Solutions to exercises on metric and Hilbert spaces
An invitation to functional analysis

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1. Metric and topological spaces

Metrics

Solution 1.1. We need to show that

$$-d(x,t) \leqslant d(x,y) - d(t,y) \leqslant d(x,t).$$

One application of the triangle inequality gives

$$d(x,y) \leqslant d(x,t) + d(t,y) \qquad \Rightarrow \qquad d(x,y) - d(t,y) \leqslant d(x,t).$$

Another application gives

$$d(t,y) \leqslant d(t,x) + d(x,y) \qquad \Rightarrow \qquad -d(x,t) \leqslant d(x,y) - d(t,y).$$

Solution 1.2. We have

$$\begin{aligned} |d(x,y) - d(s,t)| &= |d(x,y) - d(y,s) + d(y,s) - d(s,t)| \\ &\leqslant |d(x,y) - d(y,s)| + |d(y,s) - d(s,t)| \\ &\leqslant d(x,s) + d(y,t) \end{aligned}$$

after one application of the triangle inequality and two applications of Exercise 1.1.

Solution 1.3. We have

(a) $d(x,y) = \|x-y\| = \sqrt{(x-y)\cdot(x-y)} = \sqrt{(-1)^2\,(y-x)\cdot(y-x)} = \|y-x\| = d(y,x)$;

(b) Let $u = x - t$ and $v = t - y$, then we are looking to show that $\|u+v\| \leqslant \|u\| + \|v\|$. But:

$$\begin{aligned} \|u+v\|^2 = (u+v)\cdot(u+v) &= \|u\|^2 + 2\,u\cdot v + \|v\|^2 \leqslant \|u\|^2 + 2\,|u\cdot v| + \|v\|^2 \\ &\leqslant \|u\|^2 + 2\,\|u\|\,\|v\| + \|v\|^2 = \big(\|u\| + \|v\|\big)^2, \end{aligned}$$

where the last inequality sign comes from the Cauchy–Schwarz inequality.

(c) $d(x,y) = 0$ iff $(x-y)\cdot(x-y) = 0$ iff $x - y = 0$ iff $x = y$.

Solution 1.4. The Manhattan unit open ball is the interior of the square with vertices $(1,0)$, $(0,-1)$, $(-1,0)$, and $(0,1)$.

The Euclidean unit open ball is the interior of the unit circle centred at $(0,0)$.

The sup metric unit open ball is the interior of the square with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$.

Solution 1.5. It is clear from the definition that $d(y,x) = d(x,y)$ and that $d(x,y) = 0$ iff $x = y$.

For the triangle inequality, take $x, y, t \in X$ and consider the different cases:

$x=y$	$x=t$	$t=y$	$d(x,y)$	$d(x,t)+d(t,y)$
True	True	True	0	$0+0=0$
True	False	False	0	$1+1=2$
False	True	False	1	$1+0=1$
False	False	True	1	$0+1=1$
False	False	False	1	$1+1=2$

In all cases we see that $d(x,y) \leqslant d(x,t)+d(t,y)$.

Solution 1.6. Do this from scratch if you want to, but I prefer to deduce it from other examples we have seen.

First look at the case $n=1$, $X=\mathbf{F}_2$. Then $d(x,y)$ is precisely the discrete metric on $\mathbf{F}_2$ (see Exercise 1.5), in particular it is a metric. I'll denote it $d_{\mathbf{F}_2}$ for a moment to minimise confusion.

Back in the arbitrary $n \in \mathbf{N}$ case, note that $d(x,y)$ defined above can be expressed as

$$d(x,y) = d_{\mathbf{F}_2}(x_1,y_1) + \cdots + d_{\mathbf{F}_2}(x_n,y_n),$$

which is a special case of Example 2.3, therefore also a metric.

Solution 1.7. It is clear from the definition that $d'(x,y)=d'(y,x)$ and that $d'(x,y)=0$ iff $d(x,y)=0$ iff $x=y$.

For the triangle inequality, apply the inequality in the hint with $c=d(x,y)$, $a=d(x,t)$, $b=d(t,y)$.

Solution 1.8. Let $u \in U$, then $u \neq x$ so $r := d(u,x) > 0$. Then $x \notin \mathbf{B}_r(u)$, so $\mathbf{B}_r(u) \subseteq U$.

Solution 1.9. This is a variation on Example 2.8 and a generalisation of Exercise 1.8 (which is the case $r=0$).

Consider $C = \mathbf{D}_r(x)$ with $x \in X$, $r \in \mathbf{R}_{\geqslant 0}$. Let $y \in X \smallsetminus C$, then $d(x,y) > r$. Set $t = d(x,y) - r$ and consider the open ball $\mathbf{B}_t(y)$.

I claim that $\mathbf{B}_t(y) \subseteq \big(X \smallsetminus C\big)$: if $w \in \mathbf{B}_t(y)$ then $d(w,y) < t$ so

$$d(x,y) \leqslant d(x,w) + d(w,y) \leqslant d(x,w) + t \qquad \Rightarrow \qquad d(x,w) \geqslant d(x,y) - t = r,$$

hence $w \notin C$.

Solution 1.10. (a) Using the fundamental theorem of arithmetic (the existence of a unique prime factorisation of any natural number $\geqslant 2$), we have $m = p^{v_p(m)} m'$ and $n = p^{v_p(n)} n'$ with $p \nmid m'$ and $p \nmid n'$. Then

$$mn = p^{v_p(m)+v_p(n)} m'n' \qquad \text{and } p \nmid m'n',$$

so that $v_p(m) + v_p(n)$ is indeed the same as $v_p(mn)$.

(b) Write $x = \frac{m}{n}$, $y = \frac{a}{b}$, then

$$v_p(xy) = v_p\left(\frac{ma}{nb}\right) = v_p(ma) - v_p(nb) = v_p(m) + v_p(a) - v_p(n) - v_p(b) = v_p(x) + v_p(y).$$

For $v_p(x+y)$, without loss of generality assume $v := v_p(x) \leqslant v_p(y) =: u$ and write $x = p^v \frac{m'}{n'}$, $y = p^u \frac{a'}{b'}$. Then

$$x + y = p^v \frac{m'}{n'} + p^u \frac{a'}{b'} = p^v \left(\frac{m'}{n'} + p^{u-v}\frac{a'}{b'}\right) = p^v \left(\frac{m'b' + p^{u-v}a'n'}{n'b'}\right),$$

so that (since p does not divide $n'b'$)

$$v_p(x+y) = v + v_p\big(m'b' + p^{u-v}a'n'\big).$$

Since v_p of the quantity in parentheses is non-negative, we conclude that $v_p(x+y) \geqslant v = \min\big\{v_p(x), v_p(y)\big\}$.

Moreover, if $v < u$ then the quantity in parentheses has valuation zero, so that $v_p(x+y) = v = \min\big\{v_p(x), v_p(y)\big\}$.

(c) Direct from the previous part and $|x|_p = p^{-v_p(x)}$.

(d) We have

i. Clearly $v_p(y-x) = v_p(-1) + v_p(x-y) = v_p(x-y)$, so $d_p(y,x) = d_p(x,y)$.

ii. Letting $u = x-t$ and $v = t-y$, we want to prove that $|u+v|_p \leqslant |u|_p + |v|_p$. But we have already seen that

$$|u+v|_p \leqslant \max\big\{|x|_p, |y|_p\big\},$$

and the latter is clearly $\leqslant |x|_p + |y|_p$.

iii. If $x \in \mathbf{Q} \neq 0$, then $v_p(x) \in \mathbf{Z}$ so $|x|_p = p^{-v_p(x)} \in \mathbf{Q} \smallsetminus \{0\}$. Hence $|x|_p = 0$ iff $x = 0$, which implies that $d_p(x,y) = 0$ iff $x = y$.

Solution 1.11. (a) We have

$$\left\{2, 5, -7, \frac{4}{5}\right\} \subseteq \mathbf{B}_1(2)$$

$$\left\{3, 30, -24, \frac{39}{4}\right\} \subseteq \mathbf{B}_{1/9}(3).$$

(b) Recall that in the proof of the triangle inequality for the p-adic metric in Exercise 1.10, the following stronger result was shown:

$$d_p(x,y) \leqslant \max\{d_p(x,t), d_p(t,y)\}.$$

with equality holding if $d_p(x,t) \neq d_p(t,y)$. But this precisely says that if $d_p(x,t) \neq d_p(t,y)$, then $d_p(x,y)$ has to be equal to the largest of $d_p(x,t)$ and $d_p(t,y)$.

(c) First $x \in \mathbf{B}_r(c)$ iff $c \in \mathbf{B}_r(x)$ (this is true for any metric space). So it suffices to show that $x \in \mathbf{B}_r(c)$ implies $\mathbf{B}_r(x) \subseteq \mathbf{B}_r(c)$. Let $y \in \mathbf{B}_r(x)$, then $d_p(y,x) < r$, so that

$$d_p(y,c) \leqslant \max\big\{d_p(y,x), d_p(x,c)\big\} < r,$$

in other words $y \in \mathbf{B}_r(c)$.

(d) Consider two open balls $\mathbf{B}_r(x)$ and $\mathbf{B}_t(y)$. Without loss of generality $r \leqslant t$. Suppose that the balls are not disjoint and let $z \in \mathbf{B}_r(x) \cap \mathbf{B}_t(y)$. By part (c) this implies that $\mathbf{B}_r(z) = \mathbf{B}_r(x)$ and $\mathbf{B}_t(z) = \mathbf{B}_t(y)$, so that

$$\mathbf{B}_r(x) = \mathbf{B}_r(z) \subseteq \mathbf{B}_t(z) = \mathbf{B}_t(y).$$

Solution 1.12. Any open ball in any metric space is an open set (Example 2.8). Let's show that an arbitrary p-adic open ball $\mathbf{B}_r(c)$ is closed.

Let $U = \mathbf{Q} \smallsetminus \mathbf{B}_r(c)$. Given $u \in U$, we have $|u-c|_p \geqslant r$.

I claim that $\mathbf{B}_r(u) \subseteq U$, which would imply that U is open, so that $\mathbf{B}_r(c)$ is closed.

Suppose, on the contrary, that there exists $t \in \mathbf{B}_r(u) \cap \mathbf{B}_r(c)$. Then $|u-t|_p < r$ and $|t-c|_p < r$, so that

$$|u-c|_p = |(u-t)+(t-c)|_p \leqslant \max\{|u-t|_p, |t-c|_p\} < r,$$

contradicting the fact that $|u-c|_p \geqslant r$.

Topological spaces and continuous functions

Solution 1.13. Let $n \in \mathbf{N}$ and let $C_1, \ldots, C_n$ be closed subsets of X. Let

$$C = \bigcup_{i=1}^{n} C_i,$$

then the complement of C is

$$X \smallsetminus C = X \smallsetminus \left(\bigcup_{i=1}^{n} C_i \right) = \bigcap_{i=1}^{n} (X \smallsetminus C_i).$$

For each $i = 1, \ldots, n$, C_i is closed so $X \smallsetminus C_i$ is open, therefore $X \smallsetminus C$ is the intersection of finitely many open sets, hence is itself open by the topology axioms. We conclude that C is closed.

For the second statement, let $\{C_i \colon i \in I\}$ be a collection of closed subsets of X, indexed by a set I. Let

$$C = \bigcap_{i \in I} C_i,$$

then the complement of C is

$$X \smallsetminus C = X \smallsetminus \left(\bigcap_{i \in I} C_i \right) = \bigcup_{i \in I} (X \smallsetminus C_i).$$

For each $i \in I$, C_i is closed so $X \smallsetminus C_i$ is open, hence $X \smallsetminus C$ is the union of a collection of open sets, so is itself open by the topology axioms. We conclude that C is closed.

Solution 1.14. One direction is obvious: if U is open in X, then given any $u \in U$ we can take $V_u = U$ as an open neighbourhood contained in U.

In the other direction, suppose U has the given property at every $u \in U$. Then

$$U = \bigcup_{u \in U} V_u,$$

therefore U is open, since it is the union of the collection $\{V_u \colon u \in U\}$ of open sets.

Solution 1.15. If U is open, then it is an open neighbourhood of its elements by definition.

Conversely, suppose U is a neighbourhood of every element of itself. If x is an element of U, then U contains some open neighbourhood V_x of x. Now $U = \bigcup_{x \in U} V_x$, so U is open.

Solution 1.16. Let $f \colon X \longrightarrow Y$ be a function. The only open subsets of Y are $\varnothing$ and Y. Since $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$, it follows that f is continuous.

Solution 1.17.

(a) We have $x \in f^{-1}(S)$ iff $f(x) \in S$ iff $f(x) \notin \big(Y \smallsetminus S\big)$ iff $x \notin f^{-1}(Y \smallsetminus S)$.

(b) Suppose f is continuous and $C \subseteq Y$ is closed. By part (a) we have

$$f^{-1}(C) = X \smallsetminus f^{-1}(Y \smallsetminus C).$$

Then $\big(Y \smallsetminus C\big) \subseteq Y$ is open and f is continuous, so $f^{-1}(Y \smallsetminus C) \subseteq X$ is open, therefore $f^{-1}(C)$ is closed.

Conversely, suppose the inverse image of any closed subset is closed. Let $V \subseteq Y$ be open, then by part (a) we have

$$f^{-1}(V) = X \smallsetminus f^{-1}(Y \smallsetminus V).$$

So $\big(Y \smallsetminus V\big) \subseteq Y$ is closed, so $f^{-1}(Y \smallsetminus V) \subseteq X$ is closed, hence $f^{-1}(V)$ is open. We conclude that f is continuous.

Solution 1.18. Suppose $f\colon X \longrightarrow Y$ is continuous. If x is a point in X and N is a neighbourhood of $f(x)$, then N contains some open neighbourhood U of $f(x)$, whose inverse image $f^{-1}(U)$ is an open neighbourhood of x because of continuity. Since $f^{-1}(U) \subseteq f^{-1}(N)$, it follows that $f^{-1}(N)$ is a neighbourhood of x.

Conversely, suppose $f\colon X \longrightarrow Y$ is continuous at every point of X. If U be an open subset of Y, then $f^{-1}(U)$ is a neighbourhood of every element of itself. By Exercise 1.15, this implies $f^{-1}(U)$ is open. Hence f is continuous.

Solution 1.19. (a)⇔(c): Since $f^{-1}(S) = S$ for any subset S of X, we have:

($\mathcal{T}_2$ is coarser then $\mathcal{T}_1$) if and only if (if $U \in \mathcal{T}_2$ then $U \in \mathcal{T}_1$) if and only if (if $U \in \mathcal{T}_2$ then $f^{-1}(U) \in \mathcal{T}_1$) if and only if ($f$ is continuous).

(a)⇒(b): trivial, since if $x \in U_x^2$ and $U_x^2 \in \mathcal{T}_2 \subseteq \mathcal{T}_1$, we can take $U_x^1 = U_x^2$ and we are done.

(b)⇒(a): Let $U \in \mathcal{T}_2$. We use Exercise 1.14 to prove that $U \in \mathcal{T}_1$. Let $x \in U$, then setting $U_x^2 = U$ we have that U_x^2 is a $\mathcal{T}_2$-open neighbourhood of x, so by (b) there exists a cT_1-open neighbourhood U_x^1 of x such that $U_x^1 \subseteq U$. By Exercise 1.14 we conclude that U is open in the topology $\mathcal{T}_1$.

Solution 1.20. Let X and Y be topological spaces. Pick a point y in Y and define $f\colon X \longrightarrow Y$ to be the constant function sending every element of X to y. If U is an open subset of Y, then

$$f^{-1}(U) = \begin{cases} X & \text{if } y \in U, \\ \varnothing & \text{otherwise.} \end{cases}$$

Hence $f^{-1}(U)$ is open.

Solution 1.21. If U is an open subset of X, then $\iota^{-1}(U) = U \cap S$, which is open in S by the definition of the subspace topology. Hence ι is continuous.

The identity function is the special case $S = X$.

Solution 1.22. The 'only if' part follows directly from the definition of continuity.

Conversely, suppose that the inverse image of every member of S is open. It follows that the final topology $\mathcal{T}_Y'$ induced by f (see Tutorial Question 2.7) contains S, and is thus finer than $\mathcal{T}_Y$ by Tutorial Question 2.4. By part (b) of Tutorial Question 2.7, this implies that f is continuous.

Solution 1.23. (a) We start with proving that $\mathcal{T}_X$ is a topology:

- Since $\varnothing = f^{-1}(\varnothing)$ and $X = f^{-1}(Y)$, it follows that $\mathcal{T}_X$ contains $\varnothing$ and X.
- If $\{f^{-1}(U_i)\colon i \in I\}$ is a collection of members of $\mathcal{T}_X$, then

$$\bigcup_{i\in I} f^{-1}(U_i) = f^{-1}\Big(\bigcup_{i\in I} U_i\Big) \in \mathcal{T}_X.$$

- If $f^{-1}(U_1), \dots, f^{-1}(U_n)$ are members of $\mathcal{T}_X$, then

$$\bigcap_{i=1}^{n} f^{-1}(U_i) = f^{-1}\Big(\bigcap_{i=1}^{n} U_i\Big) \in \mathcal{T}_X.$$

If $\mathcal{T}$ is a topology on X such that f is continuous, then $f^{-1}(U) \in \mathcal{T}$ for every member U of $\mathcal{T}_Y$, and thus $\mathcal{T}_X \subseteq \mathcal{T}$. Therefore, $\mathcal{T}_X$ is the coarsest topology such that f is continuous.

(b) The 'only if' part has been proven in part (a), so it suffices to prove the 'if' part.

Suppose $\mathcal{T}$ is finer than $\mathcal{T}_X$. If U is a member of $\mathcal{T}_Y$, then $f^{-1}(U) \in \mathcal{T}_X \subseteq \mathcal{T}$. Hence f is continuous.

(c) Let $\mathcal{T}'_X$ be the topology on X generated by the set

$$\{f^{-1}(U) : U \in S\}.$$

Since the topology $\mathcal{T}_X$ contains $f^{-1}(U)$ for every member U of S, it follows from Tutorial Question 2.4 that $\mathcal{T}'_X \subseteq \mathcal{T}_X$. By Exercise 1.22, f is continuous when the topology on X is $\mathcal{T}'_X$, so part (a) implies that $\mathcal{T}_X \subseteq \mathcal{T}'_X$. Hence $\mathcal{T}'_X = \mathcal{T}_X$.

Solution 1.24. Let $f : X \times \{y\} \longrightarrow X$ be the map $f(x, y) = x$ and let $g : X \longrightarrow X \times \{y\}$ be the map $g(x) = (x, y)$. It is clear that g is the inverse of f. Since f is simply the projection onto the first factor of the product, it is continuous by Proposition 2.18. To show that g is continuous, consider a rectangle in $X \times \{y\}$: this is either $\varnothing$ or $U \times \{y\}$ for some open set $U \subseteq X$. Then $g^{-1}(U \times \{y\}) = U$ is open in X.

Solution 1.25.

(a) We need to check that $f^{-1} : Y \longrightarrow X$ is continuous; let $U \subseteq X$ be open, then $\left(f^{-1}\right)^{-1}(U) = f(U)$ is open in Y since f is an open map.

(b) One direction is trivial. For the other direction, we are told that every open subset U of X is of the form

$$U = \bigcup_{i \in I} U_i, \qquad U_1 \in S'.$$

Then

$$f(U) = \bigcup_{i \in I} f(U_i).$$

By assumption each $f(U_i)$ is open in Y, so their union must also be an open subset.

(c) By part (b) and Example 2.17, we only need to check the open condition on open rectangles $U_1 \times U_2 \subseteq X_1 \times X_2$: we have $\pi_1(U_1 \times U_2) = U_1$, clearly open in X_1. Same for π_2.

Solution 1.26. Let $U = X \setminus \{x\}$ and let $u \in U$. Then $u \neq x$, so by the Hausdorff property of X, there exist open neighbourhoods V_1 of u and V_2 of x such that $V_1 \cap V_2 = \varnothing$. In particular, $x \notin V_1$, so $V_1 \subseteq U$. As we have exhibited an open neighbourhood contained in U around every element of U, we conclude by Exercise 1.14 that U is open, so its complement $\{x\}$ is closed.

Interior and closure

Solution 1.27. Take $X = \{0, 1\}$ with the discrete metric, $x = 0$ and $\varepsilon = 1$. Then

$$\overline{\mathbf{B}_1(0)} = \overline{\{0\}} = \{0\} \neq \{0, 1\} = \mathbf{D}_1(0).$$

Solution 1.28.

(a) Since A and B are closed in X and Y respectively, their complements $X \setminus A$ and $Y \setminus B$ are open in X and Y respectively, and therefore $(X \setminus A) \times Y$ and $X \times (Y \setminus B)$ are open in $X \times Y$. It follows that

$$(X \times Y) \setminus (A \times B) = \big((X \setminus A) \times Y\big) \cup \big(X \times (Y \setminus B)\big)$$

is closed in $X \times Y$.

(b) By part (a), $\overline{A}\times\overline{B}$ is closed in $X\times Y$. Since $A\times B\subseteq\overline{A}\times\overline{B}$, it follows that $\overline{A\times B}\subseteq\overline{A}\times\overline{B}$. It remains to prove the other inclusion.

Given an element x of A, define $\iota_x\colon Y\longrightarrow X\times Y$ by $\iota_x(y)=(x,y)$. Let $\pi_X\colon X\times Y\longrightarrow X$ and $\pi_Y\colon X\times Y\longrightarrow Y$ be the projections. The composite function $\pi_X\circ\iota_x$ is the constant function sending every element of Y to x, which is continuous by Exercise 1.16; while $\pi_Y\circ\iota_x$ is the identity function of Y, which is continuous by Exercise 1.21. it then follows from Tutorial Question 3.8 that ι_x is continuous.

Since $\overline{A\times B}$ is closed in $X\times Y$, it follows from Exercise 1.17 that $\iota_x^{-1}(\overline{A\times B})$ is closed. Now $B\subseteq\iota_x^{-1}(\overline{A\times B})$ implies $\overline{B}\subseteq\iota_x^{-1}(\overline{A\times B})$; in other words, $\{x\}\times\overline{B}\subseteq\overline{A\times B}$. Since x is an arbitrary point in A, this implies $A\times\overline{B}\subseteq\overline{A\times B}$.

Following similar reasoning for points in $\overline{B}$, we can show that $\overline{A}\times\overline{B}\subseteq\overline{A\times B}$.

Solution 1.29. These are of course not the only possible answers (well, except for the last one).

(a) $x\longmapsto x$;

(b) $x\longmapsto e^x$;

(c) $x\longmapsto -e^x$;

(d) $x\longmapsto -x^2$;

(e) $x\longmapsto \sin(x)$;

(f) $x\longmapsto \min\{e^x,1\}$;

(g) $x\longmapsto \max\{-e^x,-1\}+1$;

(h) $x\longmapsto \arctan(x)$;

(i) $x\longmapsto 0$.

Solution 1.30. Since $A^\circ\subseteq A$, we have $\big(X\smallsetminus A\big)\subseteq\big(X\smallsetminus A^\circ\big)$. But A° is open, so $X\smallsetminus A^\circ$ is a closed set containing $X\smallsetminus A$, hence

$$\overline{X\smallsetminus A}\subseteq X\smallsetminus A^\circ.$$

For the opposite inclusion, note that $(X\smallsetminus A)\subseteq\overline{X\smallsetminus A}$, so

$$X\smallsetminus\overline{X\smallsetminus A}\subseteq X\smallsetminus(X\smallsetminus A)=A,$$

therefore $X\smallsetminus\overline{X\smallsetminus A}$ is an open set contained in A, so that

$$X\smallsetminus\overline{X\smallsetminus A}\subseteq A^\circ,$$

which implies that $X\smallsetminus A^\circ\subseteq\overline{X\smallsetminus A}$.

Solution 1.31. First we show that $\overline{\mathbf{Z}}=\mathbf{Z}$: letting $U=\mathbf{R}\smallsetminus\mathbf{Z}$, we have

$$U=\bigcup_{n\in\mathbf{Z}}(n-1,n),$$

so U is a union of open subsets, hence open.

Now we note that $\mathbf{Z}^\circ=\varnothing$: if $V\subseteq\mathbf{R}$ is a nonempty open subset, then V contains a nonempty open interval, hence is uncountable, so it cannot be contained in $\mathbf{Z}$.

Solution 1.32.

(a) Let $N \subseteq X$ be nowhere dense and let $M \subseteq N$. Then $\overline{M} \subseteq \overline{N}$ by part (a) of Tutorial Question 3.1, so $\left(\overline{M}\right)^\circ \subseteq \left(\overline{N}\right)^\circ = \varnothing$ by part (a) of Tutorial Question 3.1.

(b) Suppose N is nowhere dense and let $U \subseteq X$ be nonempty and open. If $U \cap \left(X \setminus \overline{N}\right) = \varnothing$, then $U \subseteq \overline{N}$, so $U \subseteq \left(\overline{N}\right)^\circ = \varnothing$, contradicting the non-emptiness of U. So it must be that U intersects $X \setminus \overline{N}$ nontrivially, hence $X \setminus \overline{N}$ is dense.

Conversely, suppose $X \setminus \overline{N}$ is dense but N is not nowhere dense, that is there exists a nonempty open $U \subseteq \overline{N}$. Then $U \cap \left(X \setminus \overline{N}\right) = \varnothing$, contradicting the denseness of $X \setminus \overline{N}$.

(c) It suffices to prove the case of two nowhere dense sets M and N. Let $L = M \cup N$. Then by part (b) of Tutorial Question 3.1 we have $\overline{L} = \overline{M} \cup \overline{N}$ so $X \setminus \overline{L} = \left(X \setminus \overline{M}\right) \cap \left(X \setminus \overline{N}\right)$. As $X \setminus \overline{L}$ is the intersection of two dense open subsets, it is dense and open by Tutorial Question 3.2, hence L is nowhere dense.

Metric topologies

Solution 1.33. (a) i. Put $X = \{0,1\}$, $Y = \{1\}$, $\mathcal{T}_Y = \mathcal{P}(Y)$. Let $f\colon X \longrightarrow Y$ be the function sending both 0 and 1 to 1. It follows that $\mathcal{T}_X = \{\varnothing, \{0,1\}\}$. The topology $\mathcal{T}_Y$ is defined by the discrete metric (see Tutorial Question 2.1), but $\mathcal{T}_X$ is not metrisable (see Tutorial Question 2.3).

ii. Put $X = \{1\}$, $Y = \{0,1\}$, $\mathcal{T}_Y = \left\{\varnothing, \{1\}, \{0,1\}\right\}$. Let $f\colon X \longrightarrow Y$ be the inclusion function, which sends 1 to 1. It follows that $\mathcal{T}_X = \mathcal{P}(X)$. The topology $\mathcal{T}_X$ is defined by the discrete metric (see Tutorial Question 2.1), but $\mathcal{T}_Y$ is not metrisable (see Tutorial Question 2.3).

(b) i. Let $(X, \mathcal{T}_X)$ be the set of real numbers equipped with the Euclidean topology. Put $Y = \{0,1\}$. If $f\colon X \longrightarrow Y$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

then $\mathcal{T}_Y = \left\{\varnothing, \{1\}, \{0,1\}\right\}$. The topology $\mathcal{T}_X$ is defined by the Euclidean metric, but $\mathcal{T}_Y$ is not metrisable (see Tutorial Question 2.3).

ii. Put $X = \{0,1\}$, $Y = \{1\}$, $\mathcal{T}_X = \left\{\varnothing, \{1\}, \{0,1\}\right\}$. Let $f\colon X \longrightarrow Y$ be the function sending both 0 and 1 to 1. It follows that $\mathcal{T}_Y = \{\varnothing, \{0,1\}\}$. The topology $\mathcal{T}_Y$ is defined by the discrete metric (see Tutorial Question 2.1), but $\mathcal{T}_X$ is not metrisable (see Tutorial Question 2.3).

Solution 1.34. Let $x \in X$. Given $\varepsilon > 0$, if $x' \in \mathbf{B}_\varepsilon(x)$ then $d_X(x,x') < \varepsilon$, so

$$d_Y\big(f(x), f(x')\big) = d_X(x,x') < \varepsilon,$$

hence $f(x') \in \mathbf{B}_\varepsilon(f(x))$.

Solution 1.35.

(a) Let $\mathcal{T}_1$ be the topology defined by d_1, $\mathcal{T}_2$ the topology defined by d_2. We know that each topology is generated by the corresponding open balls.

Consider an open ball $\mathbf{B}_r^{d_2}(x)$ of $\mathcal{T}_2$. I claim that the open ball $\mathbf{B}_{r/M}^{d_1}(x)$ of $\mathcal{T}_1$ is contained in $\mathbf{B}_r^{d_2}(x)$: if $y \in \mathbf{B}_{r/M}^{d_1}(x)$ then $d_1(x,y) < r/M$, so that

$$d_2(x,y) \leqslant M\, d_1(x,y) < r.$$

So $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

Now consider an open ball $\mathbf{B}_r^{d_1}(x)$ of $\mathcal{T}_1$. I claim that the open ball $\mathbf{B}_{rm}^{d_2}(x)$ of $\mathcal{T}_2$ is contained in $\mathbf{B}_r^{d_1}(x)$: if $y \in \mathbf{B}_{rm}^{d_2}(x)$ then $d_2(x,y) < rm$, so that

$$d_1(x,y) \leqslant \frac{1}{m} d_2(x,y) < r.$$

So $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, in conclusion $\mathcal{T}_1 = \mathcal{T}_2$.

(b) Let $X = \mathbf{Z}$. Let d_1 be the discrete metric on $\mathbf{Z}$. Let d_2 be the induced Euclidean metric from $\mathbf{R}$, that is $d_2(x,y) = |x-y|$ for all $x, y \in \mathbf{Z}$.

First we note that d_1 and d_2 are equivalent metrics. It suffices to show that every singleton $\{x\} \subseteq \mathbf{Z}$ is open with respect to d_2:

$$\mathbf{B}_1^{d_2}(x) = \{y \in \mathbf{Z} : |y-x| < 1\} = \{y \in \mathbf{Z} : x-1 < y < x+1\} = \{x\}.$$

Suppose that d_1 and d_2 satisfy Equation (1.1) for some $m, M > 0$. In particular, if $x \neq y$ we would have

$$m \leqslant |x-y| \leqslant M \qquad \text{for all } x \neq y \in \mathbf{Z},$$

which is blatantly false (take $y = 0$, $x = \lceil M \rceil + 1$).

Solution 1.36. The inequalities involving d_1 and d_∞ follow simply from

$$\frac{a+b}{2} \leqslant \max\{a,b\} \leqslant a+b \leqslant 2\max\{a,b\},$$

which hold for any $a, b \in \mathbf{R}_{\geqslant 0}$.

The inclusions between open balls now follow by the same reasoning as in part (a) of Exercise 1.35.

Solution 1.37.

(a) We have

$$\mathbf{B}_r^X(y) = \{x \in X : d(x,y) < r\}$$
$$\mathbf{B}_r^Y(y) = \{x \in Y : d(x,y) < r\},$$

so that

$$\mathbf{B}_r^X(y) \cap Y = \{x \in X : d(x,y) < r\} \cap Y = \{x \in Y : d(x,y) < r\} = \mathbf{B}_r^Y(y).$$

(b) In one direction, suppose A is open in Y; by Tutorial Question 3.4 we have some indexing set I such that

$$A = \bigcup_{i \in I} \mathbf{B}_{r_i}^Y(a_i),$$

with $r_i > 0$ and $a_i \in A$ for all $i \in I$. We can then let

$$U = \bigcup_{i \in I} \mathbf{B}_{r_i}^X(a_i),$$

which by Tutorial Question 3.4 is an open in X. It is clear that $A = U \cap Y$ by part (a).

Conversely, suppose $A = U \cap Y$ with U open in X. Let $a \in A$, then $a \in U$ so there exists an open (in X) ball $\mathbf{B}_r^X(a)$ such that $\mathbf{B}_r^X(a) \subseteq U$. Consider $\mathbf{B}_r^Y(a) = \mathbf{B}_r^X(a) \cap Y \subseteq U \cap Y = A$. So every point $a \in A$ is contained in an open (in Y) ball, hence A is open in Y.

Solution 1.38. Let $\mathcal{T}$ denote the product topology on $X \times Y$ and $\mathcal{T}_d$ the topology defined by the metric d.

We start by proving that any open rectangle $U \times V \in \mathcal{T}$ is also open in $\mathcal{T}_d$, which will imply that $\mathcal{T} \subseteq \mathcal{T}_d$. Consider an arbitrary element $(u,v) \in U \times V$. Since U is open in X, there exists $s > 0$ such that $\mathbf{B}_s(u) \subseteq U$. Similarly, there exists $t > 0$ such that $\mathbf{B}_t(v) \subseteq V$. Let $r = \min\{s,t\} > 0$. I claim that the d-open ball $B := \mathbf{B}_r\big((u,v)\big) \subseteq U \times V$. Why? If $(x,y) \in B$ then since d is conserving,

$$\max\left\{d_X(x,u), d_Y(y,v)\right\} = d_\infty\big((x,y),(u,v)\big) \leqslant d\big((x,y),(u,v)\big) < r,$$

so $d_X(x,u) < r \leqslant s$ hence $x \in U$, and $d_Y(y,v) < r \leqslant t$ hence $y \in V$.

Now we prove that any d-open ball $B := \mathbf{B}_\varepsilon((x,y))$ is also open in the product topology $\mathcal{T}$, which will imply that $\mathcal{T}_d \subseteq \mathcal{T}$. Let $w = (u,v) \in B$, then there exists $r > 0$ such that $\mathbf{B}_r(w) \subseteq B$. Let U_w be the d_X-open ball $\mathbf{B}_{r/2}(u) \subseteq X$, and let V_w be the d_Y-open ball $\mathbf{B}_{r/2}(v) \subseteq Y$. I claim that $U_w \times V_w \subseteq \mathbf{B}_r(w) \subseteq B$. Why? If $(s,t) \in U_w \times V_w$, since d is conserving,

$$d\big((s,t),(u,v)\big) \leqslant d_X(s,u) + d_Y(t,v) < \frac{r}{2} + \frac{r}{2} = r.$$

Solution 1.39. We need to show that d induces the discrete topology on X. It suffices to prove that any singleton $\{x\} \subseteq X$ is an open set with respect to the metric d.

Fix $x \in X$. The set of distances $d(x,y)$ with $y \neq x$ is finite, so it has a minimum element, wihich is > 0; call it D, so that $d(x,y) \leqslant D$ for all $y \in X$. Then $\mathbf{B}_D(x) = \{x\}$, which is therefore an open set.

Connectedness

Solution 1.40. By definition D is a disconnected subset of X if and only if it is a disconnected topological space in the induced topology. The latter is by definition: there exist U', V' open subsets of D such that

$$D = U' \cup V', \qquad U' \cap V' = \varnothing, \qquad U' \neq \varnothing, \qquad V' \neq \varnothing.$$

But U', V' are open in D if and only if there exist open subsets U, V of X such that $U' = U \cap D$, $V' = V \cap D$, from which the claim follows.

Solution 1.41. It follows from Example 2.28 that X is connected if it is a singleton.

Conversely, if $x_1 \neq x_2$ are elements of X, then $\{x_1\}$ and $X \setminus \{x_1\}$ are two disjoint non-empty open subsets of X such that their union is X, so X is disconnected.

Solution 1.42. Let $f\colon \bigcup_{n\in\mathbf{N}} C_n \longrightarrow \{0,1\}$ be a continuous function, where $\{0,1\}$ is given the discrete topology. Pick an element x_0 of C_0. We use induction to prove that $f(C_n) = \{f(x_0)\}$ for every natural number n.

The base case when $n = 0$ follows from the connectedness of C_0 and Proposition 2.31.

For the induction step, suppose the statement is true for a natural number n and consider an element x of C_{n+1}. Since $C_n \cap C_{n+1} \neq \varnothing$, we can pick an element x' of $C_n \cap C_{n+1}$. By the induction hypothesis, we have $f(x') = f(x_0)$. It then follows from the connectedness of C_{n+1} and Proposition 2.31 that $f(x) = f(x') = f(x_0)$.

Hence f is constant, which implies that $\bigcup_{n\in\mathbf{N}} C_n$ is connected.

Solution 1.43. Let $x \in D$.

Let $f\colon X \longrightarrow \{0,1\}$ be a continuous function, where $\{0,1\}$ is given the discrete topology. Since f is continuous, it follows from Exercise 1.17 that $f^{-1}\big(f(x)\big)$ is closed. By

Proposition 2.31, the restriction of f to D is constant, so $D \subseteq f^{-1}\big(f(x)\big)$, and therefore $X = \overline{D} \subseteq f^{-1}\big(f(x)\big)$. Hence f is constant, which implies that X is connected.

(Alternative solution): Suppose X is disconnected, so $X = U \cup V$ with U, V open, non-empty, and disjoint. Then $D \subseteq U \cup V$ with $D \cap U \neq \varnothing$, $D \cap V \neq \varnothing$ (because D is dense), and of course $D \cap U \cap V = \varnothing$, implying that D is a disconnected subset of X by Exercise 1.40.

Solution 1.44. Let $f\colon A \cup \bigcup_{i \in I} C_i \longrightarrow \{0,1\}$ be a continuous function, where $\{0,1\}$ is given the discrete topology. Pick an element a of A and consider an arbitrary element x of $A \cup \bigcup_{i \in I} C_i$. If $x \in A$, then the connectedness of A and Proposition 2.31 imply $f(x) = f(a)$. If $x \in C_i$ for some $i \in I$, then it follows from Tutorial Question 3.6 and Proposition 2.31 that $f(x) = f(a)$. Hence f is constant, which implies $A \cup \bigcup_{i \in I} C_i$ is connected.

Solution 1.45. Suppose $X \times Y$ is connected. Recall from Proposition 2.18 that the projections $\pi_X\colon X \times Y \longrightarrow X$ and $\pi_Y\colon X \times Y \longrightarrow Y$ are continuous. It then follows from Proposition 2.32 that $X = \pi_X(X \times Y)$ and $Y = \pi_Y(X \times Y)$ are connected.

Conversely, suppose that both X and Y are connected. Let $f\colon X \times Y \longrightarrow \{0,1\}$ be a continuous function, where $\{0,1\}$ is given the discrete topology. Consider two elements (x_1, y_1) and (x_2, y_2) of $X \times Y$. It follows from Exercise 1.24 that $\{x_1\} \times Y$ is homeomorphic to Y, and is therefore connected. This implies that f is constant when restricted to $\{x_1\} \times Y$. Similarly, f is constant when restricted to $X \times \{y_2\}$ because Y is connected. Hence

$$f(x_1, y_1) = f(x_1, y_2) = f(x_2, y_2),$$

and therefore $X \times Y$ is connected.

Solution 1.46. Let X be a totally separated space and let S be a subset of X with two distinct points x and y. It follows from total separatedness that there exists disjoint clopen neighbourhoods U and V of x and y respectively. Since U is clopen and does not contain y, it follows that $X \smallsetminus U$ is a clopen neighbourhood of y. Moreover, $S \cap U$ and $S \cap (X \smallsetminus U)$ are two disjoint open sets in S such that their union is S. Hence S is not connected, and therefore the only connected subsets of X are the singletons; in other words, X is totally disconnected.

Solution 1.47. We will prove all of them are totally separated, which implies total disconnectedness by Exercise 1.46.

(a) Let x and y be two distinct rational number. Without loss of generality, we assume that $x < y$. The denseness of $\mathbf{R} \smallsetminus \mathbf{Q}$ (see Example 2.23) implies that there exists an irrational number z such that $x < z < y$. The open sets $\mathbf{Q} \cap (-\infty, z)$ and $\mathbf{Q} \cap (z, \infty)$ are open in $\mathbf{Q}$, and their intersection is empty while their union is $\mathbf{Q}$, so $\mathbf{Q} \cap (-\infty, z)$ is an clopen neighbourhood of x in $\mathbf{Q}$ and $\mathbf{Q} \cap (z, \infty)$ is a clopen neighbourhood of y. Hence $\mathbf{Q}$ is totally disconnected when equipped with the Euclidean topology.

(b) Let X be a discrete space and let x and y be two points in X. It follows from the definition of the discrete topology that $\{x\}$ and $\{y\}$ are clopen, so they are disjoint clopen neighbourhoods of x and y respective. Hence X is totally separated.

Compactness

Solution 1.48. Suppose X is compact and $\{C_i : i \in I\}$ is a collection of closed sets with the finite intersection property. Suppose that

$$\bigcap_{i \in I} C_i = \varnothing.$$

Then

$$X = \bigcup_{i \in I} U_i, \qquad \text{where } U_i := X \setminus C_i,$$

is an open covering of X. Since X is compact, there exists a finite subset $J \subseteq I$ such that

$$X = \bigcup_{j \in J} U_j,$$

which implies that

$$\bigcap_{j \in J} C_j = \varnothing,$$

contradicting the finite intersection property of the collection $\{C_i : i \in I\}$.

Conversely, suppose every collection of closed sets of X with the finite intersection property has nonempty intersection. Suppose that X is not compact, so there exists an open cover of X:

$$X = \bigcup_{i \in I} U_i$$

with no finite subcover.

For each $i \in I$, let $C_i = X \setminus U_i$. Then for every finite $J \subseteq I$, $\{U_i : i \in J\}$ is not a cover of X, which means that the collection $\{C_i : i \in J\}$ has nonempty intersection. Hence the collection $\{C_i : i \in I\}$ has the finite intersection property, but note that the collection itself has empty intersection, since $\{U_i : i \in I\}$ is a cover of X, so we have reached a contradiction.

Solution 1.49.

(a) We know that $t \longmapsto 2\pi t$, $t \longmapsto \cos(t)$ and $t \longmapsto \sin(t)$ are continuous, so by Tutorial Question 3.8 so is f.

(b) Suppose $t_1 \neq t_2 \in [0,1)$ are such that $f(t_1) = f(t_2)$. Then $\cos(2\pi t_1) = \cos(2\pi t_2)$, which implies that $t_2 = 1 - t_1$. In that case $\sin(2\pi t_2) = \sin(2\pi - 2\pi t_1) = \sin(-2\pi t_1) = -\sin(2\pi t_1)$. But we also have $\sin(2\pi t_2) = \sin(2\pi t_1)$, so $\sin(2\pi t_1) = 0$, hence $t_1 = 0$ and $t_2 = 1 - t_1 = 1$, contradicting $t_2 \in [0,1)$.

We conclude that f is injective.

For surjectivity, let $(x,y) \in \mathbf{S}^1$, in other words $x^2 + y^2 = 1$. Define $\theta \in [0, 2\pi)$ by

$$\theta = \begin{cases} \arccos(x) & \text{if } y \geqslant 0 \\ 2\pi - \arccos(x) & \text{if } y < 0. \end{cases}$$

Letting $t = \theta/(2\pi)$, we have $f(t) = (x,y)$.

(c) At this point we know that f is a homeomorphism iff $f^{-1} : \mathbf{S}^1 \longrightarrow [0,1)$ is continuous. Note that $\mathbf{S}^1 \subseteq \mathbf{R}^2$ is compact: it is clearly bounded as any two points are at distance at most 2 of each other, so we just need to check that it is a closed subset of $\mathbf{R}^2$.

But $\mathbf{S}^1 = \mathbf{D}_1\big((0,0)\big) \cap C$ is the intersection of two closed sets, where

$$C = \{x, y \in \mathbf{R} : x^2 + y^2 \geqslant 1\} = \mathbf{R}^2 \setminus \mathbf{B}_1\big((0,0)\big).$$

Since $\mathbf{S}^1$ is compact, if f^{-1} were continuous then $[0,1) = f^{-1}(\mathbf{S}^1)$ would be compact, hence closed in $\mathbf{R}$. This is a contradiction, because 1 is an accumulation point of $[0,1)$ but does not lie in the set.

Solution 1.50. Y is not compact since it is not closed in $\mathbf{R}^2$, for instance the point $(0,1)$ is in the closure of Y but not in Y. On the other hand, Z is compact since it is closed and bounded in $\mathbf{R}^2$. Similarly, X is compact.

So Y and Z are not homeomorphic, and X and Y are not homeomorphic.

Suppose $f\colon X \longrightarrow Z$ is a homeomorphism. Let $x \in X^\circ$, then $f(x) \in Z^\circ$. The restriction of f to $X \smallsetminus \{x\} \longrightarrow Z \smallsetminus \{f(x)\}$ is then also a homeomorphism, but this is impossible since $X \smallsetminus \{x\} = [-1,x) \cup (x,1]$ is disconnected, while $Z \smallsetminus \{f(x)\}$ is connected.

Solution 1.51.

(a) No: removing an interior point of $[0,1]$ gives a disconnected set, but removing any point from the unit circle gives a set that is connected.

(b) No: $[0,1]$ is compact, being closed and bounded in $\mathbf{R}$, while $(0,1)$ is not compact, since it is not closed in $\mathbf{R}$.

(c) Yes: $f\colon [0,1] \longrightarrow [0,2]$ given by $f(x) = 2x$ is clearly a homeomorphism.

Solution 1.52. TODO: maybe include a direct proof of this first direction as an alternative?

Suppose K is compact as a topological space with the subspace topology from X. Let $\iota\colon K \longrightarrow X$ be the inclusion function, which is continuous by Exercise 1.21. It then follows from Proposition 2.39 that $\iota(K)$ is a compact subset of X.

Conversely, suppose K is a compact subset of X. Let $\{U_i\colon i \in I\}$ be an open cover of K in the subspace K. By the definition of the subspace topology, for every U_i there exists an open subset V_i of X such that $U_i = K_i \cap K$. Since $\{U_i\colon i \in I\}$ is an open cover of K in the subspace K, it follows that $\{V_i \in i \in I\}$ is an open cover of K in X. The compactness of K as a subset of X then implies there exists a finite subset J of I such that $K \subseteq \bigcup_{j\in J} V_j$, and thereforoe

$$K = K \cap \Big(\bigcup_{j\in J} V_j\Big) = \bigcup_{j\in J}(K \cap V_j) = \bigcup_{j\in J} U_j.$$

Hence $\{U_i\colon i \in I\}$ has a finite sub-cover, which implies K is compact as a subspace of X.

Sequences

Solution 1.53. The reflexivity $(x_n) \sim (x_n)$ and symmetry $(x_n) \sim (y_n) \iff (y_n) \sim (x_n)$ are very clear. For the transitivity, suppose $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$. Let $\varepsilon > 0$. There exists $N_1 \in \mathbf{N}$ such that $d(x_n, y_n) < \varepsilon/2$ for all $n \geqslant N_1$. There exists $N_2 \in \mathbf{N}$ such that $d(y_n, z_n) < \varepsilon/2$ for all $n \geqslant N_2$. Letting $N = \max\{N_1, N_2\}$ we have (by the triangle inequality)

$$d(x_n, z_n) \leqslant d(x_n, y_n) + d(y_n, z_n) < \varepsilon \quad \text{for all } n \geqslant N.$$

So $(x_n) \sim (z_n)$.

Solution 1.54. Suppose x and x' are two limits of a sequence (x_n). For any $\varepsilon > 0$, there exist $N, N' \in \mathbf{N}$ such that

$$x_n \in \mathbf{B}_{\varepsilon/2}(x) \qquad \text{for all } n \geqslant N \quad \text{and} \quad x_n \in \mathbf{B}_{\varepsilon/2}(x') \qquad \text{for all } n \geqslant N'.$$

Therefore, for $n = \max\{N, N'\}$ we have $x_n \in \mathbf{B}_{\varepsilon/2}(x) \cap \mathbf{B}_{\varepsilon/2}(x')$, which (via the triangle inequality) implies that $d(x, x') < \varepsilon$.

Since this holds for all $\varepsilon > 0$, we conclude that $d(x, x') = 0$ so that $x = x'$.

Solution 1.55.

(a) It is clear that $\varnothing$ and $\mathbf{N}^*$ belong to $\mathcal{T}$.

Suppose $\{U_i\colon i \in I\}$ is a collection of members of $\mathcal{T}$. If $\{U_i\colon i \in I\} \subseteq \mathcal{P}(\mathbf{N})$, then $\bigcup_{i\in I} U_i \in \mathcal{P}(\mathbf{N}) \subseteq \mathcal{T}$. Otherwise, there exists a member V of $\{U_i\colon i \in I\}$ such that $\infty \in V$. It then follows from

$$\mathbf{N}^* \smallsetminus \Big(\bigcup_{i\in I} U_i\Big) \subseteq \mathbf{N}^* \smallsetminus V$$

that $\mathbf{N}^* \smallsetminus \big(\bigcup_{i\in I} U_i\big)$ is finite, and therefore $\bigcup_{i\in I} U_i \in \mathcal{T}$.

For closure under finite intersection, it suffices to prove it for any two members U and V of $\mathcal{T}$. If at most one of U and V contains ∞, then $U \cap V \in \mathcal{P}(\mathbf{N})$. Otherwise, it then follows from

$$\mathbf{N}^* \smallsetminus (U \cap V) = (\mathbf{N}^* \smallsetminus U) \cup (\mathbf{N}^* \smallsetminus V)$$

that $\mathbf{N}^* \smallsetminus (U \cap V)$ is finite, and therefore $U \cap V \in \mathcal{T}$.

(b) Let $\{U_i\colon i \in I\}$ be an open cover of $\mathbf{N}^*$. Pick a member V of the open cover such that $\infty \in V$. Since $V \in \mathcal{T}$, it follows that $\mathbf{N}^* \smallsetminus V$ is finite. For each element x of $\mathbf{N}^* \smallsetminus V$, pick a member V_x of the open cover such that $x \in V_x$. It follows that $\{V\} \cup \{V_x\colon x \in \mathbf{N}^* \smallsetminus V\}$ is a finite sub-cover of $\{U_i \in i \in I\}$. Hence $\mathbf{N}^*$ is compact.

(c) Suppose f is continuous. It follows that for every positive real number ε, the inverse image $f^{-1}\big(\mathbf{B}_\varepsilon(f(\infty))\big)$ is open, and therefore $\mathbf{N}^* \smallsetminus f^{-1}\big(\mathbf{B}_\varepsilon(f(\infty))\big)$ is finite. Hence there exists a natural number N such that $n \geqslant N$ implies $f(n) \in \mathbf{B}_\varepsilon(f(\infty))$.

Conversely, suppose $\big(f(n)\big)$ converges to $f(\infty)$. The space $\mathbf{N}$ is discrete as a subspace of $\mathbf{N}^*$, so $f|_{\mathbf{N}}$ is continuous; this implies f is continuous at every natural number by Exercise 1.18. To apply Exercise 1.18, it remains to prove f is continuous at ∞. Let M be a neighbourhood of ∞ and pick a positive real number ε such that $\mathbf{B}_\varepsilon(f(\infty)) \subseteq M$. Since $f(n) \longrightarrow f(\infty)$ as $n \longrightarrow \infty$, there exists a natural number N such that $n \geqslant N$ implies $f(n) \in \mathbf{B}_\varepsilon(f(\infty))$. This implies

$$\mathbf{N}^* \smallsetminus f^{-1}\big(\mathbf{B}_\varepsilon(f(\infty))\big) \subseteq \{1, \ldots, N\},$$

so $f^{-1}\big(\mathbf{B}_\varepsilon(f(\infty))\big)$ is open. Since $f^{-1}\big(\mathbf{B}_\varepsilon(f(\infty))\big) \subseteq f^{-1}(M)$, it follows that $f^{-1}(M)$ is a neighbourhood of ∞, so f is continuous at ∞. Now apply Exercise 1.18 to f, we see that f is continuous.

(d) Define a function $f\colon \mathbf{N}^* \longrightarrow X$ by

$$f(n) = \begin{cases} x_n & \text{if } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$

By part (c), f is continuous, so it follows from Proposition 2.39 that

$$\{x\} \cup \{x_n\colon n \in \mathbf{N}\} = f(\mathbf{N}^*)$$

is compact.

Solution 1.56. First note that for any n, m we have by the triangle inequality:

$$d(x_n, y_n) \leqslant d(x_n, x_m) + d(x_m, y_n) \leqslant d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n),$$

so

$$d(x_n, y_n) - d(x_m, y_m) \leqslant d(x_n, x_m) + d(y_m, y_n).$$

Similarly:

$$d(x_m, y_m) \leqslant d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m)$$

so that

$$-\big(d(x_m, x_n) + d(y_n, y_m)\big) \leqslant d(x_n, y_n) - d(x_m, y_m).$$

We can summarise this as

$$\big|d(x_n, y_n) - d(x_m, y_m)\big| \leqslant d(x_m, x_n) + d(y_n, y_m).$$

Let $\varepsilon > 0$. There exists $N_1 \in \mathbf{N}$ such that $d(x_n, x_m) < \varepsilon/2$ for all $m, n \geqslant N_1$. There exists $N_2 \in \mathbf{N}$ such that $d(y_n, y_m) < \varepsilon/2$ for all $m, n \geqslant N_2$. Let $N = \max\{N_1, N_2\}$, then for all $n, m \geqslant N$ we have:

$$\big|d(x_n, y_n) - d(x_m, y_m)\big| \leqslant d(x_n, x_m) + d(y_m, y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\big(d(x_n, y_n)\big)$ is a Cauchy sequence in $\mathbf{R}$.

Solution 1.57. It suffices to prove that (x_n) being Cauchy implies (y_n) is Cauchy.

Let $\varepsilon > 0$. As $(y_n) \sim (x_n)$, there exists $N_1 \in \mathbf{N}$ such that $d(y_n, x_n) < \varepsilon/3$ for all $n \geqslant N_1$. As (x_n) is Cauchy, there exists $N_2 \in \mathbf{N}$ such that $d(x_n, x_m) < \varepsilon/3$ for all $n, m \geqslant N_2$. Let $N = \max\{N_1, N_2\}$, then for all $n, m \geqslant N$ we have

$$d(y_n, y_m) \leqslant d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) < \varepsilon.$$

Uniform continuity and completeness

Solution 1.58.

(a) Let $f', g' \colon \overline{D} \longrightarrow X$ denote the restrictions of f and g to $\overline{D}$ respectively. Since D is dense in $\overline{D}$, and the functions f' and g' agree on D, property H implies that $f' = g'$. Hence the result follows.

(b) Let X be a topological space, Y a Hausdorff topological space, D a dense subset of Y. Consider two continuous functions $f, g \colon Y \longrightarrow X$ that agree on D.

Let $(f, g) \colon Y \longrightarrow X \times X$ be the function defined by $(f, g)(y) = (f(y), g(y))$ (see Tutorial Question 3.8). Since both f and g are continuous, it follows from Tutorial Question 3.8 that (f, g) is continuous.

Let Δ denote the diagonal function of X, defined in Tutorial Question 3.9. By part (c) of Tutorial Question 3.9, the Hausdorffness of X implies that $\Delta(X)$ is closed in $X \times X$. It then follows from Exercise 1.17 that $(f, g)^{-1}\big(\Delta(X)\big)$ is closed.

Since $f(y) = g(y)$ for every element of y of D, it follows that $(f, g)(D) \subseteq \Delta(X)$, and thus $D \subseteq (f, g)^{-1}\big(\Delta(X)\big)$. However, $(f, g)^{-1}\big(\Delta(X)\big)$ is closed, so it contains the closure Y of D. This implies that $(f(y), g(y)) \in \Delta(X)$ for every element y of Y; in other words, $f(y) = g(y)$ for every element y of Y.

(c) In part (b), we have shown that Hausdorffness implies property H, so it suffices to prove the other direction.

Let X be a topological space with property H and Δ the diagonal function of X, defined in Tutorial Question 3.9. Define two projections $\pi_1 \colon X \times X \longrightarrow X$ and $\pi_2 \colon X \times X \longrightarrow X$ by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. It follows from Proposition 2.18 that π_1 and π_2 are both continuous. The projections π_1 and π_2 agree on $\Delta(X)$ by definition, so they agree on the closure $\overline{\Delta(X)}$ of $\Delta(X)$ by part (a). Therefore, if $(x,y) \in \overline{\Delta(X)}$, then

$$x = \pi_1(x,y) = \pi_2(x,y) = y,$$

so $(x,y) \in \Delta(X)$. It follows that $\overline{\Delta(X)} = \Delta(X)$.

(d) Follows immediately from the above.

Solution 1.59. (a) Let ε be a positive real number. Put $\delta = \varepsilon$. If elements x_1 and x_2 of S satisfy $d_S(x_1,x_2) < \delta$, then

$$d_X\big(\iota_S(x_1), \iota_S(x_2)\big) = d_S(x_1,x_2) < \varepsilon.$$

Hence ι_S is uniformly continous.

(b) If f is uniformly continuous, then $\iota_S \circ f$ is uniformly continuous because of part (a) and Tutorial Question 5.2.

Conversely, suppose $\iota_S \circ f$ is uniformly continuous. Let ε be a positive real number. Pick a positive real number δ such that $d_Y(y_1,y_2) < \delta$ implies $d_X\big((\iota_S \circ f)(y_1), (\iota_S \circ f)(y_2)\big) < \varepsilon$. It follows that $d_Y(y_1,y_2) < \delta$ implies

$$d_S\big(f(y_1), f(y_2)\big) = d_X\big((\iota_S \circ f)(y_1), (\iota_S \circ f)(y_2)\big) < \varepsilon.$$

Hence f is uniformly continuous.

Solution 1.60. (a) If ε is a positive real number, then $d\big((y_1,z_1),(y_2,z_2)\big) < \varepsilon$ implies

$$d_Y\big(\pi_Y(y_1,z_1), \pi_Y(y_2,z_2)\big) = d_Y(y_1,y_2) \leqslant d\big((y_1,z_1),(y_2,z_2)\big) < \varepsilon$$

and similarly $d\big(\pi_Z(y_1,z_1), \pi_Z(y_2,z_2)\big) < \varepsilon$. Hence π_Y and π_Z are uniformly continuous.

(b) If f is uniformly continuous, then it follows from Tutorial Question 5.2 and part (a) that both $\pi_Y \circ f$ and $\pi_Z \circ f$ are uniformly continuous.

Conversely, suppose both $\pi_Y \circ f$ and $\pi_Z \circ f$ are uniformly continuous. Let ε be a positive real number. It follows from the uniform continuity of $\pi_Y \circ f$ and $\pi_Z \circ f$ that there exist positive real numbers δ_Y resp. δ_Z such that $d_X(x_1,x_2) < \delta_Y$, resp. $d_X(x_1,x_2) < \delta_Z$ imply

$$d_Y\big((\pi_Y \circ f)(x_1), (\pi_Y \circ f)(x_2)\big) < \varepsilon/2 \quad \text{resp.} \quad d_Z\big((\pi_Z \circ f)(x_1), (\pi_Z \circ f)(x_2)\big) < \varepsilon/2.$$

Let $\delta = \min\{\delta_Y, \delta_Z\}$. It follows that $d_X(x_1,x_2) < \delta$ implies

$$d\big(f(x_1), f(x_2)\big) \leqslant d_Y\big((\pi_Y \circ f)(x_1), (\pi_Y \circ f)(x_2)\big) + d_Z\big((\pi_Z \circ f)(x_1), (\pi_Z \circ f)(x_2)\big) < \varepsilon,$$

so f is uniformly continuous.

Solution 1.61. Let $g\colon Y \longrightarrow X$ denote the inverse of f.

(a) By Proposition 2.52, (x_n) being Cauchy implies $\big(f(x_n)\big)$ is Cauchy, while $\big(f((x_n)\big)$ being Cauchy implies $(x_n) = \Big(g\big(f(x_n)\big)\Big)$ is Cauchy.

(b) Since X and Y are interchangeable, it suffices to prove one direction. Suppose X is complete and (y_n) is a Cauchy sequence in Y. It follows that $\big(g(y_n)\big)$ is Cauchy in X, and therefore converges to some point x in X By Theorem 2.44, $(y_n) = \Big(f\big(g(y_n)\big)\Big)$ converges to $f(x)$. Hence Y is complete.

(c) Since f has an inverse $\tan\colon (-\pi/2, \pi/2) \longrightarrow \mathbf{R}$, and both f and tan are continuous. Hence f is a homeomorphism.

Given $x_1 < x_2$, apply the Mean Value Theorem to $f(x) = \arctan(x)$ on $[x_1, x_2]$ to get some $\xi \in (x_1, x_2)$ such that

$$|f(x_2) - f(x_1)| = |f'(\xi)|\,|x_2 - x_1| = \frac{1}{1+\xi^2}\,|x_2 - x_1| \leqslant |x_2 - x_1|.$$

So for any $\varepsilon > 0$ we can take $\delta = \varepsilon$ and conclude that f is uniformly continuous.

However, its inverse $\tan\colon (-\pi/2, \pi/2) \longrightarrow \mathbf{R}$ is not uniformly continuous, because $(-\pi/2, \pi/2)$ is totally bounded (since bounded in $\mathbf{R}$), but $\mathbf{R}$ is not totally bounded. (Use **??**.)

(d) The codomain $(-\pi/2, \pi/2)$ is not complete because $(\pi/2 - 1/n)$ is Cauchy but does not converge in $(-\pi/2, \pi/2)$. However, the domain $\mathbf{R}$ is complete.

Solution 1.62.

(a) Induction on n. Base case $x_1 = 1$ clear.

Fix $n \in \mathbf{N}$ and suppose $1 \leqslant x_n \leqslant 2$. Then

$$\frac{1}{2} \leqslant \frac{x_n}{2} \leqslant 1 \qquad \text{and} \qquad \frac{1}{2} \leqslant \frac{1}{x_n} \leqslant 1,$$

so $1 \leqslant x_{n+1} \leqslant 2$.

(b) Fix $n \in \mathbf{N}$. Noting that $2x_n x_{n+1} = x_n^2 + 2$, we have

$$y_n^2 = \left(x_{n+1} - x_n\right)^2 = x_{n+1}^2 - 2x_{n+1}x_n + x_n^2 = x_{n+1}^2 - 2$$
$$2x_{n+1}y_{n+1} = 2x_{n+1}\left(\frac{1}{x_{n+1}} - \frac{x_{n+1}}{2}\right) = 2 - x_{n+1}^2 = -y_n^2.$$

(c) From part (b) we have

$$|y_{n+1}| = \frac{|y_n|^2}{2x_{n+1}} \qquad \text{for all } n \in \mathbf{N}.$$

We can use this, part (a), and induction by n.

For the base case we have $y_1 = \frac{1}{2}$.

For the induction step, fix $n \in \mathbf{N}$ and suppose $|y_n| \leqslant \frac{1}{2^n}$, then

$$|y_{n+1}| = \frac{|y_n|^2}{2x_{n+1}} \leqslant \frac{|y_n|^2}{2} \leqslant \frac{1}{2^{2n+1}} \leqslant \frac{1}{2^{n+1}}.$$

(d) Let $\varepsilon > 0$ and let $N \in \mathbf{N}$ be such that $2^{N-1} > 1/\varepsilon$. If $n \geqslant m \geqslant N$ then

$$\begin{aligned}
|x_n - x_m| &= |y_{n-1} + y_{n-2} + \cdots + y_m| \\
&\leqslant |y_{n-1}| + \cdots + |y_m| \\
&\leqslant \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^m} \\
&= \left(\frac{1}{2^{n-m-1}} + \frac{1}{2^{n-m-2}} + \cdots + 1\right)\frac{1}{2^m} \\
&\leqslant \frac{2}{2^m} \leqslant \frac{1}{2^N} < \varepsilon.
\end{aligned}$$

Here we used the fact that the geometric series with ratio 1/2 sums up to 2.

(e) Thinking of (x_n) as a sequence in $\mathbf{R}$, it converges to some limit $x \in \mathbf{R}$ by the completeness of $\mathbf{R}$. We can therefore take limits as $n \longrightarrow \infty$ on both sides of the defining relation

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \quad \text{for } n \in \mathbf{N}$$

to get

$$x = \frac{x}{2} + \frac{1}{x} \Rightarrow x^2 = 2.$$

Throwing in the fact that $x \geqslant 1$, we conclude that $x = \sqrt{2}$.

The conclusion that $\mathbf{Q}$ is not complete now follows from the fact that $\sqrt{2} \notin \mathbf{Q}$.

Solution 1.63.

(a) Suppose $\big((x_n, y_n)\big)$ is a Cauchy sequence in $(X \times Y, d)$. By part (a) of Exercise 1.60, both projections $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are uniformly continuous. Hence $(x_n) = \big(\pi_X(x_n, y_n)\big)$ and $(y_n) = \big(\pi_Y(x_n, y_n)\big)$ are Cauchy because of Proposition 2.52.

Conversely, suppose (x_n) is Cauchy in X and (y_n) is Cauchy in Y. Fix $\varepsilon > 0$. Let $N_x \in \mathbf{N}$ be such that for all $m, n \geqslant N_x$ we have $d_X(x_m, x_n) < \varepsilon$. Let $N_y \in \mathbf{N}$ be such that for all $m, n \geqslant N_y$ we have $d_Y(y_m, y_n) < \varepsilon$. Let $N = \max\{N_x, N_y\}$, then for all $m, n \geqslant N$ we have

$$d\big((x_m, y_m), (x_n, y_n)\big) = \max\big\{d_X(x_m, x_n), d_Y(y_m, y_n)\big\} < \varepsilon,$$

so $\big((x_n, y_n)\big)$ is Cauchy in $X \times Y$.

(b) Let $\big((x_n, y_n)\big)$ be a Cauchy sequence in $X \times Y$. By part (a), (x_n) is Cauchy in X and (y_n) is Cauchy in Y. Since X and Y are complete, we have $(x_n) \longrightarrow x \in X$ and $(y_n) \longrightarrow y \in Y$. By Tutorial Question 4.9, $\big((x_n, y_n)\big) \longrightarrow (x, y) \in X \times Y$.

The converse also holds: suppose $X \times Y$ is complete. Let (x_n) be a Cauchy sequence in X, and fix some $y \in Y$. Then by (a) we have that $\big((x_n, y)\big)$ is Cauchy in $X \times Y$, so $\big((x_n, y)\big) \longrightarrow (x, y) \in X \times Y$, which by Tutorial Question 4.9 implies that $(x_n) \longrightarrow x \in X$. The same proof gives us that Y is complete.

Solution 1.64.

(a) Let $\varepsilon > 0$. Set $\delta = \varepsilon$. If $x, x' \in X$ satisfy $d(x, x') < \delta = \varepsilon$, then

$$|f(x) - f(x')| = |d(x, y) - d(x', y)| \leqslant d(x, x') < \varepsilon.$$

(b) Let $\varepsilon > 0$. By part (a), there exists positive real numbers δ_1 and δ_2 such that $d(x_1, x_1') < \delta_1$ and $d(x_2, x_2') < \delta_2$ imply

$$d_{\mathbf{R}}\big(d(x_1, x_2), d(x_1', x_2)\big) < \varepsilon/2 \quad \text{and} \quad d_{\mathbf{R}}\big(d(x_1', x_2), d(x_1', x_2')\big) < \varepsilon/2.$$

Set $\delta = \min\{\delta_1, \delta_2\}$. If $(x_1, x_2), (x_1', x_2') \in X \times X$ satisfy

$$\max\{d(x_1, x_1'), d(x_2, x_2')\} = D\big((x_1, x_2), (x_1', x_2')\big) < \varepsilon$$

then

$$d_{\mathbf{R}}\big(d(x_1, x_2), d(x_1', x_2')\big) \leqslant d_{\mathbf{R}}\big(d(x_1, x_2), d(x_1', x_2)\big) + d_{\mathbf{R}}\big(d(x_1', x_2), d(x_1', x_2')\big) < \varepsilon.$$

Hence d is uniformly continuous.

Solution 1.65. This uses the same approach as Proposition 2.48: we have

$$\left|d(x'_n, y'_n) - d(x_n, y_n)\right| \leqslant d(x'_n, x_n) + d(y'_n, y_n).$$

But by assumption the two distances on the RHS can be made arbitrarily small, so we conclude that $d(x'_n, y'_n)$ and $d(x_n, y_n)$ can be made arbitrarily close, hence they have the same limit.

(This explanation shouldn't keep you from writing a more rigorous proof.)

Solution 1.66. Suppose that a continuous extension $\widehat{f}\colon \mathbf{R}_{\geqslant 0} \longrightarrow \mathbf{R}_{\geqslant 0}$ exists. Consider the sequence $(x_n) = \left(\frac{1}{n}\right) \longrightarrow 0 \in \mathbf{R}_{\geqslant 0}$. By continuity of $\widehat{f}$ we must have

$$\widehat{f}(0) = \widehat{f}\left(\lim_{n\longrightarrow\infty} \frac{1}{n}\right) = \lim_{n\longrightarrow\infty} \widehat{f}\left(\frac{1}{n}\right) = \lim_{n\longrightarrow\infty} f\left(\frac{1}{n}\right) = \lim_{n\longrightarrow\infty} n.$$

But the rightmost limit does not exist (in $\mathbf{R}_{\geqslant 0}$), contradiction.

2. Normed and Hilbert spaces

A. Appendix: Prerequisites

Equivalence relations

Solution A.1.

- Given $x \in A$, we have $f(x) = f(x)$ so $x \sim x$.
- If $x \sim y$, then $f(x) = f(y)$, so $f(y) = f(x)$, that is $y \sim x$.
- If $x \sim y$ and $y \sim z$ then $f(x) = f(y)$ and $f(y) = f(z)$, so that $f(x) = f(z)$, that is $x \sim z$.

Solution A.2. Suppose π is bijective. I claim that the only way $x \sim y$ can happen is if $x = y$: if $x \sim y$ then $\pi(x) = \pi(y)$, but π is bijective so $x = y$.

We conclude that the equivalence relation on A must be given by: $x \sim y$ if and only if $x = y$.

Solution A.3. (a) We check the equivalence relation conditions:

- Given $(a,b) \in \mathbf{N} \times \mathbf{N}$, we have $a + b = b + a$ so $(a,b) \sim (a,b)$.
- If $(a,b) \sim (c,d)$ then $a + d = b + c$, so $c + b = d + a$, that is $(c,d) \sim (a,b)$.
- If $(a,b) \sim (c,d)$ and $(c,d) \sim (x,y)$ then $a + d = b + c$ and $c + y = d + x$. Adding these two equalities gives $a + d + c + y = b + c + d + x$, and cancelling out $c + d$ on both sides we get $a + y = b + x$, that is $(a,b) \sim (x,y)$.

(b) Define $g\colon (A/\sim) \longrightarrow \mathbf{Z}$ by $g\big([(a,b)]\big) = b - a$. We first need to make sure that this is a well-defined function, in other words that the value does not depend on the chosen representative (a,b) of $[(a,b)]$: suppose $(a',b') \in [(a,b)]$, then $(a',b') \sim (a,b)$ so $a' + b = b' + a$, hence $a' - b' = a - b$.

Let's show that g is injective: if $g([(a,b)]) = g([(c,d)])$ then $a - b = c - d$, so $a + d = b + c$, so $(a,b) \sim (c,d)$, so $[(a,b)] = [(c,d)]$.

Finally, to see that g is surjective, let $n \in \mathbf{Z}$. If $n \geqslant 0$ then $n = g([(n+1,1)])$; if $n < 0$ then $n = g([(1,1-n)])$.

Solution A.4. (a) Let $f\colon V \longrightarrow V$. Clearly id_V is unipotent and $f = \mathrm{id}_V \circ f$, so $f \sim f$.

(b) Suppose $f \sim g$ so that $f = u \circ g$, where $(u - \mathrm{id}_V)^k = 0$. Pick $m \in \mathbf{Z}_{\geqslant 1}$ such that $p^m > k$, and observe that

$$0 = (u - \mathrm{id}_V)^{p^m} = u^{p^m} - \mathrm{id}_V$$

as $\mathrm{End}(V)$ has characteristic p. Thus $u^{p^m-1} \circ f = g$, and u^{p^m-1} is unipotent as

$$(u^{p^m-1} - \mathrm{id}_V)^{p^m} = u^{p^m(p^m-1)} - \mathrm{id}_V = 0.$$

(c) Define $f, g \in \mathrm{End}(V)$ by

$$f(s)(1) = s(1) + s(2), \quad f(s)(j) = s(j) \ \forall\ j \neq 1$$
$$g(s)(2) = s(1) + s(2), \quad g(s)(j) = s(j) \ \forall\ j \neq 2.$$

We have $(f - \mathrm{id}_V)^2 = (g - \mathrm{id}_V)^2 = 0$, so f and g are unipotent and thus $f \sim \mathrm{id}_V$ and $g \sim \mathrm{id}_V$. But g is invertible, and

$$(f \circ g^{-1} - \mathrm{id}_V)^3(s)(j) = \begin{cases} s(j) & \text{if } j = 1, 2 \\ 0 & \text{otherwise,} \end{cases}$$

so $(f \circ g^{-1} - \mathrm{id}_V)^{3m} \neq 0$ for all $m \geqslant 1$, and thus $f \circ g^{-1}$ cannot be unipotent, meaning $f \not\sim g$.

Solution A.5. For part (a), reflexiveness follows as $v - v = 0 \in W$, symmetry follows as $v - v' \in W$ implies $-1 \times (v - v') = v' - v \in W$, and transitivity follows as $v - v', v' - v'' \in W$ imply $v - v' + v' - v'' = v - v'' \in W$.

For part (b), if $[v] = [u]$ (and hence $v - u \in W$), then $[v] + [v'] = [v + v'] = [u + v'] = [u] + [v']$ where the middle equality follows as $v + v' - (u + v') = v - u \in W$. This shows addition is well-defined. Similarly, $\lambda[v] = [\lambda v] = [\lambda u] = \lambda[u]$ where the middle equality follows as $\lambda v - \lambda u = \lambda(v - u) \in W$. This shows scalar multiplication is well-defined.

For part (c), define $g([v]) \coloneqq f(v)$, which clearly satisfies $f = g \circ \pi$. To show this is well-defined, suppose $[v] = [v']$ so that $v' - v \in W$. Then

$$g([v]) = f(v) = f(v) + 0 = f(v) + f(v' - v) = f(v + v' - v) = f(v') = g([v']).$$

Also g is linear as $g([v + v']) = f(v + v') = f(v) + f(v') = g([v]) + g([v'])$ and $g(\lambda[v]) = g([\lambda v]) = f(\lambda v) = \lambda f(v) = \lambda g([v])$. Finally to show it is unique, suppose $g_1, g_2 \colon V/W \longrightarrow U$ both satisfy $f = g_1 \circ \pi = g_2 \circ \pi$. Then subtracting these equations gives $0 = (g_1 - g_2) \circ \pi$, which implies $g_1 = g_2 = 0$ as π is surjective.

(Un)countability

Solution A.6.

- Given $S \in X$, the identity function $\mathrm{id}_S \colon S \longrightarrow S$ is bijective, so $S \sim S$.
- If $S \sim T$ then there is a bijective function $f \colon S \longrightarrow T$, so there's a bijective inverse function $f^{-1} \colon T \longrightarrow S$, that is $T \sim S$.
- If $S \sim T$ and $T \sim W$, then there are bijective functions $f \colon S \longrightarrow T$ and $g \colon T \longrightarrow W$. The composition $g \circ f \colon S \longrightarrow W$ is bijective, so $S \sim W$.

Solution A.7. Without loss of generality, we may assume that f is surjective and we want to show that Y is finite or countable.

Also without loss of generality (by pre-composing f with any bijection $\mathbf{N} \longrightarrow X$), we may assume that $f \colon \mathbf{N} \longrightarrow Y$ is surjective.

As $f \colon \mathbf{N} \longrightarrow Y$ is surjective, there exists a right inverse $g \colon Y \longrightarrow \mathbf{N}$, in other words $f \circ g \colon Y \longrightarrow Y$ is the identity function id_Y: given $y \in Y$, the pre-image $f^{-1}(y) \subseteq \mathbf{N}$ is nonempty, so it has a smallest element n_y; we let $g(y) = n_y$. For any $y \in Y$, we have $f(g(y)) = f(n_y) = y$ as $n_y \in f^{-1}(y)$. So $f \circ g = \mathrm{id}_Y$.

In particular, this forces $g \colon Y \longrightarrow \mathbf{N}$ to be injective, hence realising Y as a subset of the countable set $\mathbf{N}$. We conclude by Proposition A.6 that Y is finite or countable.

Solution A.8. Write

$$S = \bigcup_{n \in \mathbf{N}} S_n,$$

with each S_n a countable set. It is clear that S is infinite (as, say, S_1 is, and $S_1 \subseteq S$).

For each $n \in \mathbf{N}$, fix a bijection $\varphi_n : \mathbf{N} \longrightarrow S_n$. (As Chengjing rightfully points out to me, this uses the Axiom of Countable Choice.) Define a function $\psi : \mathbf{N} \times \mathbf{N} \longrightarrow S$ by:

$$\psi\big((n,m)\big) = \varphi_n(m) \in S_n \subseteq S.$$

This is surjective, and $\mathbf{N} \times \mathbf{N}$ is countable, so S is finite or countable, and we ruled out finite above.

Solution A.9. Since B is countable we can enumerate it as $B = \{b_n : n \in \mathbf{N}\}$. For each $n \in \mathbf{N}$, let $W_n = \operatorname{Span}\{b_1, \ldots, b_n\}$. Then for each $n \in \mathbf{N}$, W_n is isomorphic (as a $\mathbf{Q}$-vector space) to $\mathbf{Q}^n$, hence W_n is countable. I claim that

$$W = \bigcup_{n \in \mathbf{N}} W_n.$$

One inclusion is obvious, as $W_n \subseteq W$ for all $n \in \mathbf{N}$. For the other direction, let $w \in W = \operatorname{Span}(B)$, so there exist $n \in \mathbf{N}$, $a_1, \ldots, a_n \in \mathbf{Q}$ and $k_1, \ldots, k_n \in \mathbf{N}$ such that

$$w = a_1 b_{k_1} + \cdots + a_n b_{k_n}.$$

Let $k = \max\{k_1, \ldots, k_n\}$, then $w \in W_k$.

So W is a countable union of countable sets, hence countable by Exercise A.8.

The last claim follows directly from the fact that $\mathbf{R}$ is an uncountable set.

Linear algebra

Solution A.10. TODO

Solution A.11. TODO

Solution A.12. TODO

Solution A.13. We can write $f(\mathbf{x}) = \mathbf{x}^\top A \mathbf{x}$ where $A = (a_{ij})$ is a real matrix. Observe that we can replace A by $A' := \frac{1}{2}(A + \overline{A}^\top)$ and the equation holds true. Since A' is a Hermitian matrix, by the Spectral theorem we can write $A' = P^{-1}BP$ where P is a (unitary) matrix whose columns are orthonormal eigenvectors of A, and $B = \operatorname{diag}(b_1, \ldots, b_n)$ is a diagonal matrix with the corresponding eigenvalues. If $\mathbf{r}_i$ denotes the i^{th} row of P^{-1}, then setting $g_i(\mathbf{x}) := \mathbf{r}_i \mathbf{x}$ gives the desired result.

Comment. The result still holds true if we allow $a_{ij} \in \mathbf{C}$, but the above proof does not apply; research Takagi factorisation.

Uniform continuity and uniform convergence

Solution A.14. (a) Take (for example) $f_n(x) = e^{x-n}$, which converges pointwise to $f(x) = 0$.

(b) Suppose for the sake of contradiction that f is uniformly continuous. Let $\varepsilon > 0$ be given. By uniform convergence, there exists $N > 0$ such that $|f_N(x) - f(x)| < \varepsilon/3$ for all $x \in \mathbf{R}$. Also by the uniform continuity of f, there exists $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon/3$ whenever $|x - x'| < \delta$. Then for all $x, x' \in \mathbf{R}$ with $|x - x'| < \delta$, we have

$$\begin{aligned} |f_N(x) - f_N(x')| &\leqslant |f_N(x) - f(x)| + |f(x) - f(x')| + |f(x') - f_N(x')| \\ &= \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon, \end{aligned}$$

which contradicts the fact that f_N is not uniformly continuous.

Solution A.15. (a) Take for example $f_n(x) = e^{-x^{2n}}$, which converges pointwise to $f(x) = \mathbf{1}_{\{0\}}(x)$.

(b) Let $\varepsilon > 0$ be given. By uniform convergence, there exists some f_n such that $|f_n(x) - f(x)| < \varepsilon/3$ for all $x \in \mathbf{R}$. By the uniform continuity of f_n, there exists some $\delta > 0$ such that $|f_n(x) - f_n(x')| < \varepsilon/3$ whenever $|x - x'| < \delta$. Then by the triangle inequality, for all $x, x' \in \mathbf{R}$ satisfying $|x - x'| < \delta$, we have

$$\begin{aligned} |f(x) - f(x')| &\leqslant |f(x) - f_n(x)| + |f_n(x) - f_n(x')| + |f_n(x') - f(x')| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon. \end{aligned}$$

B. Appendix: Miscellaneous

Zorn's Lemma

Solution B.1. The fact that $\subseteq$ is a partial order follows directly from known properties of set inclusion.

If Ω has at least two distinct elements x_1 and x_2, then $\{x_1\}$ and $\{x_2\}$ are not comparable under $\subseteq$, so the latter is not a total order.

Solution B.2. We proceed by induction on n, the cardinality of X.

Base case: if $n = 1$ then $X = \{x\}$ for a single element x. Then trivially x is a maximal element of X.

For the induction step, fix $n \in \mathbf{N}$ and suppose that any poset of cardinality n has a maximal element. Let X be a poset of cardinality $n+1$ and choose an arbitrary element $x \in X$. Let $Y = X \setminus \{x\}$, then Y is a poset of cardinality n so by the induction hypothesis has a maximal element m_Y, and clearly $m_Y \neq x$.

We have two possibilities now:

- If $m_Y \leqslant x$, then x is a maximal element of X. Why? Suppose that x is not maximal in X, so that there exists $z \in X$ such that $z \neq x$ and $x \leqslant z$. Since $z \neq x$, we must have $z \in Y$. If $z = m_Y$, then $z \leqslant x$ and $x \leqslant z$ so $z = x$, contradiction. So $z \neq m_Y$, and $m_Y \leqslant x$ and $x \leqslant z$, so $m_Y \leqslant z$, contradicting the maximality of m_Y in Y.
- Otherwise, (if it is not true that $m_Y \leqslant x$), m_Y is a maximal element of X. Why? Suppose there exists $z \in X$ such that $z \neq m_Y$ and $m_Y \leqslant z$. Since $m_Y \leqslant x$ is not true, we have $z \neq x$, so $z \in Y$, contradicting the maximality of m_Y in Y.

In either case we found a maximal element for X.

An alternative approach is to proceed by contradiction: suppose $(X, \leqslant)$ is a nonempty finite poset that does not have a maximal element. Use this to construct an unbounded chain of elements of X, contradicting finiteness.

Solution B.3. If $V = \{0\}$, then $\varnothing$ is vacuously a (in fact, the only) basis of V.

Suppose $V \neq \{0\}$. If $v \in V \setminus \{0\}$, then $\{v\}$ is a linearly independent subset of V. Let X be the set of all linearly independent subsets of V, then X is nonempty. We consider the partial order $\subseteq$ on X given by inclusion of subsets.

Let C be a nonempty chain in X and define

$$U = \bigcup_{S \in C} S,$$

then clearly $S \subseteq U$ for all $S \in C$, so we'll know that U is an upper bound for C as soon as we can show that it is linearly independent (so that $U \in X$).

Suppose there exist $n \in \mathbf{N}$, $a_1, \dots, a_n \in \mathbf{F}$, and $u_1, \dots, u_n \in U$ such that

$$a_1 u_1 + \cdots + a_n u_n = 0. \tag{B.1}$$

Let $J = \{1, \dots, n\}$. For each $j \in J$, there exists $S_j \in C$ such that $u_j \in S_j$. As C is totally ordered, there exists $i \in J$ such that $S_j \subseteq S_i$ for all $j \in J$. But this means that $u_1, \dots, u_n \in S_i$, so that the linear relation of Equation (B.1) takes place in the linearly independent set S_i. Therefore $a_1 = \cdots = a_n = 0$.

We conclude that X satisfies the conditions of Zorn's Lemma, hence it has a maximal element B. I claim that B spans V, so that it is a basis of V.

We prove this last claim by contradiction: if $v \in V \smallsetminus \mathrm{Span}(B)$, then $B' := B \cup \{v\}$ is linearly independent, hence an element of X. But $B \subseteq B'$ and $B \neq B'$, contradicting the maximality of B.

Solution B.4. (a) Clearly $(A, s_A) \leqslant (A, s_A)$. Now if $(A, s_A) \leqslant (B, s_B)$ and $(B, s_B) \leqslant (A, s_A)$, then $A \subseteq B \subseteq A \implies A = B$, and thus $s_A\big|_B = s_A = s_B = s_B\big|_A$. For the last condition, if $(A, s_A) \leqslant (B, s_B)$ and $(B, s_B) \leqslant (C, s_C)$, then clearly $A \subseteq C$, and $s_C\big|_A = s_C\big|_B\big|_C = s_B\big|_A = s_A$.

(b) Let $\mathcal{C} = \{(A_i, s_{A_i})\}_{i\in I}$ be a nonempty chain in $P(f)$. Define $A := \bigcup_{i\in I} A_i$, and $s_A(y) = s_{A_i}(y)$ if $y \in A_i$. This is well-defined as if $y \in A_i \cap A_j$, then without loss of generality $A_i \leqslant A_j$, and so $s_{A_i}(y) = s_{A_j}\big|_{A_i}(y) = s_{A_j}(y)$. Observe that $A_i \subseteq A$ and $s_A\big|_{A_i} = s_{A_i}$ for all $i \in I$, so we have constructed the desired upper bound.

(c) We deduce from the previous part and Zorn's lemma that there exists a maximal element $(M, s_M) \in P(f)$. Suppose that $M \neq Y$; then there exists $y_0 \in Y \smallsetminus M$. By the surjectivity of f, there exists $x_0 \in X$ such that $f(x_0) = y_0$. Then we can define $M' = M \cup \{y_0\}$ and $s_{M'}$ by $s_{M'}\big|_M = s_M$ and $s_{M'}(y_0) = x_0$ so that $f \circ s_{M'} = \mathrm{id}_{M'}$. But this contradicts the maximality of (M, s_M), so $M = Y$ and we obtain the desired map $s = s_M$.

Linear algebra

Solution B.5. Let $S = \{e_1, e_2, \ldots\}$ and $W = \mathrm{Span}(S)$.

For each $n \in \mathbf{N}$, define

$$W_n = \mathrm{Span}\{e_1, e_2, \ldots, e_n\} \subseteq W.$$

I claim that

$$W = \bigcup_{n\in\mathbf{N}} W_n.$$

One inclusion is clear, as $W_n \subseteq W$ for all $n \in \mathbf{N}$.

For the other inclusion, let $w \in W$. Then there exist $m \in \mathbf{N}$, $a_1, \ldots, a_m \in \mathbf{R}$ and $k_1, \ldots, k_m \in \mathbf{N}$ such that

$$w = a_1 e_{k_1} + \cdots + a_m e_{k_m}.$$

Set $n = \max\{k_1, \ldots, k_m\}$, then $w \in W_n$.

Is $W = \mathbf{R}^{\mathbf{N}}$? No. Any $w \in W$ appears in a W_n for some $n \in \mathbf{N}$, therefore only the first n entries of w can be nonzero. This means, for instance, that $v = (1, 1, 1, \ldots) \notin W$. So S does not span $\mathbf{R}^{\mathbf{N}}$.

Solution B.6. This is a straightforward rewriting of the definition of algebraic: α is algebraic if and only if it satisfies a polynomial equation with coefficients in $\mathbf{Q}$, which is equivalent to a nontrivial linear relation between the powers of α, which exists if and only if T is linearly dependent.

Solution B.7. We have to prove that $\mathrm{ev}_\alpha\colon V \longrightarrow \mathbf{F}$ is linear.

If $f_1, f_2 \in \mathbf{F}[x]$, then

$$\mathrm{ev}_\alpha(f_1 + f_2) = (f_1 + f_2)(\alpha) = f_1(\alpha) + f_2(\alpha) = \mathrm{ev}_\alpha(f_1) + \mathrm{ev}_\alpha(f_2).$$

If $f \in \mathbf{F}[x]$ and $\lambda \in \mathbf{F}$, then

$$\mathrm{ev}_\alpha(\lambda f) = (\lambda f)(\alpha) = \lambda f(\alpha) = \lambda\, \mathrm{ev}_\alpha(f).$$

Solution B.8. As in Proposition B.2, we have $B = (v_1, \dots, v_n)$ and $B^\vee = (v_1^\vee, \dots, v_n^\vee)$. Write (a_{ij}) for the entries of the matrix M. For future reference, the i-th row of M is

$$\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}.$$

By the definition of matrix representations, we have

$$\begin{aligned} T(v_1) &= a_{11}v_1 + a_{21}v_2 + \dots + a_{n1}v_n \\ T(v_2) &= a_{12}v_1 + a_{22}v_2 + \dots + a_{n2}v_n \\ &\vdots \\ T(v_n) &= a_{1n}v_1 + a_{2n}v_2 + \dots + a_{nn}v_n. \end{aligned}$$

The i-th column of $M^\vee$ is given by the $B^\vee$-coordinates of the vector $T^\vee(v_i^\vee) = v_i^\vee \circ T$. To determine these, we apply $v_i^\vee \circ T$ to the basis vectors $v_1, \dots, v_n$:

$$T^\vee(v_i^\vee)(v_j) = (v_i^\vee \circ T)(v_j) = v_i^\vee(T(v_j)) = v_i^\vee(a_{1j}v_1 + a_{2j}v_2 + \dots + a_{nj}v_n) = a_{ij}.$$

This means that

$$T^\vee(v_i^\vee) = a_{i1}v_1^\vee + a_{i2}v_2^\vee + \dots + a_{in}v_n^\vee$$

and the i-th column of $M^\vee$ is

$$\begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix},$$

precisely the i-th row of M.

We conclude that $M^\vee = M^T$, the transpose of the matrix M.

Solution B.9.

(a) Given $\varphi_1, \varphi_2 \in V^\vee$, we have

$$\begin{aligned} \Gamma(\varphi_1 + \varphi_2) &= ((\varphi_1 + \varphi_2)(v_1), \dots, (\varphi_1 + \varphi_2)(v_n)) \\ &= (\varphi_1(v_1), \dots, \varphi_1(v_n)) + (\varphi_2(v_1), \dots, \varphi_2(v_n)) \\ &= \Gamma(\varphi_1) + \Gamma(\varphi_2). \end{aligned}$$

Given $\varphi \in V^\vee$ and $\lambda \in \mathbf{F}$, we have

$$\begin{aligned} \Gamma(\lambda\varphi) &= ((\lambda\varphi)(v_1), \dots, (\lambda\varphi)(v_n)) \\ &= (\lambda\varphi(v_1), \dots, \lambda\varphi(v_n)) \\ &= \lambda\Gamma(\varphi). \end{aligned}$$

(b) Suppose Γ is injective. Let $W = \mathrm{Span}\{v_1, \dots, v_n\}$. We want to prove that $W = V$.

Suppose $W \neq V$. Let $C = \{w_1, \dots, w_k\}$ be a basis of W and extend it to a basis $B = \{w_1, \dots, w_k, w_{k+1}, \dots, w_m\}$ of V.

Let $B^\vee$ be the dual basis to B and consider its last element $v_m^\vee$ given by

$$v_m^\vee(a_1w_1 + \dots + a_mw_m) = a_m.$$

Then $v_m^\vee \neq 0$ (since $v_m^\vee(w_m) = 1$, for instance) but $v_m^\vee(w) = 0$ for all $w \in W$. In particular, $v_m^\vee(v_1) = \dots = v_m^\vee(v_n) = 0$, so $\Gamma(v_m^\vee) = 0$, contradicting the injectivity of Γ.

We conclude that $W = V$, in other words $\{v_1, \dots, v_n\}$ spans V.

Conversely, suppose $\{v_1, \dots, v_n\}$ spans V. If $\varphi_1, \varphi_2 \in V^\vee$ are such that $\Gamma(\varphi_1) = \Gamma(\varphi_2)$, then $\Gamma(\varphi_1 - \varphi_2) = 0$, so setting $\varphi = \varphi_1 - \varphi_2$, we want to show that $\varphi = 0$, the constant zero function.

If $\varphi \neq 0$, then there exists $v \in V - \{0\}$ such that $\varphi(v) \neq 0$. Since $\{v_1, \dots, v_n\}$ spans V, then we can write v as

$$v = b_1 v_1 + \dots + b_n v_n.$$

But $\Gamma(\varphi) = 0$, so

$$0 \neq \varphi(v) = b_1 \varphi(v_1) + \dots + b_n \varphi(v_n) = 0,$$

which is a contradiction. So we must have $\varphi = 0$, that is $\varphi_1 = \varphi_2$. We conclude that Γ is injective.

(c) Suppose $\Gamma\colon V^\vee \longrightarrow \mathbf{F}^n$ is surjective. Let

$$a_1 v_1 + \dots + a_n v_n = 0$$

be a linear relation.

Let $i \in \{1, \dots, n\}$. Since Γ is surjective, given the standard basis vector $e_i \in \mathbf{F}^n$ (1 in the i-th entry), there exists $\varphi_i \in V^\vee$ such that $\Gamma(\varphi_i) = e_i$. If we apply φ_i on both sides of the linear relation, we get

$$a_i = 0.$$

Since this holds for all i, the relation is trivial.

Conversely, suppose $\{v_1, \dots, v_n\}$ is linearly independent. This set can be enlarged to a basis $B = \{v_1, \dots, v_n, v_{n+1}, \dots, v_m\}$ of V, with dual basis $v_1^\vee, \dots, v_m^\vee$.

Now take an arbitrary vector in $\mathbf{F}^n$:

$$w = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

Let

$$\varphi = a_1 v_1^\vee + \dots + a_n v_n^\vee,$$

then

$$\Gamma(\varphi) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = w.$$

We conclude that Γ is surjective.

Solution B.10. (a) Suppose $T^\vee(\ell) = 0$, that is $\ell \circ T$ is the zero map. But since T is surjective, this implies $\ell = 0 \in W^\vee$.

(b) Let A be the matrix representation of T with respect to some basis $B = (b_1, \dots, b_n)$; recall that A^T is the matrix representation of $T^\vee$ with respect to the basis $B^\vee = (b_1^\vee, \dots, b_n^\vee)$. Since T is injective, $\operatorname{rank}(A) = n = \dim(V)$. Then $\operatorname{rank}(A^\mathsf{T}) = n = \dim(V^\vee)$, so A^T has full-rank and thus $T^\vee$ is surjective.

(c) Let V be the vector space of finitely supported real sequences, that is

$$V = \left\{(x_1, x_2, \dots) \in \mathbf{R}^\mathbf{N} : \text{finitely many } x_i \neq 0\right\},$$

and let $W = \mathbf{R}^\mathbf{N}$ be the space of all real sequences. Clearly $V \hookrightarrow W$ is injective. But the induced map $W^\vee \longrightarrow V^\vee$ is not surjective; the functional $(x_1, x_2, \dots) \longmapsto x_1 + x_2 + \dots$ in $V^\vee$ does not extend to a functional in $W^\vee$.

Topological groups

Solution B.11.

(a) By Exercise 1.26, if G is Hausdorff then the singleton $\{e\}$ is closed.

Conversely, suppose $\{e\}$ is a closed subset of G. Consider the map $f\colon G \times G \longrightarrow G$ given by $f(g,h) = g^{-1}h$, then f is continuous and

$$f^{-1}(e) = \{(g,g)\colon g \in G\} = \Delta(G)$$

(see Tutorial Question 3.9). Since f is continuous and $\{e\}$ is closed, $\Delta(G)$ is closed in $G \times G$, so by Tutorial Question 3.9, G is Hausdorff.

(b) We have

$$Z = \{g \in G\colon gxg^{-1}x^{-1} = e \text{ for all } x \in G\} = \bigcap_{x \in G} \{g \in G\colon gxg^{-1}x^{-1} = e\}$$

which is an intersection of closed sets, since each of the sets is the inverse image of $\{e\}$ under the continuous map $g \longmapsto gxg^{-1}x^{-1}$.

(c) The assertion is immediate from $\ker(f) = f^{-1}(e)$.

Solution B.12. In this proof, we will keep using the following fact: if U is a neighbourhood of some element g of G, and if g' is another element of G, then $g'U$ is a neighbourhood of $g'g$. This follows from the equation $g'U = L_{g'^{-1}}(U)$ and the continuity of $L_{g'^{-1}}$ (see Proposition B.9).

(a)⇒(b): This follows from Exercise 1.18.

(b)⇒(c): Suppose f is continuous at some element g of G. Since f is a group homomorphism, $f(e_G) = e_H$. If U is a neighbourhood of e_H, then $f(g)U$ is a neighbourhood of g, so $f^{-1}\big(f(g)U\big)$ is a neighbourhood of g. Since

$$x \in f^{-1}(U) \iff f(x) \in U \iff f(gx) \in f(g)U \iff gx \in f^{-1}\big(f(g)U\big),$$

it follows that $f^{-1}(U) = g^{-1}f^{-1}\big(f(g)U\big)$, so $f^{-1}(U)$ is a neighbourhood of e_G.

(c)⇒(a): Using similar arguments as in the proof for (b)⇒(c), we can prove that continuity at e_G implies continuity at every element of G. Hence f is continuous by Exercise 1.18.

Solution B.13.

(a) Let $v = f(1) \in V$.

For $n \in \mathbf{N}$ we have

$$f(n) = f(1 + 1 + \cdots + 1) = f(1) + \cdots + f(1) = nv.$$

For $m \in \mathbf{N}$ we have

$$v = f(1) = f\left(\frac{1}{m} + \cdots + \frac{1}{m}\right) = mf\left(\frac{1}{m}\right),$$

so $f(1/m) = (1/m)\,v$.

Therefore, for any $n, m \in \mathbf{N}$ we have

$$f\left(\frac{n}{m}\right) = nf\left(\frac{1}{m}\right) = \frac{n}{m}v.$$

Combining this with $f(-a) = -f(a)$ and $f(0) = 0$, we conclude that $f(x) = xv = xf(1)$ for all $x \in \mathbf{Q}$.

(b) Let $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ be additive. Let $g\colon \mathbf{Q} \longrightarrow \mathbf{R}$ be the restriction of f to $\mathbf{Q} \subseteq \mathbf{R}$. Let $a = g(1) = f(1)$.

By part (b), $g(q) = q\,g(1) = qa$ for all $q \in \mathbf{Q}$. Let $x \in \mathbf{R}$. As $\mathbf{Q}$ is dense in $\mathbf{R}$, there is some sequence $(q_n) \longrightarrow x$ with $q_n \in \mathbf{Q}$; since f is continuous we have

$$f(x) = f\left(\lim_{n\longrightarrow\infty} q_n\right) = \lim_{n\longrightarrow\infty} f(q_n) = \lim_{n\longrightarrow\infty} g(q_n) = \lim_{n\longrightarrow\infty} (q_n a) = xa = xf(1).$$

Hence f is $\mathbf{R}$-linear.

(c) It follows from Exercise B.12 that f is continuous, so by part (c) f is $\mathbf{R}$-linear.

(d) Let B be a $\mathbf{Q}$-basis for $\mathbf{R}$. Exactly one element of B is a nonzero rational, and without loss of generality we may assume it is 1. Since B is uncountable, it also contains uncountably many irrationals. Let $b, c \in B \cap (\mathbf{R} \smallsetminus \mathbf{Q})$. Consider the bijective function $\sigma\colon B \longrightarrow B$ given by

$$\sigma(b) = c, \qquad \sigma(c) = b, \qquad \sigma(x) = x \text{ for all } x \in B \smallsetminus \{b, c\}.$$

Since B is a $\mathbf{Q}$-basis of $\mathbf{R}$, σ extends by $\mathbf{Q}$-linearity to a $\mathbf{Q}$-linear transformation $f\colon \mathbf{R} \longrightarrow \mathbf{R}$. In particular, f is additive.

Suppose that f is $\mathbf{R}$-linear, then:

$$c = f(b) = bf(1) = b1 = b,$$

contradicting the fact that $b \neq c$.

Solution B.14.

(a) Suppose H is open. If g is an element of G, then gH is open because $gH = L_{g^{-1}}^{-1}(H)$ and $L_{g^{-1}}$ is continuous by Proposition B.9. Now the result follows from the equation

$$G \smallsetminus H = \bigcup_{g \notin H} gH.$$

The converse does not hold. If $G = \mathbf{R}$, which is given the Euclidean topology, and if $H = \{0\}$, then H is a closed subgroup of G but it is not open.

(b) Suppose H is closed. If g is an element of G, then $L_{g^{-1}}$ is continuous by Proposition B.9, so $gH = L_{g^{-1}}^{-1}(H)$ is closed because of Exercise 1.17. Since H is of finite index, it has are only finitely many cosets $H, g_1H, \ldots, g_nH$. It follows that

$$G \smallsetminus H = \bigcup_{n=1}^{n} gH = G,$$

which is closed because it is a finite union of closed sets. Hence H is open.

The converse does not hold. Let $G = \mathbf{R}$ but endow it with the discrete topology, and let $H = \mathbf{Z}$. Then H is open in G but it is not of finite index (because if it is, then $\mathbf{R}$ is a finite union of countable sets, and is thus countable by Exercise A.8).

(c) Arguing as in part (a), we have

$$G = \bigcup_{g \in G} gH,$$

so $\{gH\colon g \in G\}$ is an open cover of G. Since G is compact, this open cover admits a finite sub-cover, which implies that H has finite index.

(d) Yes. Let G be any infinite group with the discrete topology, and let $H = \{e\}$, then H is open in G but it does not have finite index.

Solution B.15.

(a) If g is an element of S, then gT is open because $gT = L_{g^{-1}}^{-1}(T)$ and $L_{g^{-1}}$ is continuous by Proposition B.9. It then follows from

$$ST = \bigcup_{s \in S} sT$$

that ST is open.

(b) If S or T is empty, then $ST \neq \varnothing$, so it is connected. Otherwise, the product $S \times T$ is connected by Exercise 1.45, so $ST = m(S \times T)$ is connected by Proposition 2.32.

(c) The product $S \times T$ is compact by Theorem 2.41, so $ST = m(S \times T)$ is compact by Proposition 2.39.

(d) Since inversion is a homeomorphism, it follows from Proposition 2.39 that S^{-1} is compact. The inclusion $j\colon S^{-1} \times G \longrightarrow G \times G$ is continuous by Exercise 1.21. Since T is closed, it follows from Exercise 1.17 that $m^{-1}(T) \subseteq G \times G$ is closed and then $j^{-1}\big(m^{-1}(T)\big) \subseteq S^{-1} \times G$ is closed.

We now claim that

$$ST = \pi_2\Big(j^{-1}\big(m^{-1}(T)\big)\Big);$$

and this implies ST is closed because π_2 is closed by Theorem 2.41 (here we crucially need S^{-1} to be compact). To prove this equation, we start with an element g of ST. Since $g \in ST$, there exists an element s of S and an element t of T such that $g = st$. It follows that $(s^{-1}, g) \in j^{-1}\big(m^{-1}(T)\big)$, so

$$g \in \pi_2\Big(j^{-1}\big(m^{-1}(T)\big)\Big).$$

For the other inclusion, suppose $(s', g) \in j^{-1}\big(m^{-1}(T)\big)$, It follows that $s'g \in T$, so $g \in s'^{-1}T$, which implies $g \in ST$ because $s' \in S^{-1}$. Hence the equation holds.

(e) Since $\mathbf{Z} + \pi\mathbf{Z} = \bigcup_{n \in \mathbf{Z}}(n + \pi\mathbf{Z})$, it follows from Exercise A.8 that $\mathbf{Z} + \pi\mathbf{Z} \neq \mathbf{R}$; but we know it is dense in $\mathbf{R}$, so it cannot be closed. Hence $\mathbf{Z}$ and $\pi\mathbf{Z}$ are closed in $\mathbf{R}$, but $\mathbf{Z} + \pi\mathbf{Z}$ is not closed.