C$ then

$$AB^{-1}, BC^{-1} \in \Gamma_0(N) \Longrightarrow AC^{-1} = \left(AB^{-1}\right)\left(BC^{-1}\right) \in \Gamma_0(N).$$

(c) We start by noting that

$$[I] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \Gamma_0(p),$$

since $A \in [I] \Leftrightarrow AI^{-1} \in \Gamma_0(p) \Leftrightarrow A \in \Gamma_0(p)$.

Suppose now that we have two matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right)$ that are not in $[I]$, that is $c, y \not\equiv 0 \pmod{p}$. We have $A \sim B$ if and only if $cz - dy \equiv 0 \pmod{p}$ if and only if $z/y \equiv d/c \pmod{p}$. Therefore we get one equivalence class for each value of d/c in $\{0, 1, \dots, p-1\}$, and every equivalence class $\neq [I]$ is obtained in this manner.

We conclude that there are $p + 1$ equivalence classes in total.

This proof also gives a set of $p + 1$ representatives:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -1 \\ 1 & p-1 \end{pmatrix}.$$

□

1.2. Use the Jordan Normal Form to compute the *matrix exponential*

$$e^M := I + \frac{1}{1!}M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots$$

where

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

Solution. The characteristic polynomial of M is easily computed to be

$$\det(xI - M) = (x-3)^3.$$

Since

$$(M - 3I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq 0,$$

the minimal polynomial is $(x-3)^3$, so we have a single Jordan block of size 3:

$$J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = 3I + N \quad \text{with} \quad N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The first column v_1 of P should satisfy $(M-3I)v_1 = 0$, so by solving for $\ker(M-3I)$ we deduce we can pick $v_1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^\mathsf{T}$. The second column v_2 of P should satisfy $(M-3I)v_2 = v_1$, whence we choose $v_2 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^\mathsf{T}$. Finally, the third column of P should satisfy $(M-3I)v_3 = v_2$, whence we choose $v_3 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^\mathsf{T}$. One can check that

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

indeed satisfies $M = PJP^{-1}$. Computing the exponential, we get

$$\begin{aligned} e^M &= e^{PJP^{-1}} \\ &= Pe^J P^{-1} \\ &= Pe^{3I+N}P^{-1} \\ &= Pe^{3I}e^N P^{-1} \\ &= Pe^3\left(I + \frac{1}{1!}N + \frac{1}{2!}N^2\right)P^{-1} && \text{(since } N^3 = 0\text{)} \\ &= e^3 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ &= \frac{e^3}{2} \begin{pmatrix} 0 & 0 & 2 \\ 1 & 2 & 1 \\ -2 & 0 & 4 \end{pmatrix}. \end{aligned}$$

□

1.3. Let $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ be inner products on $\mathbf{C}^n$, with respective norms $\|\cdot\|_1$ and $\|\cdot\|_2$.
Suppose that

$$\|v\|_1 \geqslant \|v\|_2 \quad \text{for all } v \in \mathbf{C}^n.$$

Suppose also that there exists a basis $v_1, \dots, v_n$ of $\mathbf{C}^n$ with the property that

$$\|v_i\|_1 = \|v_i\|_2 \quad \text{for } i = 1, 2, \dots, n.$$

Use the Spectral Theorem for $\mathbf{C}^n$ to prove that $\|\cdot\|_1 = \|\cdot\|_2$.

Solution. (Note: The danger is to try and somehow argue using linearity, but $\|\cdot\|_1 - \|\cdot\|_2$ is not a priori known to be linear.)

We know that $\langle\cdot,\cdot\rangle_1$ is of the form

$$\langle v, w\rangle_1 = \overline{v}^\mathsf{T} A_1 w,$$

where A_1 is a Hermitian matrix that is positive-definite: $\overline{A_1}^\mathsf{T} = A_1$ and $\overline{v}^\mathsf{T} A_1 v \in \mathbf{R}_{>0}$ for all $v \neq 0$. Similarly, $\langle\cdot,\cdot\rangle_2$ is given by a positive-definite Hermitian matrix A_2.

Consider

$$[v, w] := \langle v, w\rangle_1 - \langle v, w\rangle_2 = \overline{v}^\mathsf{T}(A_1 - A_2)w.$$

The matrix $A_1 - A_2$ is Hermitian, hence unitarily diagonalisable by the Spectral Theorem, i.e. $A_1 - A_2 = \overline{U}^\mathsf{T} DU$ with U unitary and $D = \operatorname{diag}(d_1, \dots, d_n)$.

For any nonzero $v \in \mathbf{C}^n$ we have

$$\overline{v}^\mathsf{T}(A_1 - A_2)v = \overline{v}^\mathsf{T} A_1 v - \overline{v}^\mathsf{T} A_2 v = \|v\|_1^2 - \|v\|_2^2 \geqslant 0$$

by the first assumption in the statement. On the other hand

$$\overline{v}^\mathsf{T}(A_1 - A_2)v = \overline{Uv}^\mathsf{T} DUv = d_1|z_1|^2 + \dots d_n|z_n|^2$$

where $Uv = \begin{pmatrix} z_1 & \dots & z_n \end{pmatrix}^\mathsf{T}$, so we conclude that $d_i \geqslant 0$ for $i = 1, \dots, n$.

Then

$$[v_i, v_i] = \overline{Uv_i}^\mathsf{T} D(Uv_i) = 0 \qquad \text{for all } i = 1, \dots, n.$$

Let z_{ij} denote the j^{th} coordinate of Uv_i: $Uv_i = \begin{pmatrix} z_{i1} & \dots & z_{in} \end{pmatrix}^\mathsf{T}$.

Fix j. Since U is invertible,$\{Uv_1, \dots, Uv_n\}$ is linearly independent, so there exists some i for which $z_{ij} \neq 0$. We have

$$d_1|z_{i1}|^2 + \dots + d_n|z_{in}|^2 = 0 \qquad \text{and } d_k \geqslant 0 \text{ for all } k = 1, \dots, n,$$

forcing $d_j = 0$.

As the argument works for all j, we conclude that $D = 0$ and thus $A_1 - A_2 = 0$, so that indeed $\|\cdot\|_1 = \|\cdot\|_2$. □

1.4.

(a) Let $f\colon X \longrightarrow \mathbf{R}$ be a function with domain $X \subseteq \mathbf{R}$. Prove that f **is not** uniformly continuous on X if and only if there exist $\varepsilon > 0$ and sequences (x_n) and (y_n) in X such that

$$\lim_{n\longrightarrow\infty} |x_n - y_n| = 0 \qquad \text{but} \qquad |f(x_n) - f(y_n)| \geqslant \varepsilon \quad \text{for all } n \in \mathbf{N}.$$

(b) Use the criterion from the previous part to prove that the function $f\colon (0,1) \longrightarrow \mathbf{R}$ given by $f(x) = \sin(1/x)$ is not uniformly continuous on $(0,1)$.

Solution.

(a) If the condition given in the statement holds, then consider the given ε and sequences (x_n), (y_n).

Take any $\delta > 0$ and choose $m \in \mathbf{N}$ such that $|x_m - y_m| < \delta$ (this of course is possible because $|x_n - y_n|$ converges to 0). Then $|f(x_m) - f(y_m)| \geqslant \varepsilon$, so this δ does not fulfil the role from the definition of uniform continuity. Since $\delta > 0$ was chosen to be arbitrary, we conclude that no δ works, so f is not uniformly continuous.

Conversely, suppose f is not uniformly continuous. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist $x, y \in X$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geqslant \varepsilon$.

In particular, for any $n \in \mathbf{Z}_{\geqslant 1}$ we can let $\delta = 1/n$ and obtain $x_n, y_n \in X$ such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geqslant \varepsilon$. Varying n, we obtain sequences (x_n) and (y_n) in X such that $|x_n - y_n| \longrightarrow 0$.

(b) Let $x_n := 1/\big(\pi/2 + 2\pi n\big)$, then

$$f(x_n) = \sin\big(\pi/2 + 2\pi n\big) = \sin\big(\pi/2\big) = 1.$$

Similarly, for $y_n := 1/\big(3\pi/2 + 2\pi n\big)$ we have

$$f(y_n) = \sin\big(3\pi/2 + 2\pi n\big) = \sin\big(3\pi/2\big) = -1.$$

Therefore $|f(x_n) - f(y_n)| = 2$ for all n, so we can take $\varepsilon = 2$ and appeal to the criterion in the previous part (since both sequences (x_n) and (y_n) converge to 0, so their difference also converges to 0). □