

MAST30026 Assignment 2

Due Friday 19 September at 20:00 on Canvas and Gradescope

Some guidelines:

- Your answers to this assignment can be handwritten (on physical paper and scanned, or on a tablet or other device), or typeset using a system that can produce professional-quality mathematical documents (e.g. $\text{\LaTeX}$ or Typst, but not Microsoft Word).

If you are writing by hand, make sure that your writing can appear clearly enough on the document you upload to Gradescope. This is usually achieved by writing legibly with a very readable writing implement.

- Please indicate clearly which question you are writing about at the top of each page. (Ideally, start a new question on a new page.)

When you upload your document to Gradescope, please mark which pages correspond to which questions.

- The quality of the exposition will be assessed alongside the correctness of the approach. There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you submit is a complete explanation.

“Completeness” of the explanation is somewhat subjective, but: results from the lectures, tutorials, exercises can be used (without having to re-prove them). Make sure you say clearly what result(s) you are using, though.

- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).
- As a large language model, I do not have an opinion about your use of generative AI to complete this assignment.

Actually... I do have an opinion.

Whatever resource you tap into, use it in a smart way: know its limitations, and do the work of really understanding what it is that you are submitting. This is true of your mate who is smart but tends to make arithmetic mistakes, of your favourite linear algebra or analysis book that uses completely different notation to ours, or of the chatbot that sounds impressive but hallucinates references or gives you a proof that relies on lots of results we have not seen in the subject (and that’s the best case scenario). Do your job: be paranoid, double-check everything, take it apart and put it back together until it makes sense to you.

- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.
- It is possible that not all questions will have the same weight in the assessment.
- No Chegg or anything similar. At all. Please.

This assignment consists of 6 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

2.1.

- (a) Suppose X is a Hausdorff topological space and give $Y \subset X$ the subspace topology. Prove that Y is Hausdorff.
- (b) Suppose X_1 and X_2 are Hausdorff topological spaces. Prove that $X_1 \times X_2$ is Hausdorff.
- (c) Let X be a Hausdorff topological space and let K be a compact subset of X . Let $\ell \in X \setminus K$. Prove that there exist open sets U, V in X such that $K \subseteq U$, $\ell \in V$, and $U \cap V = \emptyset$.
- (d) Let X be a Hausdorff topological space and let K, L be compact subsets of X such that $K \cap L = \emptyset$. Prove that there exist open sets U, V in X such that $K \subseteq U$, $L \subseteq V$, and $U \cap V = \emptyset$.

[**Hint:** Use (c).]

Solution.

- (a) Let $y_1 \neq y_2 \in Y$. Then $y_1 \neq y_2 \in X$ and X is Hausdorff so there exist open subsets U_1 and U_2 in X such that $y_1 \in U_1$, $y_2 \in U_2$, and $U_1 \cap U_2 = \emptyset$. Let $V_1 = U_1 \cap Y$, $V_2 = U_2 \cap Y$, then V_1 and V_2 are open in Y such that $y_1 \in V_1$, $y_2 \in V_2$, and $V_1 \cap V_2 \subseteq U_1 \cap U_2 = \emptyset$.
- (b) Let $(y_1, y_2) \neq (z_1, z_2) \in X_1 \times X_2$. Either $y_1 \neq z_1$ or $y_1 = z_1$.
 - If $y_1 = z_1$ then $y_2 \neq z_2 \in X_2$. Since X_2 is Hausdorff, there exist open subsets U and V of X_2 with $y_2 \in U$, $z_2 \in V$, and $U \cap V = \emptyset$. Then $X_1 \times U$ and $X_1 \times V$ are open subsets of $X_1 \times X_2$ such that $(y_1, y_2) \in X_1 \times U$, $(z_1, z_2) \in X_1 \times V$, and

$$(X_1 \times U) \cap (X_1 \times V) = X_1 \times (U \cap V) = X_1 \times \emptyset = \emptyset.$$

- If $y_1 \neq z_1 \in X_1$, we proceed in the same manner but within X_1 instead of X_2 .
- (c) For $k \in K$, we have $k \neq \ell$ and X is Hausdorff so there exist open subsets U_k and V_k of X such that $k \in U_k$, $\ell \in V_k$, and $U_k \cap V_k = \emptyset$. This gives us an open cover of K :

$$K \subseteq \bigcup_{k \in K} U_k.$$

Since K is compact, this has a finite subcover

$$K \subseteq \bigcup_{i=1}^n U_{k_i} =: U.$$

Letting

$$V := \bigcap_{i=1}^n V_{k_i},$$

we get open subsets U and V of X with the desired property.

- (d) Fix $\ell \in L$. By part (c), there exist open subsets U_ℓ and V_ℓ of X such that $K \subseteq U_\ell$, $\ell \in V_\ell$, and $U_\ell \cap V_\ell = \emptyset$. This gives us an open cover of L :

$$L \subseteq \bigcup_{\ell \in L} V_\ell.$$

Since L is compact, this has a finite subcover

$$L \subseteq \bigcup_{j=1}^m V_{\ell_j} =: V.$$

Letting

$$U := \bigcap_{j=1}^m U_{\ell_j},$$

we get open subsets U and V of X with the desired property.

□

2.2. Let X, Y be topological spaces.

- (a) Let $f: X \rightarrow Y$ be a continuous function and let $S \subseteq X$ be given the subspace topology. Let $g := f|_S: S \rightarrow Y$ be the restriction of f to S . Prove that g is a continuous function.
- (b) Let $f: X \rightarrow Y$ be a function. Let $\{U_i: i \in I\}$ be an open cover of X . For every $i \in I$, let $f_i := f|_{U_i}: U_i \rightarrow Y$ be the restriction of f to U_i .
Prove that f is continuous if and only if f_i is continuous for every $i \in I$.
- (c) Suppose you are given an open cover $\{U_i: i \in I\}$ of X and, for each $i \in I$, a continuous function $f_i: U_i \rightarrow Y$, such that for all $i, j \in I$ we have

$$f_i(x) = f_j(x) \quad \text{for all } x \in U_i \cap U_j.$$

Prove that there exists a unique continuous function $f: X \rightarrow Y$ such that for every $i \in I$, $f|_{U_i} = f_i$.

Solution.

- (a) Let V be an open subset of Y , then $f^{-1}(V)$ is an open subset of X (since f is continuous). Therefore (by the definition of the subspace topology on S) we have $g^{-1}(V) = f^{-1}(V) \cap S$ is an open subset of S . We conclude that $g: S \rightarrow Y$ is continuous.
- (b) By part (a), if f is continuous then for every $i \in I$ the restriction $f_i: U_i \rightarrow Y$ is continuous.

Conversely, suppose f_i is continuous for every $i \in I$. I claim that for every subset $V \subseteq Y$ we have

$$f^{-1}(V) = \bigcup_{i \in I} f_i^{-1}(V).$$

Assuming this for now, if $V \subseteq Y$ is open, then $f_i^{-1}(V) \subseteq X$ is open, so their union $f^{-1}(V) \subseteq X$ is also open and we are done.

For the proof of the claim:

- If $x \in f^{-1}(V)$ then $f(x) \in V$. Let $i \in I$ be such that $x \in U_i$, then $f_i(x) = f(x) \in V$, so $x \in f_i^{-1}(V)$, so $x \in \bigcup_{i \in I} f_i^{-1}(V)$.
- If $x \in \bigcup_{i \in I} f_i^{-1}(V)$ then there exists $i \in I$ such that $x \in f_i^{-1}(V)$, so $f_i(x) \in V$, so $f(x) \in V$, so $x \in f^{-1}(V)$.

- (c) Given $x \in X$, let $i \in I$ be such that $x \in U_i$. Set

$$f(x) := f_i(x).$$

To check that this is well-defined, suppose $j \in I$ is such that $x \in U_j$, then $x \in U_i \cap U_j$ so $f_i(x) = f_j(x)$.

For each $i \in I$: we have $f(x) = f_i(x)$ for all $x \in U_i$, in other words $f|_{U_i} = f_i$. Therefore f is continuous by the result of part (b).

Finally, we prove the uniqueness of the function f . Suppose $g: X \rightarrow Y$ is a function such that $g|_{U_i} = f_i$ for all $i \in I$. Let $x \in X$, then there exists $i \in I$ such that $x \in U_i$, so that

$$g(x) = g|_{U_i}(x) = f_i(x) = f(x).$$

We conclude that $g = f$. □

2.3.

- (a) Let X be a disconnected topological space and let U, V be non-empty open subsets of X such that $X = U \cup V$ and $U \cap V = \emptyset$.

Prove that if $C \subseteq X$ is connected, then $C \subseteq U$ or $C \subseteq V$.

- (b) Let X be a topological space and $C \subseteq X$ a connected subset. Suppose that B satisfies $C \subseteq B \subseteq \overline{C}$. Prove that B is connected.

[**Hint:** Use (a).]

- (c) Let X be an infinite set with the cofinite topology. Prove that X is connected.

- (d) Suppose X, Y are connected topological spaces and $A \subsetneq X$, $B \subsetneq Y$. Prove that the set $(X \times Y) \setminus (A \times B)$ is connected.

[**Hint:** Use [Exercise 1.44](#).]

Solution.

- (a) Consider $U' := C \cap U$ and $V' := C \cap V$. There are open subsets of C , they are disjoint and their union is C . If both are non-empty, then C is disconnected, contradiction. So either $U' = \emptyset$, in which case $C \subseteq V$, or $V' = \emptyset$, in which case $C \subseteq U$.

- (b) Suppose B is disconnected and let U, V be open subsets of X such that

$$U \cap B \neq \emptyset, V \cap B \neq \emptyset, B \subseteq U \cup V, U \cap B \cap V = \emptyset.$$

By part (a), $C \subseteq U \cap B$ or $C \subseteq V \cap B$. If $C \subseteq U \cap B$ then $B \subseteq \overline{C} \subseteq \overline{U}$. But $\overline{U} \cap V = \emptyset$, so $V \cap B = \emptyset$, contradiction. The other case is similar.

(**Alternative proof**) $C \subseteq B \subseteq \overline{C}$ implies that C is dense in the subspace B of X , so B is connected by [Exercise 1.43](#).

- (c) Let $f: X \rightarrow \{0, 1\}$ be a continuous function. Let $U = f^{-1}(0)$ and $V = f^{-1}(1)$, then $U \cap V = \emptyset$, and U, V are open. But in the cofinite topology the intersection of two open sets is empty only if one of the two sets is empty, which in our case implies that the function f must be constant.

- (d) Let

$$Z := (X \times Y) \setminus (A \times B) = \{z = (x, y) \in X \times Y : x \notin A \text{ or } y \notin B\}.$$

Fix $x_0 \notin A$ and $y_0 \notin B$. Then the set

$$C := \{x_0\} \times Y \cup X \times \{y_0\}$$

is connected, since $\{x_0\} \times Y$ is homeomorphic to Y (hence connected), $X \times \{y_0\}$ is homeomorphic to X (hence connected), and the two sets intersect, hence their union C is connected by [Tutorial Question 3.6](#).

For each $z = (x, y) \in Z$, let

$$C_z := \begin{cases} \{x\} \times Y & \text{if } x \notin A \\ X \times \{y\} & \text{otherwise.} \end{cases}$$

Then C_z is connected and

$$C_z \cap C = \begin{cases} (x, y_0) & \text{if } x \notin A \\ (x_0, y) & \text{otherwise} \end{cases}$$

is non-empty. By [Exercise 1.44](#), the union

$$C \cup \bigcup_{z \in Z} C_z = Z$$

is connected.

(Alternative proof) Let $f: (X \times Y) \setminus (A \times B) \longrightarrow \{0, 1\}$ be a continuous function. Fix $x_0 \in X \setminus A$ and $y_0 \in Y \setminus B$.

Let (x, y) be an element of $(X \times Y) \setminus (A \times B)$. It follows that either $x \notin A$ or $y \notin B$.

If $x \notin A$, then $\{x\} \times Y$ is a subset of $(X \times Y) \setminus (A \times B)$, but it is connected since it is homeomorphic to Y , so the restriction of f to $\{x\} \times Y$ is constant. Similarly, $X \times \{y_0\}$ is a connected subset of $(X \times Y) \setminus (A \times B)$, so

$$f(x, y) = f(x, y_0) = f(x_0, y_0).$$

If $y \notin B$, then one can show in a similar way that $f(x, y) = f(x_0, y) = f(x_0, y_0)$.

Hence f is constant, which implies that $(X \times Y) \setminus (A \times B)$ is connected. $\square$

2.4. Let X be a topological space.

- (a) Prove that, if X is Hausdorff and K and L are compact subsets of X , then $K \cap L$ is compact.
- (b) Give an example of a topological space X and a subset K of X such that K is compact but K is not closed in X .
- (c) Let $Y = \{0, 1\}$ with the trivial (aka indiscrete) topology, and let $X = Y \times \mathbf{R}$, where $\mathbf{R}$ is given its usual (Euclidean) topology.

Keep in mind the projection map $\pi: X = Y \times \mathbf{R} \longrightarrow \mathbf{R}$ and make good use of it in the following.

- i. What do the open sets of X look like?
- ii. Prove that the topological space X is not Hausdorff.
- iii. Define the following two subsets of X :

$$K = (\{0\} \times [0, 2)) \cup (\{1\} \times [2, 3])$$

$$L = (\{0\} \times (1, 3]) \cup (\{1\} \times [0, 1]).$$

Prove that K is compact (and convince yourself that your proof can be adapted to show that L is compact).

- iv. Prove that $K \cap L$ is not compact.

Solution.

- (a) Since X is Hausdorff and K, L are compact subsets, we know that K and L are closed subsets of X (by [Proposition 2.37](#)). Therefore $K \cap L$ is a closed subset of X , hence $K \cap L = K \cap (K \cap L)$ is a closed subset of K (by the definition of the induced topology on K). But K is compact, so $K \cap L$ is compact by [Proposition 2.38](#).
- (b) Let $X = \{0, 1\}$ with the topology $\{\emptyset, \{0\}, X\}$ and let $K = \{0\}$. Then K is a finite topological space, hence compact by [Tutorial Question 4.4](#), but it is not closed since its complement $\{1\}$ is not open.

(c)

- i. The open subsets of X are $\emptyset$ and the sets of the form $Y \times U$ where $U \subseteq \mathbf{R}$ is open.
- ii. Consider the points $(0, 0) \neq (1, 0)$ of X . By part (a), any open neighbourhood of $(0, 0)$ is of the form $Y \times U$, where U is a neighbourhood of 0 in $\mathbf{R}$. But then $(1, 0) \in Y \times U$. Since every open neighbourhood of $(0, 0)$ is also an open neighbourhood of $(1, 0)$, it is impossible to separate these two points by opens. Hence X is not Hausdorff.
- iii. By part (a), any open cover of K is of the form

$$K \subseteq \bigcup_{i \in I} (Y \times U_i) = Y \times \left(\bigcup_{i \in I} U_i \right), \quad U_i \subseteq \mathbf{R} \text{ open.}$$

Then we get an open cover of

$$[0, 3] = \pi(K) \subseteq \bigcup_{i \in I} U_i.$$

(Here we used the fact that the projection map π is open, see [Exercise 1.25](#).)

Since $[0, 3]$ is compact, this open cover has a finite subcover:

$$[0, 3] = \pi(K) \subseteq \bigcup_{j=1}^n U_{i_j}$$

for some $n \in \mathbf{N}$ and $i_1, \dots, i_n \in I$. This gives us a finite subcover of the original cover of K :

$$K \subseteq \bigcup_{j=1}^n (Y \times U_{i_j}) = Y \times \left(\bigcup_{j=1}^n U_{i_j} \right).$$

So K is compact.

(Alternative proof:) We are going to show that the restriction

$$\pi: K \longrightarrow [0, 3]$$

of the projection $Y \times [0, 3] \longrightarrow [0, 3]$ is a homeomorphism, which immediately implies that K is compact since $[0, 3]$ is. As a restriction of a continuous function, π is automatically continuous, and it is obviously bijective, so [Exercise 1.25](#) implies that it suffices to show that π is open.

By part i, the open subsets of X are $\emptyset$ and the sets of the form $Y \times U$ where $U \subseteq \mathbf{R}$ is open. The image of $\emptyset$ is still $\emptyset$, which is open in $[0, 3]$. If $U \subseteq \mathbf{R}$ is open, then

$$(Y \times U) \cap K = (\{0\} \times (U \cap [0, 2))) \cup (\{1\} \times (U \cap [2, 3])).$$

It follows that

$$\pi((Y \times U) \cap K) = (U \cap [0, 2)) \cup (U \cap [2, 3]) = U \cap [0, 3],$$

which is open in $[0, 3]$. Hence π is open.

- iv. We have $K \cap L = \{0\} \times (1, 2)$. Via the projection $\pi: Y \times \mathbf{R} \longrightarrow \mathbf{R}$, $K \cap L$ is homeomorphic to $(1, 2) \subseteq \mathbf{R}$. Since $(1, 2)$ is not compact (it is not a closed subset of the Hausdorff space $\mathbf{R}$), we conclude that $K \cap L$ is not compact. $\square$

2.5. Let (X, d) be a metric space and define $d': X \times X \rightarrow \mathbf{R}$ by

$$d'(x, y) := \min\{1, d(x, y)\}.$$

Note that $d'(x, y) \leq 1$ for all $x, y \in X$.

- (a) Prove that d' is a metric on X .
- (b) Prove that d and d' are topologically equivalent.
- (c) Give an example to show that (X, d) and (X, d') are not necessarily isometric.
- (d) Prove that any sequence (x_n) is Cauchy in (X, d) if and only if it is Cauchy in (X, d') .
- (e) Let $x \in X$. Prove that any sequence (x_n) converges to x with respect to d if and only if it converges to x with respect to d' .
- (f) Deduce that (X, d) is complete if and only if (X, d') is complete.

Solution.

- (a) The symmetry and non-degeneracy of d' are immediate from the symmetry and non-degeneracy of d . For the triangle inequality we have for all $x, y, z \in X$:

$$\begin{aligned} d'(x, y) + d'(y, z) &= \min\{1, d(x, y)\} + \min\{1, d(y, z)\} \\ &= \min\{2, 1 + d(x, y), 1 + d(y, z), d(x, y) + d(y, z)\} \\ &\geq \min\{2, 1, 1, d(x, z)\} \\ &= \min\{1, d(x, z)\} \\ &= d'(x, z). \end{aligned}$$

- (b) We write $\mathbf{B}$ for open balls with respect to d and $\mathbf{B}'$ for open balls with respect to d' , that is

$$\begin{aligned} \mathbf{B}_r(x) &= \{y \in X : d(x, y) < r\} \\ \mathbf{B}'_r(x) &= \{y \in X : d'(x, y) < r\}. \end{aligned}$$

For any $x \in X$, if $r \leq 1$ then $\mathbf{B}'_r(x) = \mathbf{B}_r(x)$. If $r > 1$ then $\mathbf{B}'_r(x) = \mathbf{B}'_1(x) = \mathbf{B}_1(x)$. Since the topology on a metric space is generated by open balls, we conclude that d and d' give the same topology on X .

- (c) Let $X = \mathbf{R}$ and let d be the Euclidean metric. If f is an isometric bijection from (X, d) to (X, d') , then

$$1 \geq d'(f(0), f(2)) = d(0, 2) = 2,$$

contradiction.

- (d) Suppose (x_n) is Cauchy with respect to d and let $\varepsilon > 0$. Let $N \in \mathbf{N}$ be such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. Then for all $n, m \geq N$ we have

$$d'(x_n, x_m) \leq d(x_n, x_m) < \varepsilon,$$

so (x_n) is Cauchy with respect to d' .

Conversely, suppose (x_n) is Cauchy with respect to d' and let $\varepsilon > 0$. Let $\varepsilon' = \min\{1, \varepsilon\} > 0$. Let $N \in \mathbf{N}$ be such that $d'(x_n, x_m) < \varepsilon'$. Then from the definition of d' we have

$$d(x_n, x_m) = d'(x_n, x_m) < \varepsilon' \leq \varepsilon,$$

so (x_n) is Cauchy with respect to d .

- (e) This is similar to the previous part. Suppose $(x_n) \rightarrow x$ with respect to d and let $\varepsilon > 0$. Let $N \in \mathbf{N}$ be such that $d(x_n, x) < \varepsilon$ for all $n \geq N$, then

$$d'(x_n, x) \leq d(x_n, x) < \varepsilon,$$

so $(x_n) \rightarrow x$ with respect to d' .

Conversely, suppose $(x_n) \rightarrow x$ with respect to d' and let $\varepsilon > 0$. Let $\varepsilon' = \min\{1, \varepsilon\}$. Let $N \in \mathbf{N}$ be such that $d'(x_n, x) < \varepsilon'$, then

$$d(x_n, x) = d'(x_n, x) < \varepsilon' \leq \varepsilon,$$

so $(x_n) \rightarrow x$ with respect to d .

- (f) Follows immediately from the last two parts: if (X, d) is complete, let (x_n) be Cauchy with respect to d' , then (x_n) is Cauchy with respect to d , so (x_n) converges to some $x \in X$ with respect to d , so (x_n) converges to x with respect to d' ; therefore (X, d') is complete. The converse is similar. $\square$

2.6.

- (a) Let (X, d) be a metric space such that $d(x, y) \leq 1$ for all $x, y \in X$. Prove that if $(\widehat{X}, \widehat{d})$ is a completion of (X, d) , then $\widehat{d}(\widehat{x}, \widehat{y}) \leq 1$ for all $\widehat{x}, \widehat{y} \in \widehat{X}$.
- (b) For each $n \in \mathbf{N}$, let (X_n, d_n) be a metric space such that $d_n(x, y) \leq 1$ for all $x, y \in X_n$.

Consider the product

$$X := \prod_{n=0}^{\infty} X_n.$$

Define $d: X \times X \rightarrow \mathbf{R}$ by

$$d(x, y) := \sum_{n=0}^{\infty} \frac{d_n(x_n, y_n)}{2^n}.$$

- i. Prove that (X, d) is a metric space.
- ii. For every n , fix a completion $(\widehat{X}_n, \widehat{d}_n)$ of (X_n, d_n) with isometry $\iota_n: X_n \rightarrow \widehat{X}_n$. Define

$$\widetilde{X} := \prod_{n=0}^{\infty} \widehat{X}_n$$

endowed with a metric $\widetilde{d}$ defined in the same way as above:

$$\widetilde{d}(x, y) := \sum_{n=0}^{\infty} \frac{\widehat{d}_n(x_n, y_n)}{2^n}.$$

Check that the function $\iota: X \rightarrow \widetilde{X}$ given by

$$\iota := \prod_{n=0}^{\infty} \iota_n$$

is an isometry, and prove that $(\widetilde{X}, \widetilde{d})$ is a completion of (X, d) .

Solution.

- (a) Let $\widehat{x}, \widehat{y} \in \widehat{X}$. We prove that for any $\varepsilon > 0$ we have $\widehat{d}(\widehat{x}, \widehat{y}) < 1 + \varepsilon$.

Since $\iota(X)$ is dense in $\widehat{X}$, given $\varepsilon > 0$ there exist $x, y \in X$ such that $\widehat{d}(\widehat{x}, \iota(x)) < \varepsilon/2$ and $\widehat{d}(\widehat{y}, \iota(y)) < \varepsilon/2$. Then by the triangle inequality we have

$$\widehat{d}(\widehat{x}, \widehat{y}) \leq \widehat{d}(\iota(x), \iota(y)) + \widehat{d}(\widehat{x}, \iota(x)) + \widehat{d}(\widehat{y}, \iota(y)) < d(x, y) + \varepsilon \leq 1 + \varepsilon.$$

- (b) i. The infinite series defining $d(x, y)$ converges absolutely since it is bounded above by the geometric series $\sum 1/2^n$.

The symmetry of d follows immediately from the definition of d and the symmetry of d_n for all n . The non-degeneracy is also fairly immediate: if $x \neq y$ then there exists $n \in \mathbf{N}$ such that $x_n \neq y_n$, so $d_n(x_n, y_n) > 0$, hence $d(x, y) > 0$.

For the triangle inequality we have

$$d(x, y) + d(y, z) = \sum_{n=0}^{\infty} \frac{d_n(x_n, y_n) + d_n(y_n, z_n)}{2^n} \geq \sum_{n=0}^{\infty} \frac{d_n(x_n, z_n)}{2^n} = d(x, z).$$

ii. It is easy to see that ι is an isometry since each ι_n is an isometry:

$$\tilde{d}(\iota(x), \iota(y)) = \sum_{n=0}^{\infty} \frac{\widehat{d}_n(\iota_n(x_n), \iota_n(y_n))}{2^n} = \sum_{n=0}^{\infty} \frac{d_n(x_n, y_n)}{2^n} = d(x, y).$$

Next we prove that $\tilde{X}$ is complete. Let $(x^m)_m = (x^1, x^2, \dots)$ be a Cauchy sequence of elements in $\tilde{X}$. Fix $n \in \mathbb{N}$ and consider the sequence $(x_n^m)_m$.

We claim that this sequence is Cauchy in $\widehat{X}_n$. Indeed, given any $\varepsilon > 0$ there exists N such that $\tilde{d}(x^{m_1}, x^{m_2}) < \varepsilon/2^n$ for all $m_1, m_2 > N$. In particular $d_n(x_n^{m_1}, x_n^{m_2})/2^n < \varepsilon/2^n$ for all $m_1, m_2 > N$.

As $\widehat{X}_n$ is complete, there exists some $x_n \in \widehat{X}_n$ such that $(x_n^m) \rightarrow x_n$ as $m \rightarrow \infty$. We claim that $(x^m) \rightarrow x := (x_1, x_2, \dots) \in \tilde{X}$. Given any $\varepsilon > 0$, fix an integer $M > 2 + \log_2(1/\varepsilon)$. Let N_n be such that $\widehat{d}_n(x_n^m, x_n) < 2^n \varepsilon / 2(M-1)$ for all $m > N_n$, and set $N := \max\{N_1, \dots, N_{M-1}\}$. Then for all $m > N$,

$$\begin{aligned} \tilde{d}(x^m, x) &= \sum_{n=0}^{\infty} \frac{\widehat{d}_n(x_n^m, x_n)}{2^n} \\ &\leq \sum_{n=0}^{M-1} \frac{\widehat{d}_n(x_n^m, x_n)}{2^n} + \sum_{n=M}^{\infty} \frac{1}{2^n} \\ &\leq \sum_{n=0}^{M-1} \frac{\varepsilon}{2(M-1)} + \frac{1}{2} \cdot \frac{1}{2^{M-2}} \\ &< \frac{\varepsilon}{2} + \frac{1}{2} \cdot \frac{1}{1/\varepsilon} \\ &= \varepsilon. \end{aligned}$$

It remains to show that $\iota(X)$ is dense in $\tilde{X}$. Let $x := (x_1, x_2, \dots) \in \overline{\tilde{X}}$. For all n , there exists a sequence $(y_n^1, y_n^2, \dots)$ in X_n such that $(\iota_n(y_n^m)) \rightarrow x_n$ as $m \rightarrow \infty$. Then by setting $x_n^m := \iota_n(y_n^m)$ (and hence $x^m = \iota(y^m)$), we can use exactly the same argument as above to deduce that $(x^m) \rightarrow x$ in $\tilde{X}$, where x^m is in the image of the embedding $\iota: X \rightarrow \tilde{X}$ for all m . $\square$